Spectral Characterization of Generalized Cocktail-Party Graphs

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Abstract In the paper, we prove that all generalized cocktail-party graphs with order at least 23 are determined by their adjacency spectra.

Keywords adjacency matrix; spectral characterization; generalized line graph; generalized cocktail-party graphs.

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1. Introduction

In this paper, we are concerned only with undirected simple graphs (loops and multiple edges are not allowed). All notions on graphs that are not defined here can be found in [4]. For a graph \( G = (V(G), E(G)) \), let \( n(G) = n \), \( m(G) \), \( \ell(G) \) and \( A = A(G) \) be respectively the order, size, line graph and adjacency matrix of \( G \). For some vertex \( v_i \in V(G) \), let \( d_i = d(v_i) \) stand for the degree of \( v_i \). We denote the characteristic polynomial \( \det(\lambda I - A) \) of \( G \) by \( \phi(G, \lambda) \) or simply \( \phi(G) \). The adjacency spectrum of \( G \), denoted by \( \text{Spec}(G) \), is the multiset of eigenvalues of \( A(G) \). Since \( A(G) \) is symmetric, its eigenvalues are real and we set \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \lambda_n(G) \). The maximum eigenvalue \( \lambda_1(G) \) of \( G \) is called the spectral radius (or index) of \( G \) and it is often denoted by \( \rho(G) \).

Two graphs \( G \) and \( H \) are said to be \( A \)-cospectral if the corresponding adjacency spectra are the same. A graph is said to be determined by the \( A \)-spectrum (or simply a DAS-graph) if there is no other non-isomorphic graph \( A \)-cospectral to it, i.e., \( \phi(G) = \phi(H) \) implies \( G \cong H \). The background of the question “which graphs are determined by their spectrum? ” originates from Chemistry (in 1956, Günthard and Primas [15] raised this question in the context of Hückel’s theory). For additional remarks on the topic we refer the readers to [11, 12].

Some other notations and terminology are also needed. Let \( \Delta(G) \) be the maximum degree of a graph \( G \). Let \( G_1 \cup G_2 \) denote the disjoint union of graphs \( G_1 \) and \( G_2 \), and \( kG_1 \) the disjoint
union of $k$ copies of $G_1$. The join (or complete product) $G_1 \nabla G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of $G_1$ with every vertex of $G_2$. As usual, let $P_n$, $C_n$, $K_n$ and $K_{a_1, a_2, \ldots, a_k}$ ($a_1 + a_2 + \cdots + a_k = n$) denote the path, the cycle, the complete graph and the complete $k$-multipartite graph of order $n$, respectively.

Cvetković, Doob, Simić [5] defined a generalized cocktail-party graph, denoted by $G_{CP}$, as a complete graph with some independent edges removed. A special case of this graph is the well-known cocktail-party graph $CP(k)$ obtained from $K_{2k}$ by removing $k$ disjoint edges. Hoffman [16] introduced the generalized line graph as follows: for any graph $H$ with $n$ vertices $v_1, v_2, \ldots, v_n$ and any non-negative integers $a_1, a_2, \ldots, a_n$, then the generalized line graph $L(H; a_1, a_2, \ldots, a_n)$ is the graph consisting of disjoint copies of $\ell(H)$ and $CP(a_i)$ together with additional edges joining a vertex in $\ell(H)$ with a vertex in $CP(a_i)$ if the vertex in $\ell(H)$ corresponding to an edge in $H$ has $v_i$ as an end-vertex ($i = 1, 2, \ldots, n$). It is well-known that the generalized line graphs are related to the following famous and important theorem:

**Theorem 1.1** ([2]) Let $G$ be a connected graph with least eigenvalue at least $-2$, Then either $G$ is a generalized line graph or $G$ can be represented by vectors in the root system $E_8$.

Graphs with least eigenvalue at least $-2$ have been studied since the very beginnings of the theory of graphs spectra. Much information on this field can be found in the books [1, 3, 10, 14].

It is an interesting problem to find which graphs with least eigenvalue at least $-2$ are $A$-cospectral graphs or DAS-graphs. Here we mention some known results. An exceptional graph is a connected graph with least eigenvalue at least $-2$ which is not a generalized line graph. Cvetković and Lepović [6, 7] studied the phenomenon of $A$-cospectrality in generalized line graphs and in exceptional graphs. For the regular DAS-graphs with least eigenvalue at least $-2$, van Dam and Haemers [11] gave an almost complete answer (see their Theorem 8). Further results on $A$-cospectral graphs may be found in Section 4.2 of [10]. However, for the non-regular case, van Dam and Haemers [11] stated that the following question remains open.

