Journal of Mathematical Research with Applications
May, 2013, Vol. 33, No. 3, pp. 273–287
DOI:10.3770/j.issn:2095-2651.2013.03.002
Http://jmre.dlut.edu.cn

Primitive Non-Powerful Symmetric Loop-Free Signed Digraphs with Base 3 and Minimum Number of Arcs

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Abstract Let S be a primitive non-powerful symmetric loop-free signed digraph on even n vertices with base 3 and minimum number of arcs. In [Lihua YOU, Yuhan WU. Primitive non-powerful symmetric loop-free signed digraphs with given base and minimum number of arcs. Linear Algebra Appl., 2011, 434(5), 1215–1227], authors conjectured that D is the underlying digraph of S with $\exp(D)=3$ if and only if D is isomorphic to $ED_{n,3,3}$, where $ED_{n,3,3}=(V,A)$ is a digraph with $V=\{1,2,\ldots,n\}$, $A=\{(1,i),(i,1)\mid 3\le i\le n\}\cup\{(2i-1,2i),(2i,2i-1)\mid 2\le i\le \frac{n}{2}\}\cup\{(2,3),(3,2),(2,4),(4,2)\})$. In this paper, we show the conjecture is true and completely characterize the underlying digraphs which have base 3 and the minimum number of arcs.

Keywords primitive; symmetric; non-powerful; base; signed digraph.

MR(2010) Subject Classification 05C22; 05C50; 15B35

1. Introduction

A sign pattern matrix is a matrix each of whose entries is a sign 1, −1 or 0. For a square sign pattern matrix M, notice that in the computations of the entries of the power M^k, the "ambiguous sign" may arise when we add a positive sign 1 to a negative sign −1. Then a new symbol "#" was introduced in [1] to denote the ambiguous sign. The set $\Gamma=\{0,1,-1,\#\}$ is defined as the generalized sign set and the addition and multiplication involving the symbol # are defined as follows (the addition and multiplication which do not involve # are obvious):

$$(-1)+1=1+(-1)=\#;\quad a+\#=\#+a=\#,\quad \text{for all } a\in\Gamma,$$

$$0\cdot\#=\#\cdot 0=0;\quad b\cdot\#=\#\cdot b=\#,\quad \text{for all } b\in\Gamma\setminus\{0\}.$$

In [1, 2], the matrices with entries in the set Γ are called generalized sign pattern matrices. The addition and multiplication of generalized sign pattern matrices are defined in the usual way, then the sum and product of the generalized sign pattern matrices are still generalized sign pattern matrices. In this paper, we only consider the operations of matrices over Γ.

Received December 27, 2011; Accepted September 3, 2012
Supported by the National Natural Science Foundation of China (Grant Nos. 10901061; 11071088), Program on International Cooperation and Innovation, Department of Education, Guangdong Province (Grant No. 2012gjhz0007) and the Zhujiang Technology New Star Foundation of Guangzhou City (Grant No. 2011J2200090).
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Definition 1 ([1]) *A square generalized sign pattern matrix M is called powerful if each power of M contains no # entry.*

Definition 2 ([3]) *Let M be a square generalized sign pattern matrix of order n and $M, M^2, M^3, \ldots$ be the sequence of powers of M. Suppose M^b is the first power that is repeated in the sequence. Namely, suppose b is the least positive integer such that there is a positive integer p such that*

$$M^b = M^{b+p}. \tag{1.1}$$

Then b is called the generalized base (or simply base) of M, and is denoted by $b(M)$. The least positive integer p such that (1.1) holds for $b = b(M)$ is called the generalized period (or simply period) of M, and is denoted by $p(M)$.

We now introduce some theoretical concepts of graph.

Let $D = (V, A)$ denote a digraph on n vertices. Loops are permitted, but no multiple arcs. A $u \to v$ walk in D is a sequence of vertices $u, u_1, \ldots, u_k = v$ and a sequence of arcs $e_1 = (u, u_1), e_2 = (u_1, u_2), \ldots, e_k = (u_{k-1}, v)$, where the vertices and the arcs are not necessarily distinct. We use the notation $u \to u_1 \to u_2 \to \cdots \to u_{k-1} \to v$ to refer to this $u \to v$ walk. A closed walk is a $u \to v$ walk where $u = v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W, denoted by $l(W)$. A k-cycle is a cycle of length k, denoted by C_k.

A signed digraph S is a digraph where each arc of S is assigned a sign 1 or -1. A generalized signed digraph S is a digraph where each arc of S is assigned a sign 1, -1 or #.

The sign of the walk W in a (generalized) signed digraph, denoted by sgn$W$, is defined to be $\prod_{i=1}^{k} \mathrm{sgn}(e_i)$, where $e_1, e_2, \ldots, e_k$ is the sequence of arcs of W.

Let $M = (m_{ij})$ be a square (generalized) sign pattern matrix of order n. The associated digraph $D(M) = (V, A)$ of M (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $A = \{(i, j) | m_{ij} \neq 0\}$. The associated (generalized) signed digraph $S(M)$ of M is obtained from $D(M)$ by assigning the sign of m_{ij} to each arc (i, j) in $D(M)$, and we say $D(M)$ is the underlying digraph of $S(M)$.

Let S be a (generalized) signed digraph on n vertices. Then there is a (generalized) sign pattern matrix M of order n whose associated (generalized) signed digraph $S(M)$ is S. We say that S is powerful if M is powerful. Also the base $b(S)$ and period $p(S)$ are defined to be those of M. Namely we define $b(S) = b(M)$ and $p(S) = p(M)$.

A digraph D is said to be strongly connected if there exists a path from u to v for all $u, v \in V$, and D is called primitive if there is a positive integer k such that for each vertex x and each vertex y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the primitive exponent (or simply exponent) of D, denoted by $\exp(D)$. It is also well-known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor (simply g.c.d.) of the lengths of all the cycles of D is 1. A (generalized) signed digraph S is called primitive if the underlying digraph D is primitive, and in this case we define $\exp(S) = \exp(D)$.

A digraph D is symmetric if for every arc (u,v) in D, the arc (v,u) is also in D. A (generalized) signed digraph S is called symmetric if the underlying digraph D is symmetric. If a digraph (or a generalized signed digraph) D (or S) is symmetric, then D (or S) can be regarded as an undirected graph (possibly with loops).

A digraph D is loop-free if D has no loops. In this case, if a digraph (or a generalized signed digraph) D (or S) is symmetric and loop-free, then D (or S) can be regarded as a simple graph.

The primitive exponent and exponent set of primitive symmetric loop-free digraphs were discussed in [4, 5] and the minimum number, $h(n,k)$, of edges of primitive simple graphs $G = (V,E)$ such that $|V| = n$ and $\exp(G) = k$ were determined completely in [6, 7].

Now the concept of exponent for primitive digraphs was extended to the concept of base for primitive signed digraphs [3], a natural question is to study the minimum number of arcs of primitive symmetric loop-free signed digraphs. It was shown in [1] that if a primitive signed digraph S is powerful, then $b(S) = \exp(D)$, where D is the underlying digraph of S. Then for a primitive powerful symmetric loop-free signed digraph, [7] gives the results. In [8, 9], the case of the non-powerful is studied.

Theorem 3 ([9]) *Let $\mathbf{B}_n^\star$ be the base set of primitive non-powerful symmetric loop-free signed digraphs on n vertices. Then $\mathbf{B}_n^\star = \{2, 3, \dots, 2n-1\}$.*

Theorem 4 ([9]) *Let $H(n,k)$ be the minimum number of arcs of the primitive non-powerful symmetric loop-free signed digraphs $S = (V,A)$ such that $|V| = n$ and $b(S) = k$ for $2 \le k \le 2n-1$. Then we have*

(1) $H(n,2) = 2\lfloor \frac{5n-7}{2} \rfloor$.

(2) $H(n,3) = 2\lfloor \frac{3n-2}{2} \rfloor$.

