Generalized Jordan Derivations Associate with Hochschild 2-Cocycles on Triangular Matrices

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Abstract In this paper, we prove that every generalized Jordan derivation associate with a Hochschild 2-cocycle from the algebra of upper triangular matrices to its bimodule is the sum of a generalized derivation and an antiderivation.

Keywords generalized Jordan derivation; generalized derivation; Hochschild 2-cocycle.

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1. Introduction

Let \( R \) be a commutative ring with identity, \( \mathcal{A} \) be an algebra over \( R \) and \( \mathcal{M} \) be an \( \mathcal{A} \)-bimodule. Let \( \alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{M} \) be an \( R \)-bilinear map, that is, an \( R \)-linear map on each component. \( \alpha \) is called a Hochschild 2-cocycle if
\[
a\alpha(b, c) - \alpha(ab, c) + a\alpha(a, b) - \alpha(a, b)c = 0
\]
for all \( a, b, c \in \mathcal{A} \). Let \( \Delta, \varphi \) and \( \delta \) be \( R \)-linear maps from \( \mathcal{A} \) to \( \mathcal{M} \). \( \varphi \) is called a generalized derivation if there exists a Hochschild 2-cocycle \( \alpha \) such that
\[
\varphi(ab) = \varphi(a)b + a\varphi(b) + \alpha(a, b)
\]
for all \( a, b \in \mathcal{A} \), and \( \Delta \) is called a generalized Jordan derivation if
\[
\Delta(a^2) = \Delta(a)a + a\Delta(a) + \alpha(a, a)
\]
for all \( a \in \mathcal{A} \). We denote them by \((\varphi, \alpha)\) and \((\Delta, \alpha)\), respectively. Moreover, \( \delta \) is called an antiderivation if
\[
\delta(ab) = \delta(b)a + b\delta(a)
\]
for all \( a, b \in \mathcal{A} \).

Different types of generalized derivations as well as generalized Jordan derivations have been introduced by several authors. For instance, Brešar in [1] defined one kind of generalized derivations in the sense that if \( \varphi \) is an \( R \)-linear map from \( \mathcal{A} \) to \( \mathcal{M} \) and if there exists a derivation \( d \) from \( \mathcal{A} \) to \( \mathcal{M} \) such that \( \varphi(ab) = \varphi(a)b + ad(b) \) for all \( a, b \in \mathcal{A} \), then \( \varphi \) is a generalized derivation;
another kind of generalized derivations was introduced by Nakajima [2] as follows. An \( R \)-linear map \( \varphi \) from \( A \) to \( M \) is called a generalized derivation if there exists an element \( \omega \in M \) such that
\[
\varphi(ab) = \varphi(a)b + a\varphi(b) + a\omega b 
\] for all \( a, b \in A \). In the case that \( A \) has an identity element \( I \), this is equivalent to that \( \varphi \) satisfies
\[
\varphi(ab) = \varphi(a)b + a\varphi(b) - a\varphi(I)b 
\] for all \( a, b \in A \). Nakajima in [3] defined the type of generalized derivations associate with Hochschild 2-cocycles and pointed out that this type includes not only the generalized derivations mentioned above, but left multipliers and \((\sigma, \tau)\)-derivations as well. For more details we refer the reader to [3–6] and references therein.

Throughout this paper, by \( M_n(R) \), \( n \geq 2 \), we denote the algebra of all \( n \times n \) matrices over \( R \), by \( T_n(R) \) its subalgebra of all upper triangular matrices, and by \( D_n(R) \) its subalgebra of all diagonal matrices. Benkovič in [7] showed that every Jordan derivation from \( T_n(R) \) to its bimodule is the sum of a derivation and an antiderivation. Ji and Ma [8] extended this result to generalized Jordan derivations in the usual sense, i.e., if \( \Delta \) is a generalized Jordan derivation from \( T_n(R) \) to its bimodule in the sense
\[
\Delta(a^2) = \Delta(a)a + a\Delta(a) - a\Delta(I)a 
\]
for all \( a \in T_n(R) \), then \( \Delta \) is the sum of a generalized derivation \( \varphi \) and an antiderivation \( \delta \), where \( \varphi \) is such that
\[
\varphi(ab) = \varphi(a)b + a\varphi(b) - a\varphi(I)b 
\]
for all \( a, b \in T_n(R) \). In this note we generalize the result above to show that every generalized Jordan derivation \((\Delta, \alpha)\) associate with Hochschild 2-cocycle \( \alpha \) from \( T_n(R) \) to its bimodule is the sum of a generalized derivation \((\varphi, \alpha)\) and an antiderivation. We shall assume, without further mention, that all algebras and all modules considered in this paper is 2-torsionfree.

