The Signless Laplacian Spectral Radius of Tricyclic Graphs with \( k \) Pendant Vertices

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Abstract In this paper, we determine the unique graph with the largest signless Laplacian spectral radius among all the tricyclic graphs with \( n \) vertices and \( k \) pendant vertices.

Keywords signless Laplacian spectral radius; tricyclic graph; pendant vertex.

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1. Introduction

Let \( G = (V, E) \) be a simple connected graph with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) \). Denote by \( d(v_i) \) the degree of the graph \( G \), \( N(v_i) \) the set of vertices which are adjacent to vertex \( v_i \). Let \( A(G) \) be the adjacency matrix and \( Q(G) = D(G) + A(G) \) be the signless Laplacian matrix of the graph \( G \), where \( D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n)) \) denotes the diagonal matrix of vertex degrees of \( G \). It is well known that \( Q(G) \) is a positive semidefinite matrix. Hence the eigenvalues of \( Q(G) \) can be ordered as

\[ q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0. \]

The largest eigenvalues of \( A(G) \), \( L(G) = D(G) - A(G) \) and \( Q(G) \) are called the spectral radius, the Laplacian spectral radius and the signless Laplacian spectral radius of \( G \), respectively. The signless Laplacian spectral radius is denoted by \( q(G) \) for convenience. It is easy to see that if \( G \) is connected, then \( Q(G) \) is nonnegative irreducible matrix. By the Perron-Frobenius theory, we can see that \( q(G) \) has multiplicity one and exists a unique positive unit eigenvector corresponding to \( q(G) \). We refer to such an eigenvector as the Perron vector of \( G \).

A tricyclic graph is a connected graph with the number of edges equal to the number of vertices plus two. Denote by \( T_n^k \) the set of tricyclic graphs on \( n \) vertices and \( k \) pendant vertices.

Recently, the problem concerning graphs with maximal spectral radius or the Laplacian spectral radius of a given class of graphs has been studied by many authors. Guo [1] determined the graph
with the largest spectral radius among all the unicyclic and bicyclic graphs with \( n \) vertices and \( k \) pendant vertices. Guo [2] determined the graph with the largest Laplacian spectral radius among all the unicyclic and bicyclic graphs with \( n \) vertices and \( k \) pendant vertices. Guo and Wang [3] also determined the graph with the largest Laplacian spectral radius among all the tricyclic graphs with \( n \) vertices and \( k \) pendant vertices. Geng and Li [4] determined the graph with the largest spectral radius among all the tricyclic graphs with \( n \) vertices and \( k \) pendant vertices. In this paper, we determine the unique graph with the largest signless Laplacian spectral radius among all the tricyclic and graph with \( n \) vertices and \( k \) pendant vertices.

Denote by \( C_n \) the cycle on \( n \) vertices. And a path \( P : vv_1v_2 \cdots v_k \) is such a graph that \( v_1 \) joins \( v \) and \( v_{i+1} \) joins \( v_i \) (\( i = 1, 2, \ldots, k - 1 \)).

2. Preliminaries

Let \( G - x \) or \( G - xy \) denote the graph obtained from \( G \) by deleting the vertex \( x \in V(G) \) or the edge \( xy \in E(G) \). Similarly, \( G + xy \) is a graph obtained from \( G \) by adding an edge \( xy \), where \( x, y \in V(G) \) and \( xy \notin E(G) \). A pendant vertex of \( G \) is a vertex with degree 1. A path \( P : vv_1v_2 \cdots v_k \) in \( G \) is called a pendant path, where \( v_1 \) is adjacent to \( v_{i+1} \) (\( i = 0, 1, \ldots, k - 1 \)) and \( d(v_1) = d(v_2) = \cdots = d(v_{k-1}) = 2, d(v_k) = 1 \). If \( k = 1 \), then we say \( vv_1 \) is a pendant edge of the graph \( G \). \( k \) paths \( P_{l_1}, P_{l_2}, \ldots, P_{l_k} \) are said to have almost equal lengths if \( l_1, l_2, \ldots, l_k \) satisfy \( |l_i - l_j| \leq 1 \) for \( 1 \leq i, j \leq k \). We know, by [5], that a tricyclic graph \( G \) contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in \( G \). Then let \( T_n^k = T_n^{k,3} \cup T_n^{k,4} \cup T_n^{k,6} \cup T_n^{k,7} \), where \( T_n^{k,i} \) denotes the set of tricyclic graphs in \( T_n^k \) with exact \( i \) cycles for \( i = 3, 4, 6, 7 \).