(3) $H(n,k) = 2n$ *for* $4 \le k \le 2n-1$.

In [9], the underlying digraphs of the primitive non-powerful symmetric loop-free signed digraphs which have $H(n,k)$ arcs with $k = 2$ are completely characterized, and the case when $k = 3$ is nearly characterized.

Let $n = 2m+1$ be odd, $OD_{n,3} = (V,A)$ be a digraph, where $V = \{1, 2, \dots, 2m, 2m+1\}$, $A = \{(1,i),(i,1) \mid 2 \le i \le 2m+1\} \cup \{(2i, 2i+1), (2i+1, 2i) \mid 1 \le i \le m\}$. Clearly, $OD_{n,3}$ is a primitive symmetric loop-free digraph on n vertices with $\exp(OD_{n,3}) = 2$.

Let $n = 2m$ be even, $ED_{n,3} = (V,A)$ be a digraph, where $V = \{1, 2, \dots, 2m-1, 2m\}$, $A = \{(1,i),(i,1) \mid 2 \le i \le 2m\} \cup \{(2i, 2i+1), (2i+1, 2i) \mid 1 \le i \le m-1\} \cup \{(2m-1, 2m), (2m, 2m-1)\}$. Clearly, $ED_{n,3}$ is a primitive symmetric loop-free digraph on n vertices with $\exp(ED_{n,3}) = 2$.

Let $n = 2m$ be even, $ED_{n,3,3} = (V,A)$ a digraph, where $V = \{1, 2, \dots, 2m\}$, $A = \{(1,i),(i,1) \mid 3 \le i \le 2m\} \cup \{(2i-1, 2i), (2i, 2i-1) \mid 2 \le i \le m\} \cup \{(2,3), (3,2), (2,4), (4,2)\}$. Clearly, $ED_{n,3,3}$ is a primitive symmetric loop-free digraph on n vertices with $\exp(ED_{n,3,3}) = 3$.

Theorem 5 ([9]) *Let $n \ge 6$, $S = (V(S), A(S))$ be a primitive non-powerful symmetric loop-free signed digraph on n vertices with $|A(S)| = H(n,3)$ and $b(S) = 3$. Then one of the following conditions holds:*

(1) If n is odd, D is the underlying digraph of S if and only if D is isomorphic to the digraph $OD_{n,3}$.

(2) If n is even, D is the underlying digraph of S with $\exp(D) = 2$ if and only if D is isomorphic to the digraph $ED_{n,3}$.

(3) If n is even, there exists D which is isomorphic to $ED_{n,3,3}$ such that D is the underlying digraph of S with $\exp(D) = 3$.

Conjecture 6 ([9]) *Let $n \geq 6$ be even, $S = (V(S), A(S))$ be a primitive non-powerful symmetric loop-free signed digraph on n vertices with $|A(S)| = 3n - 2$, $\exp(S) = 3$, $b(S) = 3$. Then D is the underlying digraph of S if and only if D is isomorphic to the digraph $ED_{n,3,3}$.*

In this paper, we show the Conjecture 6 is true, and completely characterize the underlying digraphs which have base 3 and the minimum number of arcs.

2. Some preliminaries

In this section, we introduce some useful definitions and properties in the proofs of our main results. Other definitions and results not in this article can be found in [10, 11].

A subgraph $H = (V_H, E_H)$ of $K = (V_K, E_K)$, denoted by $H \subseteq K$, is a graph if $V_H \subseteq V_K$ and $E_H \subseteq E_K$. A proper subgraph $H = (V_H, E_H)$ of $K = (V_K, E_K)$, denoted by $H \subsetneq K$, is a graph if $V_H \subsetneq V_K$, $E_H \subseteq E_K$ or $V_H = V_K$, $E_H \subsetneq E_K$.

If $K_i = (V_{K_i}, E_{K_i}) \subseteq K$ for all $i \in I$, we define the union $\bigcup_{i \in I} K_i$ of K_i as the graph $(\bigcup_{i \in I} V_{K_i}, \bigcup_{i \in I} E_{K_i})$. Throughout this paper, we define some subgraphs of $K = (V_K, E_K)$ as follows:

For $u, v, w \in V_K$,

$$L(u,v) = (V_L, E_L), \text{ where } V_L = \{u,v\}, E_L = \{\{u,v\}\}$$

and

$$\triangle(u,v,w) = (V_\triangle, E_\triangle) \text{ where } V_\triangle = \{u,v,w\}, E_\triangle = \{\{u,v\},\{v,w\},\{w,u\}\}.$$

We say that $L(u,v)$ is a line segment joining u and v, and $\triangle(u,v,w)$ is a triangle with vertices u, v and w.

Definition 7 ([3]) *Two walks W_1 and W_2 in a signed digraph are called a pair of $SSSD$ walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.*

From the relation between sign pattern matrices and signed digraphs, it is easy to see that a (generalized) sign pattern matrix M is powerful if and only if the associated (generalized) signed digraph $S(M)$ contains no pairs of $SSSD$ walks. Thus for a (generalized) signed digraph S, S is powerful if and only if S contains no pairs of $SSSD$ walks.

The following result will be useful.

Theorem 8 ([3]) *Let S be a primitive non-powerful signed digraph. Then*

(1) There is an integer k such that there exists a pair of $SSSD$ walks of length k from each vertex x to each vertex y in S.

(2) If there exists a pair of $SSSD$ walks of length k from each vertex x to each vertex y, then there also exists a pair of $SSSD$ walks of length $k+1$ from each vertex x to each vertex y in S.

(3) The minimal such k (as in (1)) is just $b(S)$, the base of S.

3. Characterization of simple graphs $G = (V, E)$ with $\exp(G) = 3$ and $|E| = \frac{3|V|-2}{2}$

In this section, we characterize simple graphs $G = (V, E)$ with $\exp(G) = 3$ and $|E| = \frac{3|V|-2}{2}$, where $|V|$ is even.

In Theorem 9, the following graphs will be used.

(1) $\Lambda_1(k_1, k_2, k_3) = \triangle(u_1, u_2, u_3) \cup \triangle(u_2, u_3, u_4) \cup (\bigcup_{i=1}^{k_1} \triangle(u_1, v_{2i-1}, v_{2i}))\cup (\bigcup_{i=1}^{k_2} \triangle(u_2, w_{2i-1}, w_{2i})) \cup (\bigcup_{i=1}^{k_3} \triangle(u_3, x_{2i-1}, x_{2i}))$.

(2) $\Lambda_2(k_1, k_2, k_3) = \triangle(u_1, u_2, u_3) \cup \triangle(u_2, u_3, u_4) \cup (\bigcup_{i=1}^{k_1} \triangle(u_2, v_{2i-1}, v_{2i}))\cup (\bigcup_{i=1}^{k_2} \triangle(v_1, w_{2i-1}, w_{2i})) \cup (\bigcup_{i=1}^{k_3} \triangle(v_2, x_{2i-1}, x_{2i}))$.

(3) $\Lambda_3(k_1, k_2) = L(u, v) \cup (\bigcup_{i=1}^{k_1} \triangle(u, u_{2i-1}, u_{2i})) \cup \triangle(u, u_{2k_1}, u_{2k_1+1})\cup (\bigcup_{i=1}^{k_2} \triangle(v, v_{2i-1}, v_{2i})) \cup \triangle(v, v_{2k_2}, v_{2k_2+1})$.

(4) $\Lambda_4(k_1, k_2) = L(u, v) \cup (\bigcup_{i=1}^{k_1} \triangle(u, u_{2i-1}, u_{2i})) \cup (\bigcup_{i=1}^{k_2-1} \triangle(v, v_{2i-1}, v_{2i}))\cup \triangle(v, v_{2k_2-1}, v_i) \cup \triangle(v, v_{2k_2}, v_j)(i, j \in \{1, \ldots, 2k_2 - 2\})$.