2. Proof of the main result

Let \( A \) be an algebra with identity over \( R \) and \( M \) be an \( A \)-bimodule. As usual, we regard \( M \) as an \( R \)-bimodule by actions \( rm = mr = (rI)m = m(rI) \) for all \( r \in R \) and \( m \in M \), where \( I \) is the identity of \( A \). We begin with the following lemma which is a modification of Lemma 2 in [3]. For the sake of completeness, we present the proof here.

Lemma 2.1 Let \( \Delta \) be an \( R \)-linear map from \( A \) to its bimodule \( M \) and \( \alpha: A \times A \rightarrow M \) be a Hochschild 2-cocycle. Then the following relations are equivalent:

1) \((\Delta, \alpha)\) is a generalized Jordan derivation.

2) For all \( a, b \in A \), we have
\[
\Delta(ab + ba) = \Delta(a)b + a\Delta(b) + \Delta(b)a + ba\Delta(a) + \alpha(a, b) + \alpha(b, a). \tag{5}
\]

3) For all \( a, b \in A \), we have
\[
\Delta(aba) = \Delta(a)ba + a\Delta(b)a + ab\Delta(a) + a\alpha(b, a) + \alpha(a, ba). \tag{6}
\]

Proof 1)\(\Rightarrow\)2). Since \( \Delta(a^2) = \Delta(a)a + a\Delta(a) + \alpha(a, a) \) for all \( a \in A \), replacing \( a \) by \( a + b \) gives (5).
2)⇒3). Substituting $ab + ba$ for $b$ in (5) and using the 2-cocycle condition, we have

$$2\Delta(ab) = \Delta(a(ab + ba) + (ab + ba)a) - \Delta(a^2b + ba^2)$$

$$= 2[\Delta(a)ba + a\Delta(b)a + ab\Delta(a)] +$$

$$a[\alpha(a, b) + \alpha(b, a)] + \alpha(a, ab) + \alpha(a, ba) +$$

$$[\alpha(a, b) + \alpha(b, a)]a + \alpha(ab, a) + \alpha(ba, a) -$$

$$[\alpha(a, a)b + \alpha(a^2, b) + ba(a, a) + \alpha(b, a^2)]$$

$$= 2[\Delta(a)ba + a\Delta(b)a + ab\Delta(a)] +$$

$$[\alpha(a, b) - \alpha(b, a) + \alpha(a, ab) - \alpha(a, b)b] -$$

$$[\alpha(a, b) - \alpha(b, a) + \alpha(b, a^2) - \alpha(b, a)a] +$$

$$\alpha(a, b) + \alpha(a, ba) + \alpha(a, b)a + \alpha(ab, a)$$

$$= 2[\Delta(a)ba + a\Delta(b)a + ab\Delta(a)] +$$

$$aa(a, b) + \alpha(a, ba) + \alpha(a, b)a + \alpha(ab, a).$$

Since $\alpha(a, b) + \alpha(a, ba) = \alpha(ab, a) + \alpha(a, b)a$ and $\mathcal{M}$ is 2-torsionfree, we have the relation (6).