In order to complete the proof of our main result, we need the following lemmas.

**Lemma 1** ([6,7]) Let \( G \) be a connected graph, and \( u, v \) be two vertices of \( G \). Suppose that \( v_1, v_2, \ldots, v_s \in N(v) \setminus (N(u) \cup \{u\}) \) (\( 1 \leq s \leq d(v) \)) and \( x = (x_1, x_2, \ldots, x_n) \) is the Perron vector of \( G \), where \( x_i \) corresponds to the vertex \( v_i \) (\( 1 \leq i \leq n \)). Let \( G^* \) be the graph obtained from \( G \) by deleting the edges \( vv_i \) and adding the edges \( uw_i \) (\( 1 \leq i \leq s \)). If \( x_u \geq x_v \), then \( q(G) < q(G^*) \).

Let \( G \) be a connected graph, and \( uv \in E(G) \). The graph \( G_{uv} \) is obtained from \( G \) by subdividing the edge \( uv \), i.e., adding a new vertex \( w \) and edges \( uw, uw \) in \( G - uv \).

An internal path of a graph \( G \) is a sequence of vertices \( v_1, v_2, \ldots, v_m \) with \( m \geq 2 \) such that:

1. The vertices in the sequences are distinct (except possibly \( v_1 = v_m \));
2. \( v_i \) is adjacent to \( v_{i+1} \) (\( i = 1, 2, \ldots, m - 1 \));
3. The vertex degrees \( d(v_1) \) satisfy \( d(v_1) \geq 3, d(v_2) = \cdots = d(v_{m-1}) = 2 \) (unless \( m = 2 \)) and \( d(v_m) \geq 3 \).

By similar reasoning to that of Theorem 3.1 of [8] and Lemmas 2 and 7 of [15], we have the following result.

**Lemma 2** Let \( P : v_1v_2 \cdots v_k \) (\( k \geq 2 \)) be an internal path of a connected graph \( G \). Let \( G' \) be a graph obtained from \( G \) by subdividing some edge of \( P \). Then we have \( q(G') < q(G) \).
Let \( m_i = \frac{\sum_{v_j \in E(v_i)} d(v_j)}{d(v_i)} \) be the average of the degrees of the vertices of \( G \) adjacent to \( v_i \), which is called average 2-degree of vertex \( v_i \).

From the proof of Theorem 3 of [9] and Theorem 2.10 of [10], we have the following result.

**Lemma 3** If \( G \) is a graph, then
\[
q(G) \leq \max\{ \frac{d_u(d_u + m_u) + d_v(d_v + m_v)}{d_u + d_v} : uv \in E(G) \}
\]
with equality if and only if \( G \) is regular or semiregular bipartite.

Let \( S(G) \) be a graph obtained by subdividing every edge of \( G \). Then

**Lemma 4** ([11, 12]) Let \( G \) be a graph on \( n \) vertices and \( m \) edges, \( P_G(x) = \text{det}(xI - A(G)) \), \( Q_G(x) = \text{det}(xI - Q(G)) \). Then \( P_{S(G)} = x^{m-n}Q_G(x^2) \).

**Lemma 5** Let \( u \) be a vertex of a connected graph \( G \) and \( d(u) \geq 2 \). Let \( G_{k,l} \) \( (k, l \geq 0) \) be the graph obtained from \( G \) by attaching two pendant paths of lengths \( k \) and \( l \) at \( u \), respectively. If \( k \geq l \geq 1 \), then \( q(G_{k,l}) > q(G_{k+1,l-1}) \).

