Second Order \((F, \alpha, \rho, d, p)\)-Univexity and Duality for Minimax Fractional Programming

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Abstract In this paper, we introduce a class of generalized second order \((F, \alpha, \rho, d, p)\)-univex functions. Two types of second order dual models are considered for a minimax fractional programming problem and the duality results are established by using the assumptions on the functions involved.

Keywords second order \((F, \alpha, \rho, d, p)\)-univexity; minimax fractional programming; second order duality; optimality conditions.

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1. Introduction

In recent years, there has been a growing interest in the minimax programming problems, see for instances [1–8] and the references cited therein. Schmitendorf [1] considered a class of static minimax programming problems and established the necessary and sufficient optimality conditions under the convexity assumptions. Later, Tanimoto [2] constructed a dual problem and proved some duality results for convex minimax programming problems. Chandra and Kumar [3] constructed two dual models and obtained duality theorems for the convex differentiable minimax fractional programming problems. Liu and Wu [6] considered the minimax fractional programming problems under the framework of generalized convexity, and proved weak, strong and strict converse duality theorems.


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functions, which is an extension of \((b, F)\)-convex functions \([15]\) and second order \(F\)-convex functions \([16]\), and established second order mixed type duality theorems for multiobjective nonlinear programming problems. Ahmad and Husain \([17]\) introduced second order \((F, \alpha, \rho, d, p)\)-convex functions and their generalizations, and obtained some duality theorems for multiobjective optimization problems.

Inspired by the works \([8, 10, 14, 18]\), in this paper we will introduce the concepts of second order \((F, \alpha, \rho, d, p)\)-univexity and give some examples to show the relationships among some related concepts. Some duality results for minimax fractional programming problems will be obtained under the assumptions of second order \((F, \alpha, \rho, d, p)\)-univexity.

2. Preliminaries

In this paper, we consider the following minimax fractional programming problem:

\[
\text{(P)} \quad \begin{align*}
\text{Minimize} & \quad \sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \\
\text{subject to} & \quad g(x) \leq 0, \ x \in \mathbb{R}^n,
\end{align*}
\]

where \(Y\) is a compact subset of \(\mathbb{R}^k\), \(f, h : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}\) and \(g : \mathbb{R}^n \to \mathbb{R}^m\) are in \(C^2\). It is assumed that \(f(x, y) \geq 0\) and \(h(x, y) > 0\) on \(X \times Y\), where \(X = \{x \in \mathbb{R}^n : g(x) \leq 0\}\).

For each \(x \in \mathbb{R}^n\), we define

\[
J = \{1, 2, \ldots, m\}, \quad J(x) = \{j \in J : g_j(x) = 0\},
\]

\[
Y(x) = \{y \in Y : \frac{f(x, y)}{h(x, y)} = \sup_{y \in Y} \frac{f(x, z)}{h(x, z)}\},
\]

\[
K = \{(s, t, y) \in N \times \mathbb{R}^s_+ \times \mathbb{R}^k : 1 \leq s \leq n + 1, t = (t_1, \ldots, t_s) \in \mathbb{R}^s_+ \}
\]

with \(\sum_{i=1}^{s} t_i = 1\), and \(y = \{y_1, \ldots, y_s\}\) with \(y_i \in Y(x), i = 1, \ldots, s, x \in \mathbb{R}^n\).

**Definition 2.1** \([19]\) A function \(F : X \times X \times \mathbb{R}^n \to \mathbb{R}\) (where \(X \subset \mathbb{R}^n\)) is said to be sublinear in its third component, if for any \((x, y) \in X \times X\),

\[
F(x, y, a_1 + a_2) \leq F(x, y, a_1) + F(x, y, a_2), \quad \forall a_1, a_2 \in \mathbb{R}^n
\]

and

\[
F(x, y, va) = vF(x, y, a), \quad \forall v \in \mathbb{R}_+, \forall a \in \mathbb{R}^n.
\]

Let \(f : \mathbb{R}^n \to \mathbb{R}\) be a twice differentiable function and \(F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\) be a sublinear function in its third component. We introduce the following concept of second order \((F, \alpha, \rho, d, p)\)-univex function.