3)⇒1). Taking $a = b = I$ in (6) yields $\Delta(I) = -\alpha(I, I)$. Then putting $b = I$ in (6) gives (1), since $a\Delta(I)a = -\alpha(I, I, a) = -\alpha(I, a)$. This completes the proof. □

For any idempotent $e \in \mathcal{A}$, (3) gives

$$\Delta(e) = \Delta(e)c + e\Delta(e) + \alpha(c, e).$$

(7)

For any $a \in \mathcal{A}$ satisfying $ae = ea = 0$, Lemma 2.1 implies $0 = \Delta(\alpha e + ea) = \Delta(a)e + a\Delta(e) +$$

$$\Delta(e)a + e\Delta(a) + \alpha(a, e) + \alpha(e, a).$$

Multiplying $e$ from the right yields

$$\Delta(a)e + a\Delta(e)e + e\Delta(a)e + \alpha(a, e)e + \alpha(e, a)e = 0.$$ (8)

By the fact $0 = \Delta(\alpha e) = e\Delta(a)e + \alpha(a, e) = e\Delta(a)e + \alpha(e, a)e$, (8) becomes $\Delta(a)e + a\Delta(e)e +$$

$$\alpha(a, e)e = 0.$$

Notice that $\alpha\Delta(e) = \alpha(\Delta(e)e + e\Delta(e) + \alpha(e, e)] = a\Delta(e)e + \alpha(e, e) = a\Delta(e)e -$$

$$\alpha(a, e) + \alpha(a, e)e$, and hence we obtain

$$\Delta(a)e + a\Delta(e) + \alpha(a, e) = 0 = \Delta(e)a + e\Delta(a) + \alpha(e, a)$$ (9)

for any idempotent $e, a \in \mathcal{A}$ such that $ae = ea = 0$.

Now we assume that $(\Delta, \alpha)$ is a generalized Jordan derivation from $\mathcal{T}_n(\mathcal{R})$ to its bimodule

$\mathcal{M}$. Let $e_{ij}$ be the element in $\mathcal{M}_n(\mathcal{R})$ with entries $I$ at the position $i, j$ and 0 otherwise for any

$1 \leq i, j \leq n$. By (7) we have

$$\Delta(e_{ii}) = \Delta(e_{ii})e_{ii} + e_{ii}\Delta(e_{ii}) + \alpha(e_{ii}, e_{ii})$$ (10)

and

$$e_{ki}\Delta(e_{ii})e_{ij} = -e_{ki}\alpha(e_{ii}, e_{ii})e_{ij}$$ (11)

for all $i$ and $k \leq i \leq j$. From (5) we obtain that

$$\Delta(e_{ij}) = \Delta(\alpha(e_{ii})e_{ij} + e_{ij}e_{ii})$$

$$= \Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii}) + \alpha(e_{ii}, e_{ij}) + \alpha(e_{ij}, e_{ii})$$ (12)
whenever $1 \leq i < j \leq n$. Furthermore, (9) gives that
\begin{equation}
\Delta(e_{kj})e_{ii} + e_{kj}\Delta(e_{ii}) + \alpha(e_{kj}, e_{ii}) = 0 = \Delta(e_{ii})e_{kj} + e_{ii}\Delta(e_{kj}) + \alpha(e_{ii}, e_{kj}) \tag{13}
\end{equation}
for all $k, j \neq i$.

Now define an $\mathcal{R}$-linear map $\varphi$ from $\mathcal{T}_n(\mathcal{R})$ to $\mathcal{M}$ by
\begin{equation}
\varphi(e_{ij}) = \Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \alpha(e_{ii}, e_{ij}), \quad 1 \leq i \leq j \leq n. \tag{14}
\end{equation}
According to (10), we have $\varphi(e_{ii}) = \Delta(e_{ii})$ for all $1 \leq i \leq n$.

**Lemma 2.2** $\varphi$ is a generalized derivation associate with Hochschild 2-cocycle $\alpha$.

**Proof** It is enough to check that
\begin{equation}
\varphi(e_{ij}e_{kl}) = \varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \alpha(e_{ij}, e_{kl}) \tag{15}
\end{equation}
for all $i \leq j$ and $k \leq l$. We consider two cases.

**Case 1** $j \neq k$. Our goal is to show that $\varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \alpha(e_{ij}, e_{kl}) = 0$, for $\varphi(e_{ij}e_{kl}) = 0$.