**Proof** Let \( S_1 = S(G_{k,l}) \) and \( S_2 = S(G_{k+1,l-1}) \). It is easy to see that \( S_1 \) \((S_2)\) can be obtained from \( S(G) \) by attaching pendant paths of lengths \( 2k-1 \) \((2k+1)\) and \( 2l-1 \) \((2l-3)\) at \( u \), respectively. Then applying Theorem 5 ([13]) and Lemma 4, we have
\[
\rho(S_1) > \rho(S_2),
\]
and consequently \( q(G_{k,l}) > q(G_{k+1,l-1}) \). \( \square \)

**Lemma 6** ([6]) Let \( G \) be a simple graph on \( n \) vertices which has at least one edge. Then
\[
\triangle(G) + 1 \leq q(G) \leq 2\triangle(G),
\]
where \( \triangle(G) \) is the largest degree of \( G \). Moreover, if \( G \) is connected, then the first equality holds if and only if \( G \) is the star \( K_{1,n-1} \); and the second equality holds if and only if \( G \) is a regular graph.

**Lemma 7** ([14]) Let \( e \) be an edge of the graph \( G \). Then
\[
q_1(G) \geq q_1(G - e) \geq q_2(G) \geq q_2(G - e) \geq \cdots \geq q_n(G) \geq q_n(G - e) \geq 0.
\]

Let \( B_3(1) \) be a tricyclic graph in \( T^k_n \) obtained from the graph \( G_1 \) in Figure 1 by attaching \( k \) paths with almost equal lengths to the vertex with degree 6.

Let \( B_4(1) \) be a tricyclic graph in \( T^k_n \) obtained from the graph \( G_2 \) in Figure 1 by attaching \( k \) paths with almost equal lengths to the vertex with degree 5.

Let \( B_5(1) \) be a tricyclic graph in \( T^k_n \) obtained from the graph \( G_3 \) in Figure 1 by attaching \( k \) paths with almost equal lengths to some vertex with degree 4.

Let \( B_6(1) \) be a tricyclic graph in \( T^k_n \) obtained from \( K_4 \) by attaching \( k \) paths with almost equal lengths to a vertex of \( K_4 \).
If $G \in T_n^{k,3}$, then $G$ is obtained by attaching some trees to some vertices of graph $G'$, where $G' \in \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ (see Figure 1).

If $G \in T_n^{k,4}$, then $G$ is obtained by attaching some trees to some vertices of graph $G'$, where $G' \in \{T_8, T_9, T_{10}, T_{11}\}$ (see Figure 1).

If $G \in T_n^{k,6}$, then $G$ is obtained by attaching some trees to some vertices of graph $G'$, where $G' \in \{T_{12}, T_{13}, T_{14}\}$ (see Figure 1).

If $G \in T_n^{k,7}$, then $G$ is obtained by attaching some trees to some vertices of graph $T_{15}$ (see Figure 1).

![Graphs G1 - G3 and T1 - T15](image)

**Figure 1** Graphs $G_1 - G_3$ and $T_1 - T_{15}$

### 3. Main results

**Lemma 8** If both $B_3(1)$ and $B_4(1)$ exist, then $q(B_4(1)) < q(B_3(1))$.

**Proof** Let

\[
\begin{align*}
    t_1 &= \frac{(k+5)(k+5 + \frac{2k+2+3+2+3+2}{k+5}) + 3(3 + \frac{2k+5+2}{3})}{k+5+3}, \\
    t_2 &= \frac{(k+5)(k+5 + \frac{2k+2+3+2+3+2}{k+5}) + 2(2 + \frac{3k+5}{2})}{k+5+2}, \\
    t_3 &= \frac{(k+5)(k+5 + \frac{2k+2+3+2+3+2}{k+5}) + 2(2 + \frac{2k+15}{2})}{k+5+2}, \\
    t_4 &= \frac{(k+5)(k+5 + \frac{2k+2+3+2+3+2}{k+5}) + 2(2 + \frac{k+5}{2})}{k+5+2}, \\
    t_5 &= \frac{3(3 + \frac{2k+2+3}{3}) + 2(2 + \frac{3k+5}{2})}{3+2},
\end{align*}
\]
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\[
t_6 = \frac{2(2 + \frac{k+5+2}{2}) + 2(2 + \frac{k+5+2}{2})}{2 + 2},
\]
\[
t_7 = \frac{2(2 + \frac{k+5+2}{2}) + 2(2 + \frac{2+2}{2})}{2 + 2},
\]
\[
t_8 = \frac{2(2 + \frac{2+2}{2}) + 2(2 + \frac{2+2}{2})}{2 + 2},
\]
\[
t_9 = \frac{(k + 5)(k + 5 + \frac{2k+2+3+2+2+2}{2}) + 1(1 + \frac{k+5}{1})}{k+5+1}.
\]