**Problem 1.1** Which non-regular graphs with least eigenvalue at least $-2$ are DAS-graphs?

The rest of the paper is organized as follows: In Section 2 we cite some results of graphs with least eigenvalue at least $-2$ and define an important graph invariant which will be helpful in proving our main results. In Section 3 we investigate the spectral characterization of generalized cocktail-party graph.

### 2. Basic results and an invariant of graphs with least eigenvalue at least $-2$

**Lemma 2.1** ([4]) Let $G_i$ be an $r_i$-regular graph of order $n_i$ ($i = 1, 2$). Then

$$\phi(G_1 \nabla G_2, \lambda) = \frac{\phi(G_1, \lambda)\phi(G_2, \lambda)}{(\lambda - r_1)(\lambda - r_2) - n_1 n_2}.$$ 

Doob and Cvetković [13] characterized all connected graphs with the least eigenvalue greater than $-2$ in the theorem below:
Theorem 2.1 ([13]) Let $G$ be a connected graph with $\lambda_n(G) > -2$. Then

(i) $G \in \mathcal{G}_1 = \{\ell(T) | T \text{ is a tree}\}$;
(ii) $G \in \mathcal{G}_2 = \{L(T; 1, 0, \ldots, 0) | T \text{ is a tree}\}$;
(iii) $G \in \mathcal{G}_3 = \{\ell(H) | H \text{ is an odd-unicyclic graph}\}$;
(iv) $G \in \mathcal{G}_4 = \{20 \text{ graphs with order 6 that are represented in } E_6\}$;
(v) $G \in \mathcal{G}_5 = \{110 \text{ graphs with order 7 that are represented in } E_7\}$;
(vi) $G \in \mathcal{G}_6 = \{443 \text{ graphs with order 8 that are represented in } E_8\}$.

For convenience, set $\mathcal{L} = \{G | G \text{ is a connected graph and } \lambda_n(G) \geq -2\}$, $\mathcal{L}^+ = \{G | G \text{ is a connected graph and } \lambda_n(G) > -2\}$ and $\mathcal{L}^0 = \{G | G \text{ is a connected graph and } \lambda_n(G) = -2\}$. Clearly, $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^0$ and $\mathcal{L}^+ = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6$. A graph in $\mathcal{L}$ ($\mathcal{L}^+$ or $\mathcal{L}^0$) is called an $\mathcal{L}$-graph ($\mathcal{L}^+$-graph or $\mathcal{L}^0$-graph). For the $\mathcal{L}^+$-graphs we have the following:

Theorem 2.2 ([1]) Let $G$ be a $\mathcal{L}^+$-graph with order $n$. Then

$$
\det(2I + A(G)) = \begin{cases} 
n + 1, & \text{if } G \in \mathcal{G}_1; \\
n, & \text{if } G \in \mathcal{G}_2 \cup \mathcal{G}_4; \\
3, & \text{if } G \in \mathcal{G}_5; \\
2, & \text{if } G \in \mathcal{G}_6; \\
1, & \text{if } G \in \mathcal{G}_6.
\end{cases}
$$

The following contents of this section first appeared in [19, 20]. In order to the fullness of this paper, we will state it again. Cvetković and Lepović [8] adopted the nomenclature from lattice theory and defined

$$d_G = (-1)^n \phi(G, -2)$$

as the discriminant of an $\mathcal{L}$-graph $G$. Additionally, for an $\mathcal{L}$-graph $G$ they obtained an important graph invariant named star value and showed that its formula is

$$S = \frac{(-1)^n}{(n - k)!} \phi^{(n-k)}(G, \lambda - 2) = (-1)^n \Pi_G(0) = \prod_{i=1}^{k} (\lambda_i + 2),$$

where $\phi^{(p)}(x)$ denotes the $p$-th derivative function of $\phi(x)$, $\Pi_G(\lambda) = \prod_{i=1}^{k} (\lambda - (\lambda_i + 2))$ (it is called the principal polynomial of $G$ (see [8])) and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the eigenvalues greater than $-2$ of $G$. For the discriminant and the star value of $G$, we have the following conclusion. If $G$ is an $\mathcal{L}^0$-graph, then $d_G = 0 < S$. On the other hand, it is easy to see that $\phi(G, \lambda - 2) = \lambda^{n-k} \Pi_G(\lambda)$ and then $d_G = S$ if $G$ is an $\mathcal{L}^+$-graph.