**Definition 2.2** A function \(f\) is said to be second order \((F, \alpha, \rho, d, p)\)-univex at \(y \in \mathbb{R}^n\) with respect to \(b\) and \(\phi\), if there exist functions \(\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+/(0)\), \(b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\), \(\phi : \mathbb{R} \to \mathbb{R}\), \(d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\), a vector \(p \in \mathbb{R}^n\) and a real number \(\rho\) such that for all \(x \in \mathbb{R}^n\), we have

\[
b(x, y)\phi(f(x) - f(y)) + \frac{1}{2}p^T \nabla^2 f(y)p \geq F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) + \rho d^2(x, y).
\]
Remark 2.1  
(i) If \( b(x, y) = 1, \forall x, y \in \mathbb{R}^n \) and \( \phi \) is identity on \( R \), then the second order \((F, \alpha, \rho, d, p)\)-univexity with respect to \( b \) and \( \phi \) becomes the second order \((F, \alpha, \rho, d)\)-convexity in [17];

(ii) If \( \alpha(x, y) = 1 \) for all \( x, y \in \mathbb{R}^n \), \( \rho = 0 \) (or \( d(x, y) = 0 \), \( \forall x, y \in \mathbb{R}^n \)) and \( \phi \) is identity on \( R \), then the second order \((F, \alpha, \rho, d, p)\)-univexity with respect to \( b \) and \( \phi \) is an extension of second order \((b, F)\)-convexity introduced by Pandian and Natarajan in [14]. Furthermore, if \( b(x, y) = 1, \forall x, y \in \mathbb{R}^n \), it is an extension of second order \( F \)-convexity in [16]. In the following, we give an example to show that the extension is strict.

Example 2.1  
Let \( n = 1, f(x) = x^3 - 2x + 1, b(x, y) = \alpha(x, y) = 1, \phi(t) = 2|t|, F(x, y, z) = |z(f(x) - f(y))|, \rho = -1, d(x, y) = x - y - \sqrt{2} \). Taking \( y = 0 \), for any \( x \in R, p \in R \), we have \( b(x, y)\phi(f(x) - f(y) + \frac{1}{2}p^T \nabla^2 f(y)p) = 2|x^3 - 2x| \) and \( F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) + \rho d^2(x, y) = 2|x^3 - 2x| - (x - \sqrt{2})^2 \). It is easy to see that the function \( f \) is second order \((F, \alpha, \rho, d, p)\)-univex at \( y = 0 \) with respect to \( b \) and \( \phi \). However \( f \) is neither second order \((F, \alpha, \rho, d)\)-convex nor second order \((b, F)\)-convex at \( y = 0 \). Since every second order \( F \)-convex function is second order \((b, F)\)-convex, it follows that \( f \) is not second order \( F \)-convex at \( y = 0 \).

Definition 2.3  
A function \( f \) is said to be second order \((F, \alpha, \rho, d, p)\)-pseudo-univex at \( y \) with respect to \( b \) and \( \phi \), if there exist functions \( \alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+ \setminus \{0\}, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \phi : \mathbb{R} \to \mathbb{R}, d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \) \( a vector \ p \in \mathbb{R}^n \) and a real number \( \rho \) such that for all \( x \in \mathbb{R}^n \), we have

\[
F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) \geq -\rho d^2(x, y) \Rightarrow b(x, y)\phi(f(x) - f(y) + \frac{1}{2}p^T \nabla^2 f(y)p) \geq 0.
\]

Definition 2.4  
A function \( f \) is said to be second order strictly \((F, \alpha, \rho, d, p)\)-pseudo-univex at \( y \) with respect to \( b \) and \( \phi \), if there exist functions \( \alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+ \setminus \{0\}, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \phi : \mathbb{R} \to \mathbb{R}, d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \) \( a vector \ p \in \mathbb{R}^n \) and a real number \( \rho \) such that for all \( x \in \mathbb{R}^n \), \( x \neq y \), we have

\[
F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) \geq -\rho d^2(x, y) \Rightarrow b(x, y)\phi(f(x) - f(y) + \frac{1}{2}p^T \nabla^2 f(y)p) > 0.
\]

Definition 2.5  
A function \( f \) is said to be second order \((F, \alpha, \rho, d, p)\)-quasi-univex at \( y \) with respect to \( b \) and \( \phi \), if there exist functions \( \alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+ \setminus \{0\}, b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \phi : \mathbb{R} \to \mathbb{R}, d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \) \( a vector \ p \in \mathbb{R}^n \) and a real number \( \rho \) such that for all \( x \in \mathbb{R}^n \), \( b(x, y)\phi(f(x) - f(y) + \frac{1}{2}p^T \nabla^2 f(y)p) \leq 0 \Rightarrow F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) \leq -\rho d^2(x, y).
\]

From the definitions introduced above, one can see that second order \((F, \alpha, \rho, d, p)\)-univexity implies second order \((F, \alpha, \rho, d, p)\)-pseudo-univexity and second order \((F, \alpha, \rho, d, p)\)-quasi-univexity. In general, the notions of second order \((F, \alpha, \rho, d, p)\)-univexity and second order strict \((F, \alpha, \rho, d, p)\)-pseudo-univexity are not comparable. In order to show this, we give two examples as follows.