(5) $\Lambda_5(k_1, k_2) = L(u, v_1) \cup L(u, v_2) \cup \triangle(v_1, v_3, v_4) \cup \triangle(v_2, v_4, v_5)\cup (\bigcup_{i=1}^{k_1} \triangle(u, u_{2i-1}, u_{2i})) \cup (\bigcup_{i=1}^{k_2} \triangle(v_1, w_{2i-1}, w_{2i}))$.

(6) $\Lambda_6(k_1, k_2) = \triangle(u_1, u_2, u_3) \cup L(u_1, v_1) \cup L(u_2, v_2) \cup \triangle(v_1, v_2, v_3)\cup (\bigcup_{i=1}^{k_1} \triangle(u_1, w_{2i-1}, w_{2i})) \cup (\bigcup_{i=1}^{k_2} \triangle(u_2, x_{2i-1}, x_{2i}))$.

(7) $\Lambda_7(k_1, k_2) = \triangle(u_1, u_2, u_3) \cup L(u_1, v_1) \cup L(u_2, v_2) \cup \triangle(v_1, v_2, v_3)\cup (\bigcup_{i=1}^{k_1} \triangle(u_1, w_{2i-1}, w_{2i})) \cup (\bigcup_{i=1}^{k_2} \triangle(v_1, w_{2i-1}, w_{2i}))$.

(8) $\Lambda_8(k) = \triangle(u_1, u_2, u_3) \cup L(u_1, v_1) \cup L(u_2, v_2) \cup \triangle(v_1, v_3, v_4) \cup \triangle(v_2, v_4, v_5)\cup (\bigcup_{i=1}^{k} \triangle(v_1, w_{2i-1}, w_{2i}))$.

(9) $\Lambda_9(k_1, k_2) = \triangle(u_1, u_2, u_3) \cup \triangle(u_1, u_2, u_4) \cup L(u_1, v) \cup L(u_2, w) \cup L(v, w)\cup (\bigcup_{i=1}^{k_1} \triangle(v, v_{2i-1}, v_{2i})) \cup (\bigcup_{i=1}^{k_2} \triangle(w, w_{2i-1}, w_{2i}))$.

(10) $\Lambda_{10}(k_1, k_2) = L(u, v_1) \cup \triangle(v_1, v_2, v_3) \cup L(v_2, w) \cup L(w, u)\cup (\bigcup_{i=1}^{k_1} \triangle(u, u_{2i-1}, u_{2i})) \cup \triangle(u, u_{2k_1}, u_{2k_1+1}) \cup (\bigcup_{i=1}^{k_2} \triangle(w, w_{2i-1}, w_{2i}))$.

(11) $\Lambda_{11}(k) = \triangle(u_1, u_2, u_3) \cup \triangle(u_1, u_2, u_4) \cup L(u_1, v_1) \cup \triangle(v_1, v_2, v_3) \cup L(v_2, w)\cup L(w, u_2) \cup (\bigcup_{i=1}^{k} \triangle(w, w_{2i-1}, w_{2i}))$.

(12) $\Lambda_{12}(k) = L(u, v_1) \cup \triangle(v_1, v_2, v_3) \cup L(v_2, w_1) \cup \triangle(w_1, w_2, w_3) \cup L(w_2, u)\cup (\bigcup_{i=1}^{k} \triangle(u, u_{2i-1}, u_{2i})) \cup \triangle(u, u_{2k}, u_{2k+1})$.

(13) $\Lambda_{13} = \triangle(u_1, u_2, u_3) \cup \triangle(u_1, u_2, u_4) \cup L(u_1, v_1) \cup \triangle(v_1, v_2, v_3) \cup L(v_2, w_1)\cup \triangle(w_1, w_2, w_3) \cup L(w_2, u_2)$.

Graphs $\Lambda_1(k_1, k_2, k_3)$ and $\Lambda_2(k_1, k_2, k_3)$ are shown in Figures 1-2.

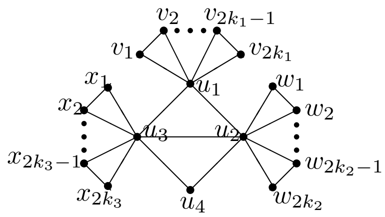

Figure 1 $\Lambda_1(k_1,k_2,k_3)$

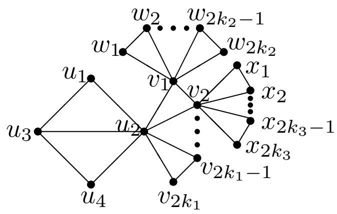

Figure 2 $\Lambda_2(k_1,k_2,k_3)$

Theorem 9 *Let $G=(V,E)$ be a primitive loop-free graph on even n vertices. If $|E|=\frac{3n-2}{2}$ and $\exp(G)=3$, then G is isomorphic to one of the following graphs:*

(1) $\Lambda_1(k_1,k_2,k_3)$, $k_1\neq 0$ or $k_2\cdot k_3\neq 0$, $k_1+k_2+k_3=\frac{n-4}{2}$.

(2) $\Lambda_2(k_1,k_2,k_3)$, $k_1\cdot k_2\neq 0$, $k_3\geq 0$, $k_1+k_2+k_3=\frac{n-4}{2}$.

(3) $\Lambda_3(k_1,k_2)$, $k_1\geq 1$, $k_2\geq 1$, $k_1+k_2=\frac{n-4}{2}$.

(4) $\Lambda_4(k_1,k_2)$, $k_1\geq 1$, $k_2\geq 2$, $k_1+k_2=\frac{n-2}{2}$.

(5) $\Lambda_5(k_1,k_2), k_1\geq 1, k_2\geq 0, k_1+k_2=\frac{n-6}{2}$.

(6) $\Lambda_6(k_1,k_2), k_1\geq 0, k_2\geq 0, k_1+k_2=\frac{n-6}{2}$.

(7) $\Lambda_7(k_1,k_2), k_1\geq 0, k_2\geq 0, k_1+k_2=\frac{n-6}{2}$.

(8) $\Lambda_8(k), k=\frac{n-8}{2}$.

(9) $\Lambda_9(k_1,k_2), k_1,k_2\geq 1, k_1+k_2=\frac{n-6}{2}$.

(10) $\Lambda_{10}(k_1,k_2), k_1,k_2\geq 1, k_1+k_2=\frac{n-6}{2}$.

(11) $\Lambda_{11}(k), k=\frac{n-8}{2}\geq 1$.

(12) $\Lambda_{12}(k), k=\frac{n-8}{2}\geq 1$.

(13) Λ_{13}.

In order to show Theorem 9, the following lemmas are needed.

Lemma 10 *Let $K=(V_K,E_K)$ be a graph such that $\exp(K)=3$. Then for any vertex $u\in V_K$, there are vertices $v,w\in V_K$, such that $\triangle(u,v,w)\subseteq K$.*

Lemma 11 ([7]) *Assume that the connected graph $K=(V_K,E_K)$ is the union of triangles. If there are triangles $\Delta_1,\ldots,\Delta_p\subseteq K$ such that $H_p=\bigcup_{i=1}^{p}\Delta_i$ is connected, then there are triangles $\Delta_{p+1},\ldots,\Delta_s\subseteq K$ such that $K=\bigcup_{i=1}^{s}\Delta_i$ and $H_t=\bigcup_{i=1}^{t}\Delta_i$ are connected for all $t=p+1,\ldots,s$.*

Lemma 12 ([7]) *Assume that the connected graph $K=(V_K,E_K)$ is the union of triangles. If $|V_K|=n$, then $|E_K|\geq\frac{3n-3}{2}$. Let us assume that $K=\bigcup_{i=1}^{s}\triangle_i$ and $H_t=\bigcup_{i=1}^{t}\triangle_i$ are connected for all $t=2,\ldots,s$.*

(1) If $|E_K|=\frac{3n-3}{2}$, then $|V_{\triangle_t}\bigcap V_{H_{t-1}}|=1$ for all $t=2,\ldots,s$.