Now we synthesize the above facts into the following definition and give a visualized notation to the star value:

Definition 2.1 Let $G$ be an $\mathcal{L}$-graph of order $n$ and $\Pi_G(\lambda)$ the principal polynomial of $G$. Then the star value of $G$ is defined as

$$\star(G) = \begin{cases} (-1)^n \Pi_G(0), & \text{if } G \in \mathcal{L}^0; \\
(-1)^n \phi(G, -2), & \text{if } G \in \mathcal{L}^+.
\end{cases}$$
Lemma 3.2 Let $G = \bigcup_{i=1}^{k} G_i$ and $H$ be two $\mathcal{L}$-graphs. Then

(i) $\star(G) = \bigcup_{i=1}^{k} \star(G_i)$;

(ii) if $G$ and $H$ are $A$-cospectral, then $\star(G) = \star(H)$.

Corollary 2.1 Let $G$ be a generalized line graph. Next we give some details for its index and least eigenvalue. Let $n, k \in \mathbb{N}$ and $\lambda(G) = \lambda(G(k))$.

The following corollary is an immediate consequence of Definition 2.1:

Corollary 2.2 Let $G$ be an $\mathcal{L}^+$-graph with order $n$. Then

$$\star(G) = \begin{cases} 
    n + 1, & \text{if } G \in \mathcal{G}_1; \\
    4, & \text{if } G \in \mathcal{G}_2 \cup \mathcal{G}_3; \\
    3, & \text{if } G \in \mathcal{G}_4; \\
    2, & \text{if } G \in \mathcal{G}_5; \\
    1, & \text{if } G \in \mathcal{G}_6.
\end{cases}$$

3. Spectral characterization of generalized cocktail-party graphs

In what follows we will directly use a well-known fact that if $G$ and $H$ are $A$-cospectral graphs, then they respectively share the same numbers of order, size and closed walks of any length.

Note that any vertex in $GCP$ is of degree $n - 1$ or $n - 2$. The following lemma indicates the reason that we adopt the notation $GCP(n, k)$ instead of $GCP$.

Lemma 3.1 A graph $G$ with order $n$ is a GCP iff $G = K_{n-2k} \Delta CP(k)$, where $k \geq 1$.

Proof The sufficiency follows from the fact that $CP(k)$ is a $(2k - 2)$-regular graph. For the necessity, by the definition of $GCP$ we know that $G$ is a bidegreed graph with vertex degree $n - 1$ or $n - 2$. Suppose that $G$ has $n - t$ vertices of degree $n - 1$. Clearly, $t \geq 2$. Thus, $G = K_{n-t} \Delta H$, where $H$ is a $(t - 2)$-regular graph. Since the size $m(H) = t(t - 2)/2$ is an integer, then $t$ is even. Set $t = 2k$. Note that the order $n(H) = t$ and $H$ is $(t - 2)$-regular. Therefore, $H$ is obtained from $K_t$ by removing $t$ pairwise disjoint edges. So, $H = CP(k)$.

From Lemma 3.1 it is easy to see $GCP = GCP(n, k) = L(K_{1,n-2k}; k, 0, \ldots, 0)$ which is a generalized line graph. Next we give some details for its index and least eigenvalue. Let $\rho_{n, k} = \rho(GCP(n, k))$ and $\lambda_{n, k} = \lambda_n(GCP(n, k))$. The following lemma is needed.

Lemma 3.2 Let $G$ be an $r$-regular graph with order $n$ and $\overline{G}$ its complement. Then

$$\phi(\overline{G}, \lambda) = (-1)^n \frac{\lambda - n + r + 1}{\lambda + r + 1} \phi(G, -\lambda - 1).$$
Lemma 3.3 Let $k \geq 1$. Then

(i) $\text{GCP}(n, k)$ has exactly one positive eigenvalue $\rho_{n,k}$.

(ii) $\rho_{n,k} = \frac{n - 3 + \sqrt{(n-3)^2 + 8(n-k-1)}}{2} \geq n - 2$ with equality if and only if $n = 2k$.