Example 2.2  
Under the conditions of Example 2.1, the function \( f \) is second order \((F, \alpha, \rho, d, p)\)-univex at \( y = 0 \) with respect to \( b \) and \( \phi \). But \( f \) is not second order strictly \((F, \alpha, \rho, d, p)\)-pseudo-univex at \( y = 0 \) with respect to \( b \) and \( \phi \), since for \( x = \sqrt{2} \) and \( y = 0 \), \( b(x, y)\phi(f(x) - f(y) + \frac{1}{2}p^T \nabla^2 f(y)p) = F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) + \rho d^2(x, y) = 0 \).
Example 2.3 Let $n = 1, f(x) = x^3 - 2x, b(x, y) = \alpha(x, y) = 1, \phi(t) = |t|$, $F(x, y, z) = |z(f(x) - f(y))|, \rho = -1$ and $d(x, y) = x - y - 2$. Taking $y = 0$, for any $x, p \in R$, we have $b(x, y)\phi(f(x) - f(y)) + \frac{d}{2}p^T\nabla^2 f(y)p = |x^3 - 2x|$ and $F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) + \rho d^2(x, y) = 2|x^3 - 2x| - (x - 2)^2$. One can easily see that $f$ is second order strictly $(F, \alpha, \rho, d, p)$-pseudo-univex at $y = 0$ with respect to $b$ and $\phi$. But $f$ is not second order $(F, \alpha, \rho, d, p)$-univex at $y = 0$ with respect to $b$ and $\phi$, because for $x = 2$ and $y = 0, b(x, y)\phi(f(x) - f(y)) + \frac{d}{2}p^T\nabla^2 f(y)p < F(x, y, \alpha(x, y)\{\nabla f(y) + \nabla^2 f(y)p\}) + \rho d^2(x, y)$.

The following necessary conditions for optimality of $(P)$ has been derived by Chandra and Kumar [3].

Theorem 2.1 Let $x^*$ be an optimal solution of $(P)$ and $\nabla g_j(x^*), j \in J(x^*)$ be linearly independent. Then, there exist $(s^*, t^*, y^*) \in K, v^* \in R_+^m$ and $\mu^* \in R_+^n$ such that

\begin{align}
\sum_{i=1}^{s^*} t_i^* (\nabla f(x^*, y_i^*)) - v^* \nabla h(x^*, y_i^*) + \nabla \sum_{i=1}^{m} \mu_j^* g_j(x^*) = 0, \\
f(x^*, y_i^*) - v^* h(x^*, y_i^*) = 0, \quad i = 1, \ldots, s^*, \\
\sum_{j=1}^{m} \mu_j^* g_j(x^*) = 0, \\
y_i^* \in Y(x^*), \quad i = 1, \ldots, s^*.
\end{align}

3. Duality theorems

In this section we consider two dual models (see [11]) to the minimax fractional programming problem $(P)$ and build weak, strong and strict converse duality for $(P)$. Now we state the first dual model as

\begin{align}
(DI) \quad \max_{(s, t, y) \in K} \sup_{(z, \mu, \lambda, p) \in H_1(s, t, y)} \lambda,
\end{align}

where $H_1(s, t, y)$ denotes the set of all $(z, \mu, \lambda, p) \in R^n \times R_+^m \times R_+ \times R^n$ satisfying

\begin{align}
\nabla \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \nabla^2 \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i))p + \\
\nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p = 0, \\
\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) - \frac{1}{2}p^T \nabla^2 \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i))p \geq 0, \\
\sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2}p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p \geq 0, \\
y_i \in Y(z), \quad i = 1, 2, \ldots, s.
\end{align}

If the set $H_1(s, t, y)$ is empty for a triplet $(s, t, y)$, we define the supremum over it to be $-\infty$. 
**Theorem 3.1** (Weak duality) Let \( x \) and \((z, \mu, \lambda, s, t, y, p)\) be feasible solutions of \((P)\) and \((DI)\), respectively. Assume that

(i) \( \sum_{i=1}^{s} t_i(f(\cdot, y_i) - \lambda h(\cdot, y_i)) \) is second order \((F, \alpha_1, \rho_1, d_1, p)\)-quasi-univex at \( z \) with respect to \( b_1 \) and \( \phi_1 \);