(2) If $|E_K|=\frac{3n-2}{2}$, then $|V_{\triangle_t}\bigcap V_{H_{t-1}}|=1$ for all $t=2,\ldots,s$, except in the case $t=q$ which satisfies $|V_{\triangle_q}\bigcap V_{H_{q-1}}|=2$.

Lemma 13 *Let $K=(V_K,E_K)$ be a connected graph which is the union of triangles. If $\exp(K)=3$, $|V_K|=n$ and $|E_K|=\frac{3n-2}{2}$, then K is isomorphic to one of the following graphs:*

(1) $K \cong \Lambda_1(k_1, k_2, k_3), k_1 \neq 0$ or $k_2 \cdot k_3 \neq 0, k_1 + k_2 + k_3 = \frac{n-4}{2}$;

(2) $K \cong \Lambda_2(k_1, k_2, k_3), k_1 \cdot k_2 \neq 0, k_3 \geq 0, k_1 + k_2 + k_3 = \frac{n-4}{2}$.

Proof We can assume $K = \bigcup_{i=1}^{s} \triangle_i$ and $H_t = \bigcup_{i=1}^{t} \triangle_i$ are connected for all $t = 2, \dots, s$. Then $|V_{\triangle_t} \bigcap V_{H_{t-1}}| = 1$ for all $t = 2, \dots, s$, except in the case $t = q$ which satisfies $|V_{\triangle_q} \bigcap V_{H_{q-1}}| = 2$ by (2) of Lemma 12.

Then let $H = \triangle(u_1, u_2, u_3) \cup \triangle(u_2, u_3, u_4)$. It is easy to see $\exp(H) = 2$, and $H \subsetneq K$. Without loss of generality, we can assume $K = H \cup (\bigcup_{i=1}^{4} K_i)$ by (2) of Lemma 12, where K_i $(i = 1, 2, 3, 4)$ is the union of triangles or $K_i = \{u_i\}$, $V_{K_i} \bigcap V_H = \{u_i\}$, and $|V_{K_i}| = n_i$, thus $n_i \geq 1$ and $n = n_1 + n_2 + n_3 + n_4$.

Clearly, $|E_{K_i}| \geq \frac{3n_i-3}{2}$ for $1 \leq i \leq 4$ by Lemma 12. Then

$$\frac{3n-2}{2} = |E_K| = |E_H| + \sum_{i=1}^{4} |E_{K_i}| \geq 5 + \sum_{i=1}^{4} \frac{3n_i - 3}{2} = \frac{3n-2}{2}.$$

Hence $|E_{K_i}| = \frac{3n_i-3}{2}$ for any $i \in \{1, 2, 3, 4\}$.

Case 1 $n_1 > 1$.

First, we claim that $n_4 = 1$. Otherwise, for any pair of vertices $x \in V_{K_1} \backslash \{u_1\}, y \in V_{K_4} \backslash \{u_4\}$, there is no walk of length 3 from vertex x to vertex y, contradicting the fact that $\exp(K) = 3$.

Secondly, we claim that $K_1 \cong \bigcup_{i=1}^{k_1} \triangle(u_1, v_{2i-1}, v_{2i})$ for some $k_1 \geq 1$. For any vertex $x \in V_{K_1} \setminus \{u_1\}$, there is a walk of length 3 from vertex x to vertex u_4, $\{x, u_1\} \in E_{K_1}$ holds. Thus the result follows from $|E_{K_1}| = \frac{3n_1-3}{2}$ and (1) of Lemma 12.

Finally, similarly to the above proof, we have $K_2 \cong \bigcup_{i=1}^{k_2} \triangle(u_2, w_{2i-1}, w_{2i})$ and $K_3 \cong \bigcup_{i=1}^{k_3} \triangle(u_3, x_{2i-1}, x_{2i})$ for some $k_2, k_3 \geq 0$, and thus $K \cong \Lambda_1(k_1, k_2, k_3)$, where $k_1 \geq 1$ and $k_1 + k_2 + k_3 = \frac{n-4}{2}$.

Case 2 $n_4 > 1$.

The proof is similar to that in Case 1.

Case 3 $n_1 = 1, n_4 = 1, n_2 > 1$.

Subcase 3.1 $n_3 > 1$.

Similarly to Case 1, we have $K_2 \cong \bigcup_{i=1}^{k_2} \triangle(u_2, w_{2i-1}, w_{2i})$ and $K_3 \cong \bigcup_{i=1}^{k_3} \triangle(u_3, x_{2i-1}, x_{2i})$ for some $k_2, k_3 \geq 1$. Thus $K \cong \Lambda_1(0, k_2, k_3)$, where $k_2 \cdot k_3 \neq 0$ and $k_2 + k_3 = \frac{n-4}{2}$.

Subcase 3.2 $n_3 = 1$.

There are two vertices $v_1, v_2 \in V_{K_2} \backslash \{u_2\}$, such that $\triangle(u_2, v_1, v_2) \subseteq K_2$ by Lemma 10 and $n_2 > 1$. If $K_2 = \bigcup_{i=1}^{k_1} \triangle(u_2, v_{2i-1}, v_{2i})$, $k_1 \geq 1$, then $\exp(K) = 2$. It is a contradiction. Hence $\bigcup_{i=1}^{k_1} \triangle(u_2, v_{2i-1}, v_{2i}) \subsetneq K_2 (k_1 \geq 1)$ and there exists triangle $\triangle' \subseteq K_2 \setminus \bigcup_{i=1}^{k_1} \triangle(u_2, v_{2i-1}, v_{2i})$. Without loss of generality, we can assume $\triangle' = \triangle(v_1, w_1, w_2)$. Then we can assume $K_2 \cong (\bigcup_{i=1}^{k_1} \triangle(u_2, v_{2i-1}, v_{2i})) \cup (\bigcup_{i=1}^{k_2} \triangle(v_1, w_{2i-1}, w_{2i})) \cup (\bigcup_{i=1}^{k_3} \triangle(v_2, x_{2i-1}, x_{2i}))$ for some $k_1, k_2 \geq 1$ by $|E_{K_2}| = \frac{3n_2-3}{2}$, $\exp(K) = 3$ and (1) of Lemma 12. Thus we have $K \cong \Lambda_2(k_1, k_2, k_3)$, where $k_1 \cdot k_2 \neq 0, k_3 \geq 0$, and $k_1 + k_2 + k_3 = \frac{n-4}{2}$.

Case 4 $n_1 = 1, n_4 = 1, n_2 = 1$.

The proof is similar to that in Subcase 3.2. $\Box$

Lemma 14 *Let $K_1 = (V_{K_1}, E_{K_1}), K_2 = (V_{K_2}, E_{K_2})$ be the unions of triangles which are connected and $V_{K_1} \bigcap V_{K_2} = \emptyset$. Let $K = L(u,v) \bigcup K_1 \bigcup K_2$, where $u \in V_{K_1}, v \in V_{K_2}$. If $|V_K| = n$, $|E_K| = \frac{3n-2}{2}$ and $\exp(K) = 3$, then K is isomorphic to one of the following graphs:*

(1) $K \cong \Lambda_3(k_1, k_2), k_1 \geq 1, k_2 \geq 1, k_1 + k_2 = \frac{n-4}{2}$;

(2) $K \cong \Lambda_4(k_1, k_2), k_1 \geq 1, k_2 \geq 2, k_1 + k_2 = \frac{n-2}{2}$.

Proof Clearly, n is even by the fact that $|E_K| = \frac{3n-2}{2}$. Suppose $|V_{K_i}| = n_i$ $(i = 1, 2)$. Then we have $n_1 + n_2 = n$, and n_1, n_2 have the same parity.

Case 1 Both n_1 and n_2 are even.

By Lemma 12, $|E_{K_i}| \geq \frac{3n_i-2}{2} (i = 1, 2)$. Then

$$\frac{3n-2}{2} = |E_K| = |E_{K_1}| + |E_{K_2}| + 1 \geq \frac{3n_1-2}{2} + \frac{3n_2-2}{2} + 1 = \frac{3n-2}{2}.$$

Hence $|E_{K_i}| = \frac{3n_i-2}{2}$ $(i = 1, 2)$.