By Lemmas 3 and 6, we can get
\[
q(B_4(1)) \leq \max\{t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9\} \leq k + 7 = \triangle(B_3(1)) + 1 < q(B_3(1)).
\]

By similar reasoning to that of Lemma 8, we can get the following lemma.

**Lemma 9** If \( B_3(1), B_6(1) \) and \( B_7(1) \) exist, then \( q(B_6(1)) < q(B_3(1)) \), \( q(B_7(1)) < q(B_3(1)) \).

**Theorem 1** Let \( G \in T_n^{k,3} \). Then \( q(G) \leq q(B_3(1)) \); the equality holds if and only if \( G \cong B_3(1) \).

**Proof** Choose \( G \in T_n^{k,3} \) such that \( q(G) \) is as large as possible. Denote by \( C_p, C_q, C_h \) the three cycles of \( G \), respectively.

We first prove that \( G \) must be obtained by attaching some trees to some vertices of \( T_1 \) in Figure 1.

Denote the vertex set of \( G \) by \( \{v_1, v_2, \ldots, v_n\} \) and the Perron vector of \( G \) by \( x = (x_1, x_2, \ldots, x_n) \), where \( x_i \) corresponds to \( v_i \).

Assume \( G \) is obtained by attaching some trees to some vertices of graph \( T_3 \) in Figure 1. If \( x_r \geq x_u \), then let \( G^* = G - uv_{i+1} - uv_{i-1} - uf_1 - \cdots - uf_z + rv_{i+1} + rv_{i-1} + rf_1 + \cdots + rf_z \), where \( uv_{i+1}, uv_{i-1} \in E(C_h) \), and \( f_1, \ldots, f_z \) are all the neighbors of \( u \) in those trees (if exist) attaching to \( u \). If \( x_r < x_u \), then let \( G^* = G - rv_{j+1} - rv_{j-1} - rq_1 - \cdots - rq_s + uv_{j+1} + uv_{j-1} + q_1 + \cdots + q_s \), where \( rv_{j+1}, rv_{j-1} \in E(C_p) \), and \( q_1, \ldots, q_s \) are all the neighbors of \( r \) in those trees (if exist) attaching to \( r \). Combining two cases above, by Lemma 1, we can see that \( q(G^*) > q(G) \) and \( G^* \in T_n^{k,3} \), a contradiction. Hence \( G \) cannot be obtained by attaching some trees to some vertices of graph \( T_3 \).

By similar reasoning, it is easy to prove that \( G \) cannot be obtained by attaching trees to some vertices of graph \( T_2, T_4, T_5, T_6, T_7 \). Hence \( G \) must be obtained by attaching some trees to vertices of \( T_1 \).

Next, we will prove that \( G \) must be obtained by attaching exactly one tree to some vertex of \( T_1 \).