(ii) \( \sum_{j=1}^{m} \mu_j g_j(\cdot) \) is second order \((F, \alpha_2, \rho_2, d_2, p)\)-quasi-univex at \( z \) with respect to \( b_2 \) and \( \phi_2 \);

(iii) \( b_1(x, z) > 0, b_2(x, z) \geq 0, \phi_1(a) > 0 \Rightarrow a \geq \phi_2(a) \leq 0 \) and \( \rho_1 d_1^2(x, z) / \alpha_1(x, z) + \rho_2 d_2^2(x, z) / \alpha_2(x, z) > 0 \).

Then

\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq \lambda.
\]

**Proof** By the feasibility of \( x \) for \((P)\), \( \mu \in R^m_+ \) and \((3.7)\), we have

\[
\sum_{j=1}^{m} \mu_j g_j(x) \leq 0 \leq \sum_{j=1}^{m} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p.
\]

The assumption (iii) implies

\[
b_2(x, z) \phi_2 \left( \sum_{j=1}^{m} \mu_j g_j(x) - \sum_{j=1}^{m} \mu_j g_j(z) + \frac{1}{2} p^T \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p \right) \leq 0.
\]

Using the hypothesis (ii) we have

\[
F \left( x, z, \alpha_2(x, z) \left\{ \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p \right\} \right) \leq -\rho_2 d_2^2(x, z).
\]

Since \( \alpha_2(x, z) > 0 \) and \( F \) is sublinear in its third component, we get

\[
F \left( x, z, \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p \right) \leq -\alpha_2^{-1}(x, z) \rho_2 d_2^2(x, z).
\]

The sublinearity of \( F \) with \((5)\) yields

\[
F \left( x, z, \nabla \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \nabla^2 \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)p \right)
\]

\[
\geq - F \left( x, z, \nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z) p \right)
\]

\[
\geq \alpha_2^{-1}(x, z) \rho_2 d_2^2(x, z) > -\alpha_1^{-1}(x, z) \rho_1 d_1^2(x, z)
\]

Therefore

\[
F \left( x, z, \alpha_1(x, z) \left\{ \nabla \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \nabla^2 \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)p \right\} \right) > -\rho_1 d_1^2(x, z).
\]

The above inequality with assumption (i) implies

\[
b_1(x, z) \phi_1 \left( \sum_{i=1}^{s} t_i(f(x, y_i) - \lambda h(x, y_i)) - \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \right.
\]
be linearly independent. Then there exist \( \lambda \in H \) such that \( \lambda > 0 \), we get
\[
\sum_{i=1}^{s} t_i(f(x, y_i) - \lambda h(x, y_i)) = \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i(f(x, y_i) - \lambda h(z, y_i)) p \geq 0.
\]

Hence
\[
\sum_{i=1}^{s} t_i(f(x, y_i) - \lambda h(x, y_i)) \geq 0 \quad \text{(by (6))}.
\]

Therefore, there exists a certain \( i_0 \) such that
\[
f(x, y_{i_0}) - \lambda h(x, y_{i_0}) \geq 0.
\]

Thus
\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq \lambda.
\]

According to Remark 2.2, we have the following corollary.

**Corollary 3.1** In Theorem 3.1, if the assumption (i) is replaced by the condition that the function \( \sum_{i=1}^{s} t_i(f(\cdot, y_i) - \lambda h(\cdot, y_i)) \) is second order \( (F, \alpha_1, \rho_1, d_1, p) \)-univex at \( z \) with respect to \( b_1 \) and \( \phi_1 \); or the assumption (ii) is replaced by the condition that \( \sum_{j=1}^{m} \mu_j g_j(\cdot) \) is second order \( (F, \alpha_2, \rho_2, d_2, p) \)-univex at \( z \) with respect to \( b_2 \) and \( \phi_2 \), then the conclusion also holds.