It follows from (2) of Lemma 12 and $\exp(K) = 3$ that $K_1 \cong (\bigcup_{i=1}^{k_1} \triangle(u, u_{2i-1}, u_{2i})) \cup \triangle(u, u_{2k_1}, u_{2k_1+1}), K_2 \cong (\bigcup_{i=1}^{k_2} \triangle(v, v_{2i-1}, v_{2i})) \cup \triangle(v, v_{2k_2}, v_{2k_2+1})$ for some $k_1, k_2 \geq 1$. Thus $K \cong \Lambda_3(k_1, k_2)$, where $k_1 \geq 1$, $k_2 \geq 1$ and $k_1 + k_2 = \frac{n-4}{2}$.

Case 2 Both n_1 and n_2 are odd.

By Lemma 12, $|E_{K_i}| \geq \frac{3n_i-3}{2} (i = 1, 2)$. For all $i = 1, 2$, if $|E_{K_i}| \geq \frac{3n_i-1}{2}$, then

$$\frac{3n-2}{2} = |E_K| = |E_{K_1}| + |E_{K_2}| + 1 \geq \frac{3n_1-1}{2} + \frac{3n_2-1}{2} + 1 = \frac{3n}{2}.$$

It is a contradiction. Hence we can assume $|E_{K_1}| = \frac{3n_1-3}{2}$ and $|E_{K_2}| = \frac{3n_2-1}{2}$. Thus we have $K_1 \cong (\bigcup_{i=1}^{k_1} \triangle(u, u_{2i-1}, u_{2i}))$ for some $k_1 \geq 1$ and $K_2 \cong (\bigcup_{i=1}^{k_2-1} \triangle(v, v_{2i-1}, v_{2i})) \cup \triangle(v, v_{2k_2-1}, v_i) \cup \triangle(v, v_{2k_2}, v_j)$ where $i, j \in \{1, \ldots, 2k_2-2\}$ for some $k_2 \geq 2$ by Lemma 12 and $\exp(K) = 3$. Therefore, $K \cong \Lambda_4(k_1, k_2)$, where $k_1 \geq 1$, $k_2 \geq 2$, $k_1 + k_2 = \frac{n-2}{2}$. $\Box$

Lemma 15 *Let K not be the union of triangles and $K = L(u_1, v_1) \bigcup L(u_2, v_2) \bigcup K_1 \bigcup K_2$, where $K_1 = (V_{K_1}, E_{K_1}), K_2 = (V_{K_2}, E_{K_2})$ are connected and are unions of triangles, $u_1, u_2 \in V_{K_1}, v_1, v_2 \in V_{K_2}$. If $|V_K| = n$, $|E_K| = \frac{3n-2}{2}$ and $\exp(K) = 3$, then K is isomorphic to one of the following graphs:*

(1) $\Lambda_5(k_1, k_2), k_1 \geq 1, k_2 \geq 0, k_1 + k_2 = \frac{n-6}{2}$.

(2) $\Lambda_6(k_1, k_2), k_1 \geq 0, k_2 \geq 0, k_1 + k_2 = \frac{n-6}{2}$.

(3) $\Lambda_7(k_1, k_2), k_1 \geq 0, k_2 \geq 0, k_1 + k_2 = \frac{n-6}{2}$.

(4) $\Lambda_8(k), k = \frac{n-8}{2}$.

Proof Clearly, n is even by the fact that $|E_K| = \frac{3n-2}{2}$. Suppose $|V_{K_1}| = n_1, |V_{K_2}| = n_2$. We have $n_1 + n_2 = n$, and n_1, n_2 have the same parity.

We claim that both n_1 and n_2 are odd. Otherwise, if both n_1 and n_2 are even, then

$|E_{K_i}| \geq \frac{3n_i-2}{2}$ for all $i=1,2$ by Lemma 12, and thus

$$\frac{3n-2}{2} = |E_K| = |E_{K_1}| + |E_{K_2}| + 2 \geq \frac{3n_1-2}{2} + \frac{3n_2-2}{2} + 2 = \frac{3n}{2}.$$

It is a contradiction. By Lemma 12,

$$\frac{3n-2}{2} = |E_K| = |E_{K_1}| + |E_{K_2}| + 2 \geq \frac{3n_1-3}{2} + \frac{3n_2-3}{2} + 2 = \frac{3n-2}{2}.$$

Then $|E_{K_i}| = \frac{3n_i-3}{2}$ for all $i=1,2$. And for any pair of triangles $\triangle_1, \triangle_2 \subseteq K_i$ $(i=1,2)$, they have at most one common vertex by (1) of Lemma 12.

For any pair of vertices $u \in E_{K_1} \setminus \{u_1,u_2\}$, $v \in E_{K_2} \setminus \{v_1,v_2\}$, there is a walk of length 3 from vertex u to vertex v by $\exp(K)=3$. Hence $\{u,u_1\} \in E_{K_1}$ (or $\{u,u_2\} \in E_{K_1}$), and $\{v,v_1\} \in E_{K_2}$ (or $\{v,v_2\} \in E_{K_2}$).

For all $i=1,2$, let $S_{u_i} = \{\triangle \subseteq K_1 | u_i \in V_\triangle\}, S_{v_i} = \{\triangle \subseteq K_2 | v_i \in V_\triangle\}$. Thus $|S_{u_i}| \geq 1$ and $|S_{v_i}| \geq 1$ by the fact that K_i $(i=1,2)$ is connected and is the union of triangles.

Case 1 $u_1 = u_2$.

Then $v_1 \neq v_2$, and $\{v_1,v_2\} \notin E_{K_2}$ since K is not the union of triangles.

Since K_2 is the union of triangles and $|E_{K_2}| = \frac{3n_2-3}{2}$, there is one pair of triangles $\triangle_1 \in S_{v_1}$, $\triangle_2 \in S_{v_2}$, such that they have exact one common vertex. Assume that $\triangle_1 = \triangle(v_1,v_3,v_4), \triangle_2 = \triangle(v_2,v_4,v_5)$.

If $|S_{v_i}| > 1$ for all $i=1,2$, then for any pair of vertices $x \in V_{S_{v_1} \setminus \triangle_1}$, $y \in V_{S_{v_2} \setminus \triangle_2}$, there is no walk of length 3 from vertex x to vertex y, contradicting the fact that $\exp(K)=3$. So we can assume $|S_{v_2}|=1$, then $K_2 \cong \triangle(v_1,v_3,v_4) \cup \triangle(v_2,v_4,v_5) \cup (\bigcup_{i=1}^{k_2} \triangle(v_1,w_{2i-1},w_{2i}))$ for some $k_2 \geq 0$.

On the other hand, by Lemma 10 and (1) of Lemma 12, we have $K_1 \cong \bigcup_{i=1}^{k_1} \triangle(u,u_{2i-1},u_{2i})$ for some $k_1 \geq 1$. Thus $k \cong \Lambda_5(k_1,k_2), k_1 \geq 1, k_2 \geq 0, k_1+k_2 = \frac{n-6}{2}$.

Case 2 $v_1 = v_2$.

Then $u_1 \neq u_2$. The proof is similar to that in Case 1.

Case 3 $u_1 \neq u_2$ and $v_1 \neq v_2$.

Subcase 3.1 $\{u_1,u_2\} \in E_{K_1}$ and $\{v_1,v_2\} \in E_{K_2}$.

By Lemma 10, $\triangle' = \triangle(u_1,u_2,u_3) \subseteq K_1, \triangle'' = \triangle(v_1,v_2,v_3) \subseteq K_2$. Then $\triangle' \in S_{u_i}, \triangle'' \in S_{v_i}$ for all $i=1,2$.