(iii) $\lambda_{n,k} \geq -2$ with equality if and only if $k > 1$.

Proof By Lemma 3.2 we get

$$\phi(\text{CP}(k)) = \lambda^k(\lambda + 2)^{k-1}(\lambda - 2k + 2).$$

From Lemma 2.1 it follows that

$$\phi(\text{GCP}(n, k)) = \phi(K_{n-2k}\nabla \text{CP}(k))$$

$$= \frac{\phi(K_{n-2k})\phi(\text{CP}(k))}{(\lambda - n + 2k + 1)(\lambda - 2k + 2)}[(\lambda - n + 2k + 1)(\lambda - 2k + 2) - 2k(n - 2k)]$$

$$= \lambda^k(\lambda + 1)^{n-2k-1}(\lambda + 2)^{k-1}[\lambda^2 - (n - 3)\lambda - 2(n - k - 1)]$$

which shows that (iii) obviously holds. Let $h_1(\lambda)$ denote the above quadratic factor. It is easy to see that $\rho_{n,k}$ is the largest root of $h_1(\lambda) = 0$. Thus, $\rho_{n,k} = \frac{n - 3 + \sqrt{(n-3)^2 + 8(n-k-1)}}{2}$. Since $n - 2k \geq 0$, we have $8(n - k - 1) \geq 4n - 8$ and so $\rho_{n,k} \geq \frac{n - 3 + \sqrt{(n-3)^2 + 4n - 8}}{2} = n - 2$ with equality if and only if $n = 2k$. Thus, (ii) holds. Let $\lambda$ be another root of $h_1(\lambda) = 0$. Then $\lambda \cdot \rho_{n,k} = -2(n - k - 1)$ and thus $\lambda$ is negative and (i) follows.

Remark 3.1 For $n = 2k$, then $G = \text{CP}(k)$ which has been shown to be a DAS-graph [11]. Thus, we always set $n - 2k > 0$ in this subsection and so $\rho_{n,k} > n - 2$.

Lemma 3.4 ([17]) A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.

Note that from Lemma 3.3(i) and Lemma 3.4 we know $\text{GCP}(n, k)$ is also a complete $(n-k)$-multipartite graph. Actually, $\text{GCP}(n, k) = K_{1,n-2k,2k}$, where $a^b$ means that $b$ is the number of parts of cardinality $a$. The following observation follows readily from Lemma 3.4:

Lemma 3.5 Let $G$ be a graph $A$-cospectral with $\text{GCP}(n, k)$. Then $G$ is the union of some isolated vertices and a complete multipartite graph.

Lemma 3.6 No two non-isomorphic generalized cocktail-party graphs are $A$-cospectral.

Proof Assume by way of contradiction that $\text{GCP}(n', k')$ and $\text{GCP}(n, k)$ are $A$-cospectral. Then $n' = n$ and $\rho_{n',k'} = \rho_{n,k}$. From Lemma 3.3(i) it follows that

$$\frac{n - 3 + \sqrt{(n-3)^2 + 8(n-k'-1)}}{2} = \rho_{n,k'} = \rho_{n,k} = \frac{n - 3 + \sqrt{(n-3)^2 + 8(n-k-1)}}{2},$$

which leads to $k' = k$ and thus $\text{GCP}(n', k') \cong \text{GCP}(n, k)$.

To prove the following theorem, we need the notion of a maximal exceptional graph: every exceptional graph is a subgraph of (a least) one such graph.

Theorem 3.1 For $n \geq 23$, $\text{GCP}(n, k)$ is a DAS-graph.
Proof If \( k = 0 \), then \( GCP(n,0) = K_n \) which has been proved to be a DAS-graph [11]. Now set \( k \geq 1 \). Let \( G \) be any graph \( A \)-cospectral to \( GCP(n,k) \). Then \( n(G) = n, \rho(G) = \rho_{n,k} \) and \( \lambda_n(G) = \lambda_{n,k} \). From the known fact that \( \Delta(G) \geq \rho(G) \), we obtain by Lemma 3.3(i) that \( \Delta(G) > n - 2 \), and so \( \Delta(G) = n - 1 \). Hence, \( G \) is a connected graph and therefore \( G \) is a multipartite graph by Lemma 3.5. From Lemma 3.3(iii) it follows that \( \lambda_n(G) = \lambda_{n,k} \geq -2 \). Consequently, \( G \) is one of the graphs described in Theorem 1.1.