**Theorem 3.2** (Weak duality) Let \( x \) and \((z, u, \lambda, s, t, y, p)\) be feasible solutions of \((P)\) and \((DI)\) respectively. Assume that
\[
\text{(i) } \sum_{i=1}^{s} t_i(f(\cdot, y_i) - \lambda h(\cdot, y_i)) \text{ is second order } (F, \alpha_1, \rho_1, d_1, p)-\text{pseudo-univex at } z \text{ with respect to } b_1 \text{ and } \phi_1;
\]
\[
\text{(ii) } \sum_{j=1}^{m} \mu_j g_j(\cdot) \text{ is second order } (F, \alpha_2, \rho_2, d_2, p)-\text{quasi-univex (or second order } (F, \alpha_2, \rho_2, d_2, p)-\text{univex)} \text{ at } z \text{ with respect to } b_2 \text{ and } \phi_2;
\]
\[
\text{(iii) } b_1(x, z) > 0, b_2(x, z) \geq 0, \phi_1(a) \geq 0 \Rightarrow a \geq 0, a \leq 0 \Rightarrow \phi_2(a) \leq 0 \text{ and } \rho_1 d_1^2(x, z)/\alpha_1(x, z) + \rho_2 d_2^2(x, z)/\alpha_2(x, z) \geq 0.
\]
Then
\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq \lambda.
\]

**Proof** The proof is similar to that of Theorem 3.1. □

**Theorem 3.3** (Strong duality) Let \( x^* \) be an optimal solution of \((P)\) and \( \nabla g_j(x^*), j \in J(x^*) \) be linearly independent. Then there exist \( (s^*, t^*, y^*) \in K \) and \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \in H_1(s^*, t^*, y^*) \) such that \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \) is a feasible solution of \((DI)\) and the corresponding values of \((P)\) and \((DI)\) are equal. In addition, if the assumptions of Theorem 3.1 (or Theorem 3.2) hold for all feasible solutions of \((DI)\) and \( x^* \), then \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \) is an optimal solution of \((DI)\).

**Proof** By Theorem 2.1, there exist \( (s^*, t^*, y^*) \in K \) and \((x^*, \mu^*, \lambda^*, p = 0) \in H_1(s^*, t^*, y^*) \) with \( \lambda^* = f(x^*, y_i^*)/h(x^*, y_i^*) \) and \( y_i^* \in Y(x^*) \) \((i = 1, 2, \ldots, s^*) \) such that \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \)
Thus, using the assumption (i) and \( x^* \) is a feasible solution of (DI), and so the corresponding values of (P) and (DI) are equal. From weak duality we can get that \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0)\) is an optimal solution of (DI).

**Theorem 3.4** (Strict converse duality) Let \( x^* \) and \((z^*, \mu^*, \lambda^*, s^*, t^*, y^*, p)\) be optimal solution of (P) and (DI), respectively. Let \( \nabla g_j(x^*) \), \( j \in J(x^*) \) be linearly independent. Assume that

(i) \( \sum_{i=1}^{s} t_i (f(\cdot, y_i^*) - \lambda h(\cdot, y_i^*)) \) is second order strictly \((F, \alpha_1, \rho_1, d_1, p)\)-pseudo-univex at \( z^* \) with respect to \( b_1, \phi_1; \)

(ii) \( \sum_{j=1}^{m} \mu_j g_j(\cdot) \) is second order \((F, \alpha_2, \rho_2, d_2, p)\)-quasi-univex at \( z^* \) with respect to \( b_2, \phi_2; \)

(iii) \( b_1(x^*, z^*) > 0, b_2(x^*, z^*) \geq 0, \phi_1(a) > 0 \Rightarrow a > 0, a \leq 0 \Rightarrow \phi_2(a) \leq 0, \) and the inequality \( \rho_1 d_1^2(x^*, z^*)/\alpha_1(x^*, z^*) + \rho_2 d_2^2(x^*, z^*)/\alpha_2(x^*, z^*) \geq 0 \) holds.

Then \( x^* = z^*; \) that is to say, \( z^* \) is an optimal solution of (P).

**Proof** We shall assume that \( x^* \neq z^* \) and reach a contradiction. From Theorem 3.3, there exists \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, 0)\) to be a feasible solution of (DI) such that

\[
\sup_{y \in Y} \frac{f(x^*, y)}{h(x^*, y)} = \lambda'.
\]

Similarly to the proof of Theorem 3.1, we can obtain

\[
F(x^*, z^*, \alpha_1(x^*, z^*)\{\nabla \sum_{i=1}^{s} t_i (f(x^*, y_i^*) - \lambda h(x^*, y_i^*)) + \nabla^2 \sum_{i=1}^{s} t_i (f(z^*, y_i^*) - \lambda h(z^*, y_i^*))p\})
\geq -\rho_1 d_1^2(x^*, z^*).
\]