By (14) we have
\begin{align*}
\varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \alpha(e_{ij}, e_{kl}) \\
= [\Delta(e_{ii})e_{ij} + e_{ii}\Delta(e_{ij}) + \alpha(e_{ii}, e_{ij})]e_{kl} + \\
e_{ij}[\Delta(e_{kk})e_{kl} + e_{kk}\Delta(e_{kl}) + \alpha(e_{kk}, e_{kl})] + \alpha(e_{ij}, e_{kl}) \\
= e_{ii}\Delta(e_{ij})e_{kl} + \alpha(e_{ii}, e_{ij})e_{kl} + e_{ij}\Delta(e_{kk})e_{kl} + e_{ij}\alpha(e_{kk}, e_{kl}) + \alpha(e_{ij}, e_{kl}). \tag{16}
\end{align*}

Since $e_{ij}\alpha(e_{kk}, e_{kl}) = -\alpha(e_{ij}, e_{kl}) + \alpha(e_{ij}, e_{kk})e_{kl} = -\alpha(e_{ij}, e_{kl}) + [e_{ii}\alpha(e_{ij}, e_{kk}) - \alpha(e_{ii}, e_{ij})e_{kk}]e_{kl}$, (16) becomes
\begin{equation}
e_{ii}\Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kk})e_{kl} + e_{ii}\alpha(e_{ij}, e_{kk})e_{kl}. \tag{17}
\end{equation}

If $i \neq k$, then (13) implies
\begin{equation}
e_{ii}[\Delta(e_{ij})e_{kk} + e_{ij}\Delta(e_{kk})]e_{kl} = 0. \tag{18}
\end{equation}

If $i = k$, then (12) gives
\begin{align*}
e_{ii}\Delta(e_{ij})e_{il} + e_{ij}\Delta(e_{ii})e_{il} + e_{ii}\alpha(e_{ij}, e_{ii})e_{il} \\
= e_{ii}\Delta(e_{ij})e_{il} + e_{ij}\Delta(e_{ii})e_{il} + [\alpha(e_{ij}, e_{ii}) + \alpha(e_{ii}, e_{ij})]e_{il}e_{il} \\
= [\Delta(e_{ii}) - \Delta(e_{ij})]e_{ij} - \Delta(e_{ij})e_{ii} - \alpha(e_{ij}, e_{ii})e_{il} + \alpha(e_{ij}, e_{ii})e_{il} \\
= -\alpha(e_{ij}, e_{ii})e_{il} + \alpha(e_{ij}, e_{ii})e_{il} = 0.
\end{align*}

Hence (15) holds true in the first case.

**Case 2** $j = k$. Now we have to show that
\[\varphi(e_{il}) = \varphi(e_{ij})e_{jl} + e_{ij}\varphi(e_{jl}) + \alpha(e_{ij}, e_{jl}).\]

Assume $i < j < l$. Then (14) gives us
\[\varphi(e_{ij})e_{jl} + e_{ij}\varphi(e_{jl}) + \alpha(e_{ij}, e_{jl}).\]
Generalized Jordan derivations associated with Hochschild 2-cocycles on triangular matrices

From (11), we have

\[ e_{ij} \Delta(e_{jj}) e_{jl} = - e_{ij} \alpha(e_{jj}, e_{jl}) e_{jl} = - e_{ij} \alpha(e_{jj}, e_{jl}), \]

and since \( \alpha(e_{ii}, e_{ij}) e_{jl} = e_{ii} \alpha(e_{jj}, e_{jl}) - \alpha(e_{ij}, e_{jl}) + \alpha(e_{ii}, e_{il}), \) (18) becomes

\[
\Delta(e_{ii}) e_{il} + e_{ii} \Delta(e_{ij}) e_{jl} + e_{ij} \Delta(e_{ji}) e_{jl} + e_{ii} \alpha(e_{ij}, e_{jl}) \\
= \varphi(e_{il}) - e_{ii} \Delta(e_{il}) + e_{ii} \Delta(e_{ij}) e_{jl} + e_{ij} \Delta(e_{ji}) e_{jl} + e_{ii} \alpha(e_{ij}, e_{jl}) \\
= \varphi(e_{il}) - e_{ii} \Delta(e_{il}) - e_{ii} \Delta(e_{ij}) e_{jl} - e_{ij} \Delta(e_{ji}) - \alpha(e_{ij}, e_{jl}) \\
= \varphi(e_{il}) - e_{ii} \Delta(e_{il}) e_{ij} + e_{ij} \Delta(e_{ji}) + \alpha(e_{ij}, e_{jl}) \\
= \varphi(e_{il}) - e_{ii} \Delta(e_{jl}) e_{ij} - e_{ii} \alpha(e_{ij}, e_{jl}),
\]