Claim 3.1 \( G \) cannot be represented by vectors in the root system \( E_8 \).

Proof Assume by contradiction that \( G \) is such a graph. In order to show the claim, we need an important result due to Cvetković, Lepović, Rowlinson and Simić [9], which is that all the maximal exceptional graphs are determined. Such a graph has order \( 22, 28, 29, 30, 31, 32, 33, 34 \) or \( 36 \). Denote by \( \rho_n \) the maximal index of a maximal exceptional graph of order \( n \). So \( \rho(G) \leq \rho_n \).

From Table 1 [9] we get the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>22</th>
<th>28</th>
<th>29</th>
<th>30</th>
<th>31</th>
<th>32</th>
<th>33</th>
<th>34</th>
<th>36</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_n )</td>
<td>14</td>
<td>17</td>
<td>19</td>
<td>18.9282</td>
<td>19</td>
<td>19.2111</td>
<td>19.5498</td>
<td>20</td>
<td>21</td>
</tr>
</tbody>
</table>

Table 1 The index of maximal exceptional graphs

It follows that an exceptional graph has index at most 21. By Remark 3.1 we have \( \rho_{n,k} > n - 2 \). Hence, for \( n \geq 23 \) we get \( \rho_{n,k} > 21 \geq \rho_n \geq \rho(G) \), a contradiction.

So, Claim 3.1 shows that \( G \) is a generalized line graph. Without loss of generality, set \( G = L(H; a_1, a_2, \ldots, a_n) \) and \( V(G) = V(\ell(H)) \cup V(CP(a_1)) \cup \cdots \cup V(CP(a_n)) \). From the definition of generalized line graph it follows that \( V(\ell(H)) \cap V(CP(a_i)) = \emptyset (1 \leq i \leq n) \) and \( V(CP(a_i)) \cap V(CP(a_j)) = \emptyset (1 \leq i \neq j \leq n) \). Since \( G \) is a complete multipartite graph and \( CP(a_i) = K_{2^{a_i}} (1 \leq i \leq n) \), \( V(CP(a_i)) \) can be partitioned into \( C_{i1} \cup C_{i2} \cup C_{ia_i} \), where each cell \( C_{ij} \) contains exactly two vertices \( (1 \leq j \leq a_i \) and \( 1 \leq i \leq n) \).

Claim 3.2 At most one of \( a_1, a_2, \ldots, a_n \) is not equal to 0.

Proof Assume that \( a_i \neq 0 \) and \( a_j \neq 0 \) \( (1 \leq i \neq j \leq n) \). Let \( u \in C_{ik} \subset V(CP(a_i)) \) \( (1 \leq k \leq a_i) \) and \( v \in C_{jl} \subset V(CP(a_j)) \) \( (1 \leq l \leq a_j) \). Since \( V(CP(a_i)) \cap V(CP(a_j)) = \emptyset \), we have \( C_{ik} \cap C_{jl} = \emptyset \) and so \( uv \notin E(G) \). On the other hand, since \( u \) and \( v \) belong to different cells, \( u \) must be adjacent to \( v \), a contradiction.

From Claim 3.2, without loss of generality, we can set \( a_2 = \cdots a_n = 0 \).

Claim 3.3 The \( A \)-cospectral graph \( G \) is equal to \( GCP(n,a_1) = K_{n-2a_1} \nabla CP(a_1) \).

Proof Since \( G \) is a multipartite graph and \( V(\ell(H)) \cap V(CP(a_1)) = \emptyset \), every vertex of \( \ell(H) \) must be adjacent to each one of \( CP(a_1) \), which shows that all vertices \( e_i \) \( (1 \leq i \leq n(\ell(H))) \) of \( \ell(H) \) must be such that all edges \( e_i \) \( (1 \leq i \leq m(H)) \) have a common vertex, say \( v_1 \), in the graph.
$H$. Hence, $H$ is a star and so $\ell(H)$ is a clique denoted by $K_t$. Since $n(G) = t + 2a_1 = n$, we have $t = n - 2a_1$ and thus $G = GCP(n, a_1)$.

From Claim 3.3 and Lemma 3.6, we obtain that $G \cong GCP(n, k)$ and we are done.

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