

Gorenstein Homological Dimensions and Auslander Categories with Respect to a Semidualizing Module

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Abstract Let R be a commutative noetherian local ring. In this paper, we study Gorenstein projective, injective and flat modules with respect to a semidualizing R -module C , and we give some connections between C -Gorenstein homological dimensions and the Auslander categories of \hat{R} .

Keywords semidualizing modules; C -Gorenstein injective modules; C -Gorenstein projective modules; C -Gorenstein flat modules; Auslander categories.

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1. Introduction

Throughout this paper, R will denote a commutative ring with nonzero identity and \hat{R} will denote the \mathfrak{M} -adic completion of a local ring (R, \mathfrak{M}) .

When R is two-sided noetherian, Auslander and Bridger [1] introduced the G-dimension, $\text{G-dim}_R M$, for every finitely generated R -module M . They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M$ is finite. Several decades later, Enochs and Jenda extended the ideas of Auslander and Bridger and introduced three homological dimensions, called the Gorenstein projective, injective and flat dimensions. They proved that these dimensions are similar to the classical homological dimensions, i.e., projective, injective and flat dimensions, respectively. The Gorenstein projective, injective and flat dimensions of a module are defined in terms of resolutions by Gorenstein projective, injective and flat modules, respectively.

Let R be a noetherian ring with a dualizing complex D . The Auslander categories $A(R)$ and $B(R)$ with respect to D are defined in [3, (3.1)]. In [6], it was shown that the modules in $A(R)$ are those of finite Gorenstein projective dimension (Gorenstein flat dimension) (see [6, Thm. 4.1]) and the modules in $B(R)$ are those of finite Gorenstein injective dimension (see [6, Thm. 4.4]). Esmkhani and Tousi [9, 10] extended the characterization of finiteness of Gorenstein projective, injective and flat dimensions in [6] to arbitrary local noetherian ring probably without a dualizing complex. They proved over a local noetherian ring R , for an R -module M , the

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Gorenstein projective dimension of M is finite if and only if the Gorenstein flat dimension of M is finite if and only if $\hat{R} \otimes_R M$ belongs to the Auslander category $A(\hat{R})$ (see [9, Thm. 3.4, Cor. 3.5]), and M is Gorenstein injective if and only if $\text{Hom}_R(\hat{R}, M)$ belongs to the Auslander category $B(\hat{R})$, M is cotorsion and $\text{Ext}_R^i(E, M) = 0$ for all injective R -modules E and all $i > 0$ (see [10, Thm. 2.5]).

Since the dualizing complexes (or modules) for a general ring usually do not exist, semidualizing complexes and modules (see Def. 2.1) have received much attention in recent years, see, for example, [5, 12, 13, 15–20, 22]. The examples of semidualizing modules include the rank 1 free module and a dualizing module, when one exists.

Let R be a noetherian ring and C a fixed semidualizing R -module (or complex). The Auslander categories $A_C(R)$ and $B_C(R)$ with respect to C are defined by Avramov, Foxby [3, (3.1)] (or Christensen [5, Def. 4.1]). In [12], Holm, Jørgensen proved that if R admits a dualizing complex D , then an R -module M is in $A_{C^+}(R)$ if and only if the C -Gorenstein projective dimension of M is finite if and only if the C -Gorenstein flat dimension of M is finite, where $C^+ = \mathbf{R}\text{Hom}_R(C, D)$. Dually, M is in $B_{C^+}(R)$ if and only if the C -Gorenstein injective dimension of M is finite [12, Thm. 4.6].

Motivated by [9, 10], the main aim of this paper is to extend the characterization of C -Gorenstein projective, injective and flat dimensions in [12] to arbitrary local noetherian ring, possibly without a dualizing complex.

Let R be a local noetherian ring and C a fixed semidualizing R -module, and let D denote the dualizing complex of \hat{R} . Then by [5, Thm. 5.6], $\hat{C} = C \otimes_R \hat{R}$ is a semidualizing module of \hat{R} , and by [5, Cor. 2.12], the complex $\hat{C}^+ = \mathbf{R}\text{Hom}_{\hat{R}}(\hat{C}, D)$ is semidualizing for \hat{R} .