where the third equality results from the fact that \( \Delta(e_{il}) = \Delta(e_{ij}) e_{jl} + \Delta(e_{jl}) e_{ij} + \alpha(e_{ij}, e_{jl}) + \alpha(e_{ij}, e_{jl}). \) By (13) we have \( \Delta(e_{ii}) e_{il} + e_{ii} \Delta(e_{il}) + \alpha(e_{ii}, e_{il}) = 0. \) Multiplying \( e_{ij} \) from the right yields \( e_{ii} \Delta(e_{il}) e_{ij} = - \alpha(e_{ii}, e_{il}) e_{ij} = - e_{ii} \alpha(e_{jl}, e_{ij}), \) whence (19) equals \( \varphi(e_{il}). \)

Next we assume that \( i = j < l, \) then \( \varphi(e_{ii}) e_{il} = \varphi(e_{il}) \) and

\[
\varphi(e_{ij}) e_{jj} + e_{ij} \varphi(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
= \Delta(e_{ij}) e_{jj} + e_{ij} \Delta(e_{jj}) e_{jl} + e_{ij} \Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
= \varphi(e_{il}) - e_{ii} \Delta(e_{il}) e_{ij} + e_{ij} \Delta(e_{ji}) + \alpha(e_{ij}, e_{jl}) \\
= \varphi(e_{il}) - e_{ii} \alpha(e_{ij}, e_{il}) e_{ij} - e_{ii} \alpha(e_{ij}, e_{il}),
\]

since \( e_{ii} \Delta(e_{ii}) e_{il} = - e_{ii} \alpha(e_{ii}, e_{il}) e_{il} = - e_{ii} \alpha(e_{ii}, e_{il}). \) Now, let \( i < j = l. \) We have \( \varphi(e_{ij}) e_{jj} = \varphi(e_{ij}) \) and

\[
\varphi(e_{ij}) e_{jj} + e_{ij} \varphi(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
= \Delta(e_{ij}) e_{jj} + e_{ij} \Delta(e_{jj}) e_{jl} + e_{ij} \Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
= \Delta(e_{ij}) e_{jj} + e_{ij} \Delta(e_{jj}) e_{jl} + e_{ij} \Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
= \Delta(e_{ij}) e_{jj} + e_{ij} \Delta(e_{jj}) e_{jl} + e_{ij} \Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}),
\]

where the last equality is due to \( \alpha(e_{ii}, e_{ij}) e_{jj} = e_{ii} \alpha(e_{ij}, e_{jj}) - \alpha(e_{ij}, e_{jj}) + \alpha(e_{ii}, e_{jj}). \) According to (14), (20) equals

\[
\varphi(e_{ij}) - e_{ii} \Delta(e_{ij}) e_{jj} + e_{ij} \Delta(e_{jj}) + \alpha(e_{ij}, e_{jj}) \\
= \varphi(e_{ij}) - e_{ii} \Delta(e_{ij}) e_{jj} - e_{ij} \Delta(e_{jj}) - \alpha(e_{ij}, e_{jj}) \\
= \varphi(e_{ij}) - e_{ii} \Delta(e_{jj}) e_{ij} + e_{ij} \Delta(e_{jj}) + \alpha(e_{jj}, e_{ij}) \\
= \varphi(e_{ij}) - e_{ii} \Delta(e_{jj}) e_{ij} - e_{ii} \alpha(e_{jj}, e_{ij}).
\]

From (13), we have \( \Delta(e_{ii}) e_{jj} + e_{ii} \Delta(e_{ij}) + \alpha(e_{ii}, e_{ij}) = 0. \) Multiplying \( e_{ij} \) from the right gives

\( e_{ii} \Delta(e_{jj}) e_{ij} = - \alpha(e_{ii}, e_{jj}) e_{ij} = - e_{ii} \alpha(e_{jj}, e_{ij}), \) whence (21) equals \( \varphi(e_{ij}). \) Finally, if \( i = j = l, \)
then (15) follows from (10) and \( \varphi(e_{ii}) = \Delta(e_{ii}) \). Therefore, (15) holds true in every case and the proof is completed. □