If $|S_{u_1}| > 1$ and $|S_{v_2}| > 1$, then for any pair of vertices $x \in V_{S_{u_1} \setminus \triangle'}, y \in V_{S_{v_2} \setminus \triangle''}$, there is no walk of length 3 from vertex x to vertex y, contradicting the fact $\exp(K)=3$. Hence $|S_{u_1}| \geq 1$ and $|S_{v_2}| = 1$ hold, or $|S_{v_2}| \geq 1$ and $|S_{u_1}| = 1$ hold.

Similarly, we have $|S_{u_2}| = 1$ when $|S_{v_1}| \geq 1$, and $|S_{v_1}| = 1$ when $|S_{u_2}| \geq 1$.

Subcase 3.1.1 $|S_{u_1}| > 1$ and $|S_{u_2}| > 1$.

Then $|S_{v_1}| = |S_{v_2}| = 1$. It is easy to see that $K \cong \Lambda_6(k_1,k_2), k_1,k_2 \geq 1, k_1+k_2 = \frac{n-6}{2}$.

Subcase 3.1.2 $|S_{u_1}| > 1$ and $|S_{u_2}| = 1$.

Then $|S_{v_2}| = 1$. It is easy to check that if $|S_{v_1}| = 1$, then $K \cong \Lambda_6(k_1, 0)$ where $k_1 = \frac{n-6}{2}$; and if $|S_{v_1}| > 1$, then $K \cong \Lambda_7(k_1, k_2)$ where $k_1, k_2 \geq 1$ and $k_1 + k_2 = \frac{n-6}{2}$.

Subcase 3.1.3 $|S_{u_1}| = 1, |S_{u_2}| > 1$.

The proof is similar to that in Subcase 3.1.2.

Subcase 3.1.4 $|S_{u_1}| = 1, |S_{u_2}| = 1$.

Then one of the following holds:

(1) If $|S_{v_1}| = |S_{v_2}| = 1$, then $K \cong \Lambda_6(0, 0)$ where $n = 6$.

(2) If $|S_{v_1}| > 1$ and $|S_{v_2}| = 1$, then $K \cong \Lambda_7(k_1, 0)$ where $k_1 = \frac{n-6}{2}$.

(3) If $|S_{v_1}| = 1$ and $|S_{v_2}| > 1$, then $K \cong \Lambda_7(0, k_2)$ where $k_2 = \frac{n-6}{2}$.

(4) If $|S_{v_1}| > 1$, $|S_{v_2}| > 1$, then $K \cong \Lambda_6(k_1, k_2)$ where $k_1, k_2 \geq 1$, $k_1 + k_2 = \frac{n-6}{2}$.

From above arguments, we have $K \cong \Lambda_6(k_1, k_2)$ or $K \cong \Lambda_7(k_1, k_2)$ where $k_1, k_2 \geq 0$ and $k_1 + k_2 = \frac{n-6}{2}$.

Subcase 3.2 $\{u_1, u_2\} \in E_{K_1}$ and $\{v_1, v_2\} \notin E_{K_2}$.

Similarly to Case 1, assume $\triangle_1 = \triangle(v_1, v_3, v_4)$, $\triangle_2 = \triangle(v_2, v_4, v_5)$, and $\triangle_3 = \triangle(u_1, u_2, u_3)$. Then $K_1 \cong \triangle_3$ and $\triangle_1, \triangle_2 \subseteq K_2$ by Lemma 10 and (1) of Lemma 12. Otherwise, there exist vertices $x \in V_{S_{u_1} \setminus \triangle_3}$ (or $x \in V_{S_{u_2} \setminus \triangle_3}$), $y = v_5$ (or $y = v_3$) such that there is no walk of length 3 from x to y, contradicting the fact $\exp(K) = 3$. Hence $K \cong \Lambda_8(k), k = \frac{n-8}{2}$.

Subcase 3.3 $\{u_1, u_2\} \notin E_{K_1}$ and $\{v_1, v_2\} \in E_{K_2}$.

It is similar to that in Subcase 3.2.

Subcase 3.4 $\{u_1, u_2\} \notin E_{K_1}$ and $\{v_1, v_2\} \notin E_{K_2}$.

There exist vertices $x \in V_{S_{u_1}} \setminus \{u_1\}, y \in V_{S_{v_2}} \setminus \{v_2\}$, such that there is no walk of length 3 from vertex x to vertex y, contradicting the fact $\exp(K) = 3$.

Combining the above arguments, we complete the proof. □

Lemma 16 *Let K not be the union of triangles and*

$$K = L(u_1, v_1) \bigcup L(v_2, w_1) \bigcup L(w_2, u_2) \bigcup K_1 \bigcup K_2 \bigcup K_3,$$

where $K_i = (V_{K_i}, E_{K_i})(i = 1, 2, 3)$ is connected and is the union of triangles, $u_1, u_2 \in V_{K_1}, v_1, v_2 \in V_{K_2}, w_1, w_2 \in V_{K_3}$. If $|V_K| = n$, $|E_K| = \frac{3n-2}{2}$ and $\exp(K) = 3$, then K is isomorphic to one of the following graphs:

(1) $\Lambda_9(k_1, k_2), k_1, k_2 \geq 1, k_1 + k_2 = \frac{n-6}{2}$.

(2) $\Lambda_{10}(k_1, k_2), k_1, k_2 \geq 1, k_1 + k_2 = \frac{n-6}{2}$.

(3) $\Lambda_{11}(k), k = \frac{n-8}{2} \geq 1$.

(4) $\Lambda_{12}(k), k = \frac{n-8}{2} \geq 1$.

(5) Λ_{13}.

Proof Note that n is even by $|E_K| = \frac{3n-2}{2}$. For $1 \geq i \geq 3$, suppose $|V_{K_i}| = n_i$. Then $n_i \geq 3$ by the fact that K_i is the union of triangles and $n_1+n_2+n_3=n$. Thus all of n_1,n_2,n_3 are even or one of n_1,n_2,n_3 is even, and the other two are odd.

We claim that one of n_1,n_2,n_3 is even, the other two are odd. Otherwise, we have $|E_{K_i}|\geq \frac{3n_i-2}{2}$ for all $i=1,2,3$ by Lemma 12, then

$$\frac{3n-2}{2}=|E_K|=\sum_{i=1}^{3}|E_{K_i}|+3\geq\sum_{i=1}^{3}\frac{3n_i-2}{2}+3=\frac{3n}{2}.$$

It is a contradiction.

Without loss of generality, we can assume n_1 is even and n_2,n_3 are odd. According to Lemma 12, we have $|E_{K_1}|\geq\frac{3n_1-2}{2}, |E_{K_i}|\geq\frac{3n_i-3}{2}(i=2,3)$. Then

$$\frac{3n-2}{2}=|E_K|=\sum_{i=1}^{3}|E_{K_i}|+3\geq\frac{3n_1-2}{2}+\frac{3n_2-3}{2}+\frac{3n_3-3}{2}+3=\frac{3n-2}{2}.$$

Thus $|E_{K_1}|=\frac{3n_1-2}{2}$ and $|E_{K_i}|=\frac{3n_i-3}{2}$ for $i=2,3$.

Let $u\in V_{K_1}\setminus\{u_1,u_2\}$ and $v\in V_{K_2}\setminus\{v_1,v_2\}$. Then there is a walk of length 3, namely, $u\to x\to y\to v$, for some $x,y\in V_K$ by the fact that $\exp(K)=3$. It is easy to check that $x=u_1, y=v_1$. Thus $\{u,u_1\}\in E_{K_1}$ for any $u\in V_{K_1}\setminus\{u_1,u_2\}$, and $\{v,v_1\}\in E_{K_2}$ for any $v\in V_{K_2}\setminus\{v_1,v_2\}$. Similarly, we have $\{u,u_2\}\in E_{K_1}$ for any $u\in V_{K_1}\setminus\{u_1,u_2\}$, $\{v,v_2\}\in E_{K_2}$ for any $v\in V_{K_2}\setminus\{v_1,v_2\}$, $\{w,w_i\}\in E_{K_3}$ for any $w\in V_{K_3}\setminus\{w_1,w_2\}$ and $i=1,2$.