We define $A'_C(R)$ to be those R -modules M such that $\hat{R} \otimes_R M \in A_{\hat{C}^+}(\hat{R})$ and $B'_C(R)$ to be those R -modules M such that $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$. In Section 3, we characterize C -Gorenstein projective, injective and flat modules in terms of the classes $A'_C(R)$ and $B'_C(R)$. Our main results are Theorems 3.5, 3.7 and 3.10 which state, respectively, that:

Theorem A *Let R be a local noetherian ring and M an R -module. Then M is C -Gorenstein injective if and only if $M \in B'_C(R)$, M is cotorsion and $\text{Ext}_R^i(\text{Hom}_R(C, E), M) = 0$ for all injective R -modules E and all $i > 0$.*

Theorem B *Let R be a local noetherian ring, M an R -module, and n a non-negative integer. Then the following conditions are equivalent:*

- (1) $\mathcal{GF}_C\text{-pd}_R(M) \leq n$.
- (2) $M \in A'_C(R)$ and $\text{Tor}_i^R(\text{Hom}_R(C, I), M) = 0$ for all injective R -modules I and all $i > n$.
- (3) $M \in A'_C(R)$ and $\text{Ext}_R^i(M, C \otimes_R L) = 0$ for all cotorsion R -modules L with finite flat dimension and all $i > n$.
- (4) $M \in A'_C(R)$ and $\text{Ext}_R^i(M, C \otimes_R F) = 0$ for all cotorsion flat R -modules F and all $i > n$.

Theorem C *Let R be a local noetherian ring, M an R -module, and n a non-negative integer. Then the following conditions are equivalent:*

- (1) $\mathcal{G}\mathcal{P}_C\text{-pd}_R(M) \leq n$.
- (2) $M \in A'_C(R)$ and $\text{Ext}_R^i(M, C \otimes_R P) = 0$ for all projective R -modules P and all $i > n$.
- (3) $\mathcal{G}\mathcal{F}_C\text{-pd}_R(M) < \infty$ and $\text{Ext}_R^i(M, C \otimes_R P) = 0$ for all projective R -modules P and all $i > n$.

2. Semidualizing modules and associated categories

The homological dimensions of interest in this paper are built from semidualizing modules and their associated projective, injective and flat classes, defined next. Semidualizing modules have been considered by many authors [5, 12, 13, 15–20, 22].

Definition 2.1 *A finitely generated R -module C is semidualizing if*

- (a) *The natural homothety morphism $R \rightarrow \text{Hom}_R(C, C)$ is an isomorphism, and*
- (b) $\text{Ext}_R^{\geq 1}(C, C) = 0$.

Let C be a semidualizing R -module. We set

- $\mathcal{P}_C(R)$ = *the subcategory of modules $C \otimes_R P$ where P is R -projective,*
- $\mathcal{F}_C(R)$ = *the subcategory of modules $C \otimes_R F$ where F is R -flat,*
- $\mathcal{F}_C^{\text{cot}}(R)$ = *the subcategory of modules $C \otimes_R F$ where F is flat and cotorsion,*
- $\mathcal{I}_C(R)$ = *the subcategory of modules $\text{Hom}_R(C, I)$ where I is R -injective.*

Modules in $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, $\mathcal{F}_C^{\text{cot}}(R)$ and $\mathcal{I}_C(R)$ are called C -projective, C -flat, C -flat C -cotorsion and C -injective, respectively. An R -module M is C -cotorsion if $\text{Ext}_R^1(C \otimes_R F, M) = 0$ for all flat R -modules F .

Lemma 2.2 (1) *Every \hat{C} -injective \hat{R} -module is C -injective as an R -module.*

- (2) *Every \hat{C} -flat \hat{R} -module is C -flat as an R -module.*

Proof (1) Let \bar{I} be any injective \hat{R} -module. Then \bar{I} is an injective R -module. By adjointness, we have

$$\text{Hom}_{\hat{R}}(\hat{C}, \bar{I}) \cong \text{Hom}_{\hat{R}}(C \otimes_R \hat{R}, \bar{I}) \cong \text{Hom}_R(C, \text{Hom}_{\hat{R}}(\hat{R}, \bar{I})) \cong \text{Hom}_R(C, \bar{I}).$$

It follows that $\text{Hom}_{\hat{R}}(\hat{C}, \bar{I})$ is a C -injective R -module.