Now set \( \delta = \Delta - \varphi \). Then \( \delta(e_{ij}) = \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii}) + \alpha(e_{ij}, e_{ii}) \) for all \( 1 \leq i < j \leq n \) and \( \delta(D_n(\mathcal{R})) = 0 \). Since
\[
\delta(e_{ij}e_{kl} + e_{kl}e_{ij}) = \Delta(e_{ij}e_{kl} + e_{kl}e_{ij}) - \varphi(e_{ij}e_{kl} + e_{kl}e_{ij})
= \Delta(e_{ij})e_{kl} + e_{ij}\Delta(e_{kl}) + \Delta(e_{kl})e_{ij} + e_{kl}\Delta(e_{ij}) + \alpha(e_{ij}, e_{kl}) + \alpha(e_{kl}, e_{ij}) - \\
[\varphi(e_{ij})e_{kl} + e_{ij}\varphi(e_{kl}) + \varphi(e_{kl})e_{ij} + e_{kl}\varphi(e_{ij}) + \alpha(e_{ij}, e_{kl}) + \alpha(e_{kl}, e_{ij})]
= \delta(e_{ij})e_{kl} + e_{ij}\delta(e_{kl}) + \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}),
\]
we have that \( \delta \) is a Jordan derivation. Moreover, we have

**Lemma 2.3** \( \delta \) is an antiderivation.

**Proof** Since \( \delta = \Delta - \varphi \), \( \varphi \) is a generalized derivation and \( \delta(D_n(\mathcal{R})) = 0 \), it follows that
\[
\delta(e_{ij}) = \Delta(e_{ij})e_{ii} + e_{ij}\Delta(e_{ii}) + \alpha(e_{ij}, e_{ii})
= [\delta(e_{ij}) + \varphi(e_{ij})]e_{ii} + e_{ij}[\delta(e_{ii}) + \varphi(e_{ii})] + \alpha(e_{ij}, e_{ii})
= \delta(e_{ij})e_{ii} + \varphi(e_{ij})e_{ii} + e_{ij}\varphi(e_{ii}) + \alpha(e_{ij}, e_{ii})
= \delta(e_{ij})e_{ii} + \varphi(e_{ij}e_{ii}) = \delta(e_{ij})e_{ii}
\]
if \( i < j \). Note that \( \delta \) is a Jordan derivation, we then have
\[
\delta(e_{ij}) = \delta(e_{ij}e_{jj} + e_{jj}e_{ij}) = \delta(e_{ij})e_{jj} + e_{ij}\delta(e_{jj}) + \delta(e_{jj})e_{ij} + e_{jj}\delta(e_{ij})
= e_{jj}\delta(e_{ij})
\]
when \( i < j \). We proved that
\[
\delta(e_{ij}) = \delta(e_{ij})e_{ii} \quad \text{and} \quad \delta(e_{ij}) = e_{jj}\delta(e_{ij}) \tag{22}
\]
whenever \( i < j \). Our goal is to prove that
\[
\delta(e_{ij}e_{kl}) = \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) \tag{23}
\]
for all \( i \leq j \) and \( k \leq l \). Again we consider two cases.

**Case 1** \( j \neq k \). We have to show that \( \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) = 0 \).

If \( i = k \) and \( j = l \), this holds true since \( \delta \) is a Jordan derivation.

If \( i \neq k \) and \( j \neq l \), it follows from (22) that \( \delta(e_{kl})e_{ij} + e_{kl}\delta(e_{ij}) = \delta(e_{kl})e_{kk}e_{ij} + e_{kl}e_{jj}\delta(e_{ij}) = 0 \).