If $u_1=u_2, v_1=v_2, w_1=w_2$, then $L(u_1,v_2)\bigcup L(v_1,w_2)\bigcup L(w_1,u_2)=\triangle(u_1,v_1,w_1)$, which contradicts the fact that K is not the union of triangle.

If $u_1\neq u_2$, then $\{u_1,u_2\}\in E_{K_1}$, and $\frac{3n_1-2}{2}=|E_{K_1}|\geq 2n_1-3$, so $n_1\leq 4$. It is clear that $n_1=4$ by the fact that n_1 is even and $n_1\geq 3$. Thus $|E_{K_1}|=\frac{3n_1-2}{2}=5$. It is easy to check that $K_1\cong\triangle(u_1,u_2,u_3)\cup\triangle(u_1,u_2,u_4)$.

If $v_1\neq v_2$, then $\{v_1,v_2\}\in E_{K_2}$, and $\frac{3n_2-3}{2}=|E_{K_2}|\geq 2n_2-3$, so $n_2\leq 3$. It is clear that $n_2=3$ by the fact that $n_2\geq 3$ and n_2 is odd. Thus $|E_{K_2}|=\frac{3n_2-3}{2}=3$. Then $K_2\cong\triangle(v_1,v_2,v_3)$.

Similarly, if $w_1\neq w_2$, then $n_3=3$, and we have $K_3\cong\triangle(w_1,w_2,w_3)$.

Case 1 $u_1\neq u_2, v_1=v_2, w_1=w_2$.

We have $K_1\cong\triangle(u_1,u_2,u_3)\cup\triangle(u_1,u_2,u_4)$. Then $K\cong\Lambda_9(k_1,k_2)$, where $k_1\geq 1, k_2\geq 1, k_1+k_2=\frac{n-6}{2}$ by the (1) of Lemma 12 and $\exp(K)=3$.

Case 2 $u_1=u_2, v_1\neq v_2, w_1=w_2$.

Then $K_2\cong\triangle(v_1,v_2,v_3)$. We have $K\cong\Lambda_{10}(k_1,k_2)$ where $k_1\geq 1, k_2\geq 1, k_1+k_2=\frac{n-6}{2}$ by Lemma 12 and $\exp(K)=3$.

Case 3 $u_1=u_2, v_1=v_2, w_1\neq w_2$.

The proof is similar to that in Case 2.

Case 4 $u_1\neq u_2, v_1\neq v_2, w_1=w_2$.

Then $K_1\cong\triangle(u_1,u_2,u_3)\cup\triangle(u_1,u_2,u_4)$ and $K_2\cong\triangle(v_1,v_2,v_3)$. Thus $K\cong\Lambda_{11}(k)$ where $k=\frac{n-8}{2}\geq 1$ by Lemma 12 and $\exp(K)=3$.

Case 5: $u_1 \neq u_2, v_1 = v_2, w_1 \neq w_2$.

It is similar to that in Case 4.

Case 6 $u_1 = u_2, v_1 \neq v_2, w_1 \neq w_2$.

Then $K_2 \cong \triangle(v_1, v_2, v_3)$ and $K_3 \cong \triangle(w_1, w_2, w_3)$. Thus $K \cong \Lambda_{12}(k)$ where $k = \frac{n-8}{2} \geq 1$ by (2) of Lemma 12 and $\exp(K) = 3$.

Case 7 $u_1 \neq u_2, v_1 \neq v_2, w_1 \neq w_2$.

Then $K_1 \cong \triangle(u_1, u_2, u_3) \cup \triangle(u_1, u_2, u_4)$, $K_2 \cong \triangle(v_1, v_2, v_3)$, $K_3 \cong \triangle(w_1, w_2, w_3)$. It is easy to check that $K \cong \Lambda_{13}$.

Combining the above arguments, we complete the proof. $\Box$

Now we are ready to prove Theorem 9.

Proof of Theorem 9 Note that $G = (V, E)$ is a primitive loop-free graph with $\exp(G) = 3$, then for any vertex $u \in V$, there are vertices $v, w \in V$ such that $\triangle(u, v, w) \subseteq G$.

Let $K = (V_K, E_K)$ be the union of all triangles of G and $K_1, \ldots, K_t$ be the components of K, where for all $1 \leq i \leq t$, $K_i = (V_{K_i}, E_{K_i})$ and $|V_{K_i}| = n_i$. Then $n_i \geq 3$ by the fact K_i is the union of triangles, and $|E_{K_i}| \geq \frac{3n_i-3}{2}$ according to Lemma 12.

Let $E' = \{\{u, v\} | u \in V_{K_i}, v \in V_{K_j}, \text{where } i \neq j, i, j \in \{1, \ldots, t\}\}$. Then $|E'| \geq t - 1$ by the fact that G is connected. Then

$$n = |V| = |V_K| = \sum_{i=1}^{t} |V_{K_i}| = \sum_{i=1}^{t} n_i,$$

and

$$\frac{3n-2}{2} = |E| = \sum_{i=1}^{t} |E_{K_i}| + |E'| \geq \sum_{i=1}^{t} \frac{3n_i - 3}{2} + |E'| = \frac{3n - 3t}{2} + |E'|.$$

Thus $|E'| \leq \frac{3t-2}{2}$.

For any $i \in \{1, \ldots, t\}$, let $\delta(i) = |\{\{u, v\} \in E' | u \in V_{K_i}\}|$, and $\delta(1) = \min\{\delta(1), \ldots, \delta(t)\}$. Then $\delta(1)t \leq \sum_{i=1}^{t} \delta(i) = 2|E'| \leq 3t - 2$, so $\delta(1) \leq 2$.

Case 1 $\delta(1) = 0$.

Lemma 13 implies the Theorem 9 because $\delta(1) = 0$ if and only if $t = 1$.

Case 2 $\delta(1) = 1$.

We claim that $t = 2$ in this case. For $t \geq 3$, without loss of generality, we can assume $\{u, v\} \in E'$, where $u \in V_{K_1}, v \in V_{K_2}$, then for any pair of vertices $u' \in V_{K_1} \setminus \{u\}, w \in \bigcup_{i=3}^{t} V_{K_i}$, such that there exists a walk of length 3, namely, $u' \to w_1 \to w_2 \to w$ for some vertices $w_1, w_2 \in V_K$ by the fact that $\exp(G) = 3$. It is easy to see $w_1 = u, w_2 = v$, and $\{v, w\} \in E'$ for any $w \in \bigcup_{i=3}^{t} V_{K_i}$. Then

$$\frac{3t-2}{2} \geq |E'| \geq 1 + \sum_{i=3}^{t} n_i \geq 1 + 3(t - 2).$$

Thus $t \leq 2$. It is a contradiction. Hence $t = 2$ and Lemma 14 implies Theorem 9.

Case 3 $\delta(1) = 2$.

Then $t \geq 2$ and $|E'| \geq t$. Without loss of generality, we can assume $\{\{u_1, v_1\}, \{u_2, v_2\}\} \subseteq E'$, where $u_1, u_2 \in V_{K_1}, v_1 \in V_{K_i}, v_2 \in V_{K_j}, 2 \leq i, j \leq t$.

If $i = j$, then $t = 2$. Otherwise, similarly to Case 2, we have

$$\frac{3t-2}{2} \geq |E'| \geq 2 + \sum_{l=2}^{t} n_l - n_i \geq 2 + 3(t-2).$$

Then $t \leq 2$, it is a contradiction. Hence $t = 2$ and Lemma 15 implies Theorem 9.

If $i \neq j$, then $t \geq 3$. Similarly to Case 2, we have

$$\frac{3t-2}{2} \geq |E'| \geq 2 + \sum_{l=2}^{t} n_l - n_i - n_j \geq 2 + 3(t-3).$$

Then $t \leq 4$. Hence $t = 3$ or 4.