Assume there exist two trees, say \( T'_1, T'_2 \) are attached to vertices \( w_1, w_2 \) of \( T_1 \), respectively. If \( x_{w_1} \leq x_{w_2} \), then let \( G^* = G - w_1u_1 - w_1u_2 - \cdots - w_1u_g + w_2u_1 + \cdots + w_2u_g \), where \( u_1, \ldots, u_g \) are all the neighbors of \( w_1 \) in \( T_1 \). If \( x_{w_1} > x_{w_2} \), then let \( G^* = G - w_2u'_1 - w_2u'_2 - \cdots - w_2u'_g + w_1u'_1 + \cdots + w_1u'_f \), where \( u'_1, \ldots, u'_f \) are all the neighbors of \( w_2 \) in \( T'_2 \). By Lemma 1, we can see that \( q(G^*) > q(G) \) and \( G^* \in T_n^{k,3} \), a contradiction. Hence \( G \) has only one tree, say \( T^* \), attached to some vertex, say \( v \), of \( T_1 \).
Thirdly, we prove that \(d(u) \leq 2\), for any \(u \in V(T^*)\), \(u \notin V(T_1)\), where \(T^*\) is a tree which attaches to some vertex of \(T_1\). If \(d(u) > 2\), denote \(N(u) = \{z_1, z_2, \ldots, z_s\}\) and \(N(v) = \{w_1, w_2, \ldots, w_t\}\), \(t \geq 3\). Let \(z_1, w_3\) belong to the path joining \(v\) and \(u\), and \(w_1\) belong to one cycle in \(G\). If \(x_v \geq x_u\), let \(G^* = G - w_2 - \cdots - zw_s + v z_3 + \cdots + v z_s\). If \(x_v < x_u\), let \(G^* = G - vw_1 + uw_1\). It is easy to see that \(G^* \in T_k^{n,3}\). By Lemma 1, we can get that \(q(G^*) > q(G)\), a contradiction. Hence, \(G\) is a graph obtained from \(T_1\) by attaching \(k\) paths.

By Lemma 5, it is easy to get that the \(k\) paths attached to \(v\) of \(T_1\) have almost equal lengths.

Let \(v_1\) be the common vertex of the three cycles of \(T_1\). Finally, we prove that \(v = v_1\).

Assume that \(v \neq v_1\). Without loss of generality, suppose that \(v \in C_p\), where \(C_p\) is some cycle of \(T_1\). Let \(P_1, P_2, \ldots, P_k\) be the \(k\) paths attached to \(v\), and \(vw_{i1} \in P_i\) \((i = 1, 2, \ldots, k)\). Denote \(v_1 v_{m-1}, v_1 v_{m+1} \in C_q, v_1 v_{j-1}, v_1 v_{j+1} \in C_h\), where \(C_q\) and \(C_h\) are the two cycles except \(C_p\) of \(T_1\).

If \(x_v \geq x_{v_1}\), then let \(G^* = G - v_1 v_{i-1} - v_1 v_{i+1} - v_1 v_{j-1} - v_1 v_{j+1} + vv_{i-1} + vv_{i+1} + vv_{j-1} + vv_{j+1}\). If \(x_v < x_{v_1}\), then let \(G^* = G - vw_{i1} - vw_{21} - \cdots - vw_{k_1} + v_1 w_{11} + v_1 w_{21} + \cdots + v_1 w_{k_1}\). Obviously, \(G^* \in T_k^{n,3}\), and by Lemma 1, we get \(q(G^*) > q(G)\), a contradiction. Hence \(v = v_1\).

By Lemmas 2 and 7, it is easy to prove that all the cycles in \(G\) have length 3. Then \(G \cong B_3(1)\). \(\Box\)

By similar reasoning to that of Theorem 1, it is not difficult to prove the following theorems.

**Theorem 2** Let \(G \in T_k^{n,4}\). Then \(q(G) \leq q(B_4(1))\), and the equality holds if and only if \(G \cong B_4(1)\).

**Theorem 3** Let \(G \in T_k^{n,6}\). Then \(q(G) \leq q(B_6(1))\), and the equality holds if and only if \(G \cong B_6(1)\).

**Theorem 4** Let \(G \in T_k^{n,7}\). Then \(q(G) \leq q(B_7(1))\), and the equality holds if and only if \(G \cong B_7(1)\).

From Lemmas 8, 9 and Theorems 1–4, we get the main result.

**Theorem 5** Let \(G \in T_k^n\), \(k \geq 1\). Then \(q(G) \leq q(B_3(1))\), and the equality holds if and only if \(G \cong B_3(1)\).

**References**


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