Next assume that \( i = k \) and \( j \neq l \). If \( i = l \), then from (22) we have \( \delta(e_{ii})e_{ij} + e_{ii}\delta(e_{ij}) = 0 \), since \( \delta \) vanishes on diagonal elements. If \( i \neq l \), then from the fact that \( \delta \) is a Jordan derivation we infer that
\[
0 = \delta(e_{ii}e_{ij} + e_{ij}e_{ii})e_{jj} = \delta(e_{ii})e_{ij} + e_{ii}\delta(e_{ij})e_{jj} + \delta(e_{ij})e_{ii}e_{jj} + e_{ij}\delta(e_{ii})e_{jj} = \delta(e_{ii})e_{ij} + e_{ii}\delta(e_{ij}).
\]
In the case \( i \neq k \) and \( j = l \) we proceed similarly as above. Now we have
\[
\delta(e_{kj})e_{ij} + e_{kj}\delta(e_{ij}) = e_{kj}\delta(e_{ij}) = e_{kk}\delta(e_{kj}e_{ij}) = e_{kk}\delta(e_{kj}e_{ij}) = 0.
\]

**Case 2** \( j = k \). The case when \( i = j = l \) is trivial, since \( \delta \) vanishes on diagonal elements.
In cases $i < j = l$ or $i = j < l$, it follows from (22) that
\[
\delta(e_{ij}) = \delta(e_{ij})e_{ij} + e_{ji}\delta(e_{ij}), \quad \delta(e_{il}) = \delta(e_{il})e_{ii} + e_{il}\delta(e_{ii}).
\]

Finally, let $i < j < l$. Then we have\[
\delta(e_{ij})e_{ij} + e_{ji}\delta(e_{ij}) = \delta(e_{jl})e_{jj} + e_{jj}\delta(e_{ji}) = 0,
\]
while\[
\delta(e_{il}) = \delta(e_{il})e_{ii} + e_{il}\delta(e_{ii})
\]
whence (23) holds. This completes the proof. □

**Theorem 2.4** Let $(\Delta, \alpha)$ be a generalized Jordan derivation associate with Hochschild 2-cocycle $\alpha$ from $T_n(\mathcal{R})$ to a $T_n(\mathcal{R})$-bimodule $\mathcal{M}$. Then there exists a generalized derivation $(\varphi, \alpha)$, associate with the same Hochschild 2-cocycle $\alpha$, and an antiderivation $\delta$ from $T_n(\mathcal{R})$ to $\mathcal{M}$ with $\delta(D_n(\mathcal{R})) = 0$ such that $\Delta = \delta + \varphi$. Moreover, $(\varphi, \alpha)$ and $\delta$ are uniquely determined.

**Proof** It suffices to prove the uniqueness. Suppose $\Delta = \delta_1 + \varphi_1 = \delta_2 + \varphi_2$, where $\varphi_1$ and $\varphi_2$ are generalized derivations associate with $\alpha$, while $\delta_1$ and $\delta_2$ are antiderivations vanishing on diagonals. Then $d = \delta_1 - \delta_2 = \varphi_2 - \varphi_1$ is a derivation and an antiderivation as well. Therefore, $d$ vanishes on commutators, which implies $d(e_{ij}) = d(e_{ij}e_{jj} - e_{jj}e_{ij}) = 0$ for all $i < j$. On the other hand, $d(D_n(\mathcal{R})) = \delta_1(D_n(\mathcal{R})) = \delta_2(D_n(\mathcal{R})) = 0$. It follows that $d(T_n(\mathcal{R})) = 0$ and this completes the proof. □

Let $m \geq n \geq 2$. We may regard $\mathcal{M}_m(\mathcal{R})$ as a $T_n(\mathcal{R})$-bimodule by the actions $AX = (A \oplus I_{m-n})X$, $XA = X(A \oplus I_{m-n})$, for all $A \in T_n(\mathcal{R})$ and $X \in \mathcal{M}_m(\mathcal{R})$, where $I_{m-n}$ is the identity of $\mathcal{M}_{m-n}(\mathcal{R})$. As a corollary to Theorem 2.4, we shall easily derive

**Corollary 2.5** Let $m \geq n \geq 2$. Then a generalized Jordan derivation from $T_n(\mathcal{R})$ to $\mathcal{M}_m(\mathcal{R})$ is a generalized derivation.

In the case $\alpha = 0$, we have

**Corollary 2.6** ([7]) Let $m \geq n \geq 2$. There are no proper Jordan derivations from $T_n(\mathcal{R})$ to $\mathcal{M}_m(\mathcal{R})$. In particular, there are no proper Jordan derivations from $T_n(\mathcal{R})$ to itself.

**References**