Subcase 3.1 $t = 3$.

Then $|E'| = 3$ by the fact that $t \leq |E'| \leq \frac{3t-2}{2}$, and $\delta(1) = \delta(2) = \delta(3) = 2$ because $\delta(1) = 2$ is the minimal. We can assume $E' = \{\{u_1, v_1\}, \{v_2, w_1\}, \{w_2, u_2\}\}$, where $u_1, u_2 \in V_{K_1}, v_1, v_2 \in V_{K_2}, w_1, w_2 \in V_{K_3}$ by the fact that G is connected. Hence Lemma 16 implies Theorem 9.

Subcase 3.2 $t = 4$.

We claim that $|E'| \neq 4$. Otherwise, we have $\delta(1) = \delta(2) = \delta(3) = \delta(4) = 2$ because $\delta(1) = 2$ is the minimal. Then we can assume $E' = \{\{u_1, v_1\}, \{v_2, w_1\}, \{w_2, x_1\}, \{x_2, u_2\}\}$, where $u_i \in V_{K_1}, v_i \in V_{K_2}, w_i \in V_{K_3}, x_i \in V_{K_4}$ $(i = 1, 2)$ by the fact that G is connected. Now for any pair of vertices x, y, where $x \in V_{K_1} \setminus \{u_1, u_2\}$, $y \in V_{K_3} \setminus \{w_1, w_2\}$, there is no walk of length 3 from vertex x to vertex y, contradicting the fact $\exp(G) = 3$.

We also claim that $|E'| \neq 5$. Otherwise, $\sum_{i=1}^{4} \delta(i) = 2|E'| = 10$ and $\delta(i) \geq \delta(1) = 2$ $(i = 2, 3, 4)$, then $\{\delta(1), \delta(2), \delta(3), \delta(4)\} = \{2, 2, 2, 4\}$ or $\{2, 2, 3, 3\}$. It is easy to check that the structure of graph G is isomorphic to one of the following graphs:

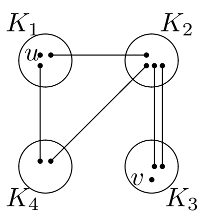

Figure 3.1 G_1

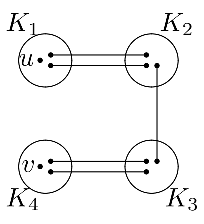

Figure 3.2 G_2

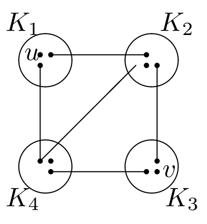

Figure 3.3 G_3

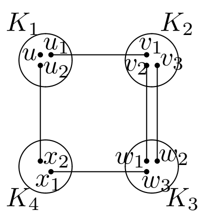

Figure 3.4 G_4

In Figures 3.1–3.3, there exist vertices $u, v \notin V_{E'}$ such that there is no walk of length 3 from vertex u to vertex v, contradicting the fact $\exp(G) = 3$.

In Figure 3.4, let $E' = \{\{u_1, v_1\}, \{v_2, w_1\}, \{v_3, w_2\}, \{w_3, x_1\}, \{x_2, u_2\}\}$, where for $i = 1, 2, j = 1, 2, 3$, $u_i \in V_{K_1}, v_j \in V_{K_2}, w_j \in V_{K_3}, x_i \in V_{K_4}$. For any vertex $u \in V_{K_1} \setminus \{u_1, u_2\}$, any vertex $w \in V_{K_3}$, there is a walk of length 3, namely, $u \to y' \to y'' \to w$ for some vertices $y', y'' \in V_K$ by the fact that $\exp(G) = 3$, then $V_{K_3} = \{w_1, w_2, w_3\}$ for w is any vertex in V_{K_3}.

If $y' = u_1$, then $y'' = v_1$, and we have $v_1 = v_2 = v_3$. Thus $u \to u_1 \to v_1 \to w_1$ (or w_2) is the walk of length 3 from $u \in V_{K_1} \setminus \{u_1, u_2\}$ to w_1(or w_2). If $y' = u_2$, then $y'' = x_2$, and we have $x_1 = x_2$. Thus $u \to u_2 \to x_1 \to w_3$ is the walk of length 3 from $u \in V_{K_1} \setminus \{u_1, u_2\}$ to w_3. But there exist vertices $v \in V_{K_2} \setminus \{v_1\}$, $x \in V_{K_4} \setminus \{x_1\}$ by the fact $|V_{K_i}| \geq 3$ for $1 \leq i \leq 4$, such that there is no walk of length 3 from vertex v to vertex x, contradicting the fact $\exp(G) = 3$.

Hence $|E'| \neq 4$ and $|E'| \neq 5$, but $|E'| = 4$ or 5 by the fact that $4 = t \leq |E'| \leq \frac{3t-2}{2} = 5$. It is a contradiction. Thus $t \neq 4$.

Combining the above arguments, we complete the proof. $\square$

4. Proof of Conjecture 6 and characterization of base 3

In this section, we show the Conjecture 6 is true and completely characterize the underlying digraphs which have base 3 and the minimum number of arcs.

The following result can be used to prove the conjecture is true.

Lemma 17 ([9]) *Let $S = (V, A)$ be a primitive non-powerful symmetric loop-free signed digraph with $\exp(S) = k$ on n vertices. If there exist two vertices i and j such that there is only one walk of length k from i to j in S, then $b(S) > k$.*

Now we show the Conjecture 6 is true.

Theorem 18 *Let $n \geq 6$ be even, $S = (V(S), A(S))$ be a primitive non-powerful symmetric loop-free signed digraph on n vertices with $|A(S)| = H(n, 3) = 3n - 2$, $\exp(S) = 3$, $b(S) = 3$. Then D is the underlying digraph of S if and only if D is isomorphic to the digraph $ED_{n,3,3}$.*

Proof Sufficiency. It follows from (3) of Theorem 5.

Necessity. Since $|V(S)| = n, |A(S)| = 3n - 2$ and $\exp(S) = 3$, D is isomorphic to one of the graphs listed in Theorem 9.

If D is isomorphic to one of the graphs $\Lambda_2, \ldots, \Lambda_{13}$ listed in Theorem 9, then $b(S) > 3$ by Lemma 17, contradicting the fact that $b(S) = 3$. Then D is isomorphic to $\Lambda_1(k_1, k_2, k_3)$, where $k_1 \neq 0$ or $k_2 \cdot k_3 \neq 0$, $k_1 + k_2 + k_3 = \frac{n-4}{2}$.

If $k_2 \cdot k_3 \neq 0$, then for any pair of vertices w, x, where $w \in \bigcup_{i=1}^{2k_2} w_i$, $x \in \bigcup_{i=1}^{2k_3} x_i$, there is only one walk of length 3 from vertex w to vertex x, and we have $b(S) > 3$ by Lemma 17, contradicting the fact that $b(S) = 3$, so $k_2 \cdot k_3 = 0$ and $k_1 \neq 0$. Thus D is isomorphic to the digraph $ED_{n,3,3}$. $\square$

By Theorems 5 and 18, we can completely characterize the underlying digraphs which have base 3 and the minimum number of arcs as follows.

Theorem 19 *Let $n \geq 6$, $S = (V(S), A(S))$ be a primitive non-powerful symmetric loop-free signed digraph on n vertices with $|A(S)| = H(n, 3)$ and $b(S) = 3$. Then one of the following conditions holds:*

(1) If n is odd, D is the underlying digraph of S if and only if D is isomorphic to the digraph $OD_{n,3}$.

(2) If n is even, D is the underlying digraph of S with $\exp(D) = 2$ if and only if D is isomorphic to the digraph $ED_{n,3}$.

(3) If n is even, D is the underlying digraph of S with $\exp(D) = 3$ if and only if D is isomorphic to the digraph $ED_{n,3,3}$.

Acknowledgments The authors would like to thank the referees for their useful comments and suggestions of this paper.

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