Using the assumption (i) and \( x^* \neq z^* \), we get

\[
b_1(x^*, z^*)\phi_1(\sum_{i=1}^{s} t_i (f(x^*, y_i^*) - \lambda h(x^*, y_i^*)) - \sum_{i=1}^{s} t_i (f(z^*, y_i^*) - \lambda h(z^*, y_i^*))) + \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} (f(z^*, y_i^*) - \lambda h(z^*, y_i^*))p > 0.
\]

Since \( b_1(x^*, z^*) > 0 \) and \( \phi_1(a) > 0 \Rightarrow a > 0, \) it follows that

\[
\sum_{i=1}^{s} t_i (f(x^*, y_i^*) - \lambda h(x^*, y_i^*)) - \sum_{i=1}^{s} t_i (f(z^*, y_i^*) - \lambda h(z^*, y_i^*)) + \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} (f(z^*, y_i^*) - \lambda h(z^*, y_i^*))p > 0.
\]

Hence

\[
\sum_{i=1}^{s} t_i (f(x^*, y_i^*) - \lambda h(x^*, y_i^*)) > 0 \; \text{(by (6))}.
\]

Therefore, there exists a certain \( i_0 \) such that

\[
f(x^*, y_{i_0}^*) - \lambda h(x^*, y_{i_0}^*) > 0.
\]

Thus

\[
\sup_{y \in Y} \frac{f(x^*, y)}{h(x^*, y)} > \lambda^*.
\]
which combined with (3.9) implies \( \lambda^* > \lambda^* \). It contradicts the assumption that \((z^*, \mu^*, \lambda^*, s^*, t^*, y^*, p)\) is an optimal solution of (DI). Hence \( x^* = z^* \).

**Corollary 3.2** In Theorem 3.4, if the assumption (ii) is replaced by the condition that \( \sum_{j=1}^{m} \mu_j g_j(\cdot) \) is second order \((F, \alpha_2, \rho_2, d_2, p)\)-univex at \( z^* \) with respect to \( b_2, \phi_2 \), then the conclusion also holds.

Next, we formulate the second dual model to (P) as

\[
\text{(DII)} \quad \max_{(s, t, y) \in K} \sup_{(z, \mu, \lambda, p) \in H_2(s, t, y)} \lambda,
\]

where \( H_2(s, t, y) \) denotes the set of all \((z, \mu, \lambda, p) \in R^n \times R^m_+ \times R_+ \times R^n \) satisfying

\[
\nabla \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \nabla^2 \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i))p = 0,
\]

\[
\nabla \sum_{j=1}^{m} \mu_j g_j(z) + \nabla^2 \sum_{j=1}^{m} \mu_j g_j(z)p = 0,
\]

\[
\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \sum_{j \in J_0} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i))p - \frac{1}{2} p^T \nabla^2 \sum_{j \in J_0} \mu_j g_j(z)p \geq 0,
\]

\[
\sum_{j \in J_l} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j \in J_l} \mu_j g_j(z)p \geq 0, \quad l = 1, 2, \ldots, r,
\]

\[
y_i \in Y(z), \quad i = 1, 2, \ldots, s,
\]

where \( J_s \subset J, \ l = 0, 1, \ldots, r \) with \( \bigcup_{l=0}^{r} J_l = J \) and \( J_1 \cap J_2 = \emptyset \) if \( l_1 \neq l_2 \). If the set \( H_2(s, t, y) \) is empty for a triplet \((s, t, y)\), we define the supremum over it to be \(-\infty\).

**Theorem 3.5** (Weak duality) Let \( x \) and \((z, \mu, \lambda, s, t, y, p)\) be feasible solutions of (P) and (DII), respectively. Assume that

(i) \( \sum_{i=1}^{s} t_i(f(\cdot, y_i) - \lambda h(\cdot, y_i)) + \sum_{j \in J_0} \mu_j g_j(\cdot) \) is second order \((F, \alpha_0, \rho_0, d_0, p)\)-quasi-univex at \( z \) with respect to \( b_0, \phi_0 \);

(ii) \( \sum_{j \in J_l} \mu_j g_j(\cdot) \), is second order \((F, \alpha_l, \rho_l, d_l, p)\)-quasi-univex at \( z \) with respect to \( b_l, \phi_l \) \((l = 1, 2, \ldots, r)\);

(iii) \( b_0(x, z) > 0, b_l(x, z) \geq 0, \phi_0(a) > a \geq 0 \Rightarrow a \geq 0 \Rightarrow \phi_l(a) \leq 0 \) \((l = 1, 2, \ldots, r)\) and \( \sum_{l=0}^{r} \rho_l d_l^2(x, z)/\alpha_l(x, z) > 0 \).

Then

\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq \lambda.
\]

**Proof** Since \( x \) is a feasible solution of (P) and \((z, \mu, \lambda, s, t, y, p)\) is a feasible solution of (DII), we have that, for any \( l \in \{1, 2, \ldots, r\} \),

\[
\sum_{j \in J_l} \mu_j g_j(x) \leq \sum_{j \in J_l} \mu_j g_j(z) - \frac{1}{2} p^T \nabla^2 \sum_{j \in J_l} \mu_j g_j(z)p.
\]
It follows from $b_l(x, z) \geq 0$, $a \leq 0 \Rightarrow \phi_l(a) \leq 0$ that
\[ b_l(x, z)\phi_l\left(\sum_{j \in J_l} \mu_j g_j(x) - \sum_{j \in J_l} \mu_j g_j(z) + \frac{1}{2} b^T \nabla^2 \sum_{j \in J_l} \mu_j g_j(z)p\right) \leq 0, \quad l = 1, 2, \ldots, r. \]

Using the hypothesis (ii), we have
\[ F\left(x, z, \alpha_l(x, z)\left\{\nabla \sum_{j \in J_l} \mu_j g_j(z) + \nabla^2 \sum_{j \in J_l} \mu_j g_j(z)p\right\}\right) \leq -\rho_0 d_l^2(x, z), \quad l = 1, 2, \ldots, r. \]

Since $\alpha_l(x, z) > 0$ and $F$ is sublinear in its third component, we get
\[ F\left(x, z, \nabla \sum_{j \in J_l} \mu_j g_j(z) + \nabla^2 \sum_{j \in J_l} \mu_j g_j(z)p\right) \leq -\alpha_l^{-1}(x, z)\rho_0 d_l^2(x, z), \quad l = 1, 2, \ldots, r. \]

The sublinearity of $F$ with (10) implies
\[ F\left(x, z, \nabla \left[\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \sum_{j \in J_{0}} \mu_j g_j(z)\right]\right] + \nabla^2 \left[\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \sum_{j \in J_{0}} \mu_j g_j(z)p\right] \geq \sum_{l=1}^{r} -F\left(x, z, \nabla \sum_{j \in J_{l}} \mu_j g_j(z) + \nabla^2 \sum_{j \in J_{l}} \mu_j g_j(z)p\right) \geq \sum_{l=1}^{r} \alpha_l^{-1}(x, z)\rho_0 d_l^2(x, z) > -\alpha_0^{-1}(x, z)\rho_0 d_0^2(x, z). \]

Therefore
\[ F\left(x, z, \alpha_0(x, z)\left\{\nabla \left[\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \sum_{j \in J_{0}} \mu_j g_j(z)\right]\right\} + \nabla^2 \left[\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \sum_{j \in J_{0}} \mu_j g_j(z)p\right]\right) \geq -\rho_0 d_0^2(x, z). \]

By using the assumption (i), we have
\[ b_0(x, z)\phi_0\left(\sum_{i=1}^{s} t_i(f(x, y_i) - \lambda h(x, y_i)) + \sum_{j \in J_{0}} \mu_j g_j(x) - \sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) - \sum_{j \in J_{0}} \mu_j g_j(z) + \frac{1}{2} b^T \nabla^2 \left[\sum_{i=1}^{s} t_i(f(z, y_i) - \lambda h(z, y_i)) + \sum_{j \in J_{0}} \mu_j g_j(z)p\right]\right) > 0. \]

Utilizing the assumption (iii) and (11), we obtain
\[ \sum_{i=1}^{s} t_i(f(x, y_i) - \lambda h(x, y_i)) \geq 0. \]

Hence, there exists $i_0$ such that
\[ f(x, y_{i_0}) - \lambda h(x, y_{i_0}) \geq 0. \]

It follows that
\[ \sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq \frac{f(x, y_{i_0})}{h(x, y_{i_0})} \geq \lambda. \]
Similarly, we have the following corollary.

**Corollary 3.3** In Theorem 3.5, if the assumption (i) is replaced by the condition that \( \sum_{i=1}^{s} t_i (f(\cdot, y_i) - \lambda h(\cdot, y_i)) + \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order \((F_0, \rho_0, d_0, p)\)-univex at \( z \) with respect to \( b_0, \phi_0 \); or the assumption (ii) is replaced by the condition that \( \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order \((F_0, \rho_0, d_0, p)\)-univex at \( z \) with respect to \( b_0, \phi_0 \), the conclusion also holds.

**Theorem 3.6** (Weak duality) Let \( (x, z, \mu, \lambda, s, t, y, p) \) be feasible solutions of \((P)\) and \((DII)\), respectively. Assume that

1. \( \sum_{i=1}^{s} t_i (f(\cdot, y_i) - \lambda h(\cdot, y_i)) + \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order \((F_0, \rho_0, d_0, p)\)-pseudo-univex at \( z \) with respect to \( b_0, \phi_0 \);
2. \( \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order \((F_0, \rho_0, d_0, p)\)-quasi-univex (or second order \((F_0, \rho_0, d_0, p)\)-univex) at \( z \) with respect to \( b_0, \phi_0 \) \( l = 1, 2, \ldots, r \);
3. \( b_0 (x, z) > 0, b_l (x, z) \geq 0, \phi_0 (a) \geq 0 \Rightarrow a \geq 0, a \leq 0 \Rightarrow \phi_l (a) \leq 0 \) \( (l = 1, 2, \ldots, r) \) and \( \sum_{l=0}^{r} \rho_l d_l^2 (x, z) / \alpha_l (x, z) \geq 0 \).

Then

\[
\sup_{y \in Y} \frac{f(x, y)}{h(x, y)} \geq \lambda.
\]

**Proof** The proof is similar to that of Theorem 3.5. \( \square \)

Similarly to Theorems 3.3 and 3.4, we can establish the following theorems.

**Theorem 3.7** (Strong duality) Let \( x^* \) be an optimal solution of \((P)\) and \( \nabla g_j (x^*) \), \( j \in J(x^*) \) be linearly independent. Then there exist \((s^*, t^*, y^*) \in K \) and \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \in H_2 (s^*, t^*, y^*) \) such that \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \) is a feasible solution of \((DII)\) and the corresponding values of \((P)\) and \((DII)\) are equal. In addition, if the assumptions of Theorem 3.5 (or Theorem 3.6) hold for all feasible solution of \((DII)\) and \( x^* \), then \((x^*, \mu^*, \lambda^*, s^*, t^*, y^*, p = 0) \) is an optimal solution of \((DII)\).

**Theorem 3.8** (Strict converse duality) Let \( x^* \) and \((z^*, \mu^*, \lambda^*, s^*, t^*, y^*, p) \) be optimal solution of \((P)\) and \((DII)\). Assume that

1. \( \sum_{i=1}^{s} t_i (f(\cdot, y_i) - \lambda h(\cdot, y_i)) + \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order strictly \((F_0, \rho_0, d_0, p)\)-pseudo-univex at \( z^* \) with respect to \( b_0, \phi_0 \);
2. \( \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order \((F_0, \rho_0, d_0, p)\)-quasi-univex at \( z^* \) with respect to \( b_l, \phi_l \) \( l = 1, 2, \ldots, r \);
3. \( b_0 (x^*, z^*) > 0, b_l (x^*, z^*) \geq 0, \phi_0 (a) > 0 \Rightarrow a > 0, a \leq 0 \Rightarrow \phi_l (a) \leq 0 \) \( (l = 1, 2, \ldots, r) \) and \( \sum_{l=0}^{r} \rho_l d_l^2 (x^*, z^*) / \alpha_l (x^*, z^*) \geq 0 \).

Then \( x^* = z^* \), that is to say, \( z^* \) is an optimal solution of \((P)\).

**Corollary 3.4** In Theorem 3.8, if the assumption (ii) is replaced by the condition that \( \sum_{j \in J_i} \mu_j g_j (\cdot) \) is second order \((F_0, \rho_0, d_0, p)\)-univex at \( z^* \) with respect to \( b_l, \phi_l \) \( (l = 1, 2, \ldots, r) \), the conclusion also holds.

4. Conclusion
In this study, a new class of generalized second order univex functions is proposed. Some examples to support the introduced concepts are given. Based on the generalized second order univex functions, the second order duality theorems for a minimax fractional programming problem are established. It is of interest to extend the present approach to establish similar results for other classes of optimization problems such as in [5]:

Minimize \[ \sup_{y \in Y} \frac{f(x,y) + (x^T B x)^{1/2}}{h(x,y) - (x^T D x)^{1/2}} \]
subject to \( g(x) \leq 0, \ x \in \mathbb{R}^n \),

where \( Y \) is a compact subset of \( \mathbb{R}^k \), \( f, h : \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R} \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R}^m \) are in \( C^2 \), and \( B \) and \( D \) are \( n \times n \) positive semidefinite matrices.

References