- (2) Let \bar{F} be any flat \hat{R} -module. Then \bar{F} is a flat R -module. So, we have

$$\hat{C} \otimes_{\hat{R}} \bar{F} \cong (C \otimes_R \hat{R}) \otimes_{\hat{R}} \bar{F} \cong C \otimes_R \bar{F}.$$

Hence $\hat{C} \otimes_{\hat{R}} \bar{F}$ is a C -flat R -module. \square

Definition 2.3 *Let \mathcal{X} be a class of R -modules and M an R -module. An \mathcal{X} -resolution of M is a complex of R -modules in \mathcal{X} of the form*

$$X = \cdots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$$

such that $H_0(X) \cong M$ and $H_n(X) = 0$ for $n \geq 1$, and the following exact sequence is the

augmented \mathcal{X} -resolution of M associated to X :

$$X^+ = \cdots \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0.$$

The \mathcal{X} -projective dimension of M is the quantity

$$\mathcal{X}\text{-pd}_R(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, one has $\mathcal{X}\text{-pd}_R(0) = -\infty$. The modules of \mathcal{X} -projective dimension 0 are the nonzero modules of \mathcal{X} .

An \mathcal{X} -resolution X of M is proper if the augmented resolution X^+ is $\text{Hom}_R(\mathcal{X}, -)$ -exact.

We define (proper) \mathcal{X} -coresolutions and \mathcal{X} -injective dimension dually. And the \mathcal{X} -injective dimension of M is $\mathcal{X}\text{-id}_R(M)$.

When \mathcal{X} is the class of projective, injective and flat R -modules, we write $\text{pd}_R M$, $\text{id}_R M$, and $\text{fd}_R M$ for the classical homological dimensions of M . Similarly, the Gorenstein projective, injective and flat dimensions of M are denoted $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$, respectively.

By $\tilde{\mathcal{P}}_C(R)$, $\tilde{\mathcal{I}}_C(R)$ and $\tilde{\mathcal{F}}_C(R)$ we denote the classes of R -modules with finite C -projective, C -injective and C -flat dimension, respectively. Note that if R is a noetherian ring of finite Krull dimension, then $\tilde{\mathcal{P}}(R) = \tilde{\mathcal{F}}(R)$, and so $\tilde{\mathcal{P}}_C(R) = \tilde{\mathcal{F}}_C(R)$ (see [21, Thm.4.2.8]). We shall use these facts without comment.

Definition 2.4 An R -module M is called C -Gorenstein injective if

- (I1) $\text{Ext}_R^{\geq 1}(\text{Hom}_R(C, I), M) = 0$ for any injective R -module I , and
- (I2) there exist injective R -modules I_0, I_1, \dots together with an exact sequence:

$$X = \cdots \rightarrow \text{Hom}_R(C, I_1) \rightarrow \text{Hom}_R(C, I_0) \rightarrow M \rightarrow 0$$

such that this sequence stays exact when we apply the functor $\text{Hom}_R(\text{Hom}_R(C, E), -)$ to it for any injective R -module E (i.e., M admits a proper $\mathcal{I}_C(R)$ -resolution).

An R -module M is called C -Gorenstein projective if

- (P1) $\text{Ext}_R^{\geq 1}(M, C \otimes_R P) = 0$ for any projective R -module P , and
- (P2) there exist projective R -modules P^0, P^1, \dots together with an exact sequence:

$$X = 0 \rightarrow M \rightarrow C \otimes_R P^0 \rightarrow C \otimes_R P^1 \rightarrow \cdots,$$

and also, this sequence stays exact when we apply the functor $\text{Hom}_R(-, C \otimes_R Q)$ to it for any projective R -module Q (i.e., M admits a proper $\mathcal{P}_C(R)$ -coresolution).

An R -module M is called C -Gorenstein flat if

- (F1) $\text{Tor}_{\geq 1}^R(\text{Hom}_R(C, I), M) = 0$ for any injective R -module I , and
- (F2) there exist flat R -modules F^0, F^1, \dots together with an exact sequence:

$$X = 0 \rightarrow M \rightarrow C \otimes_R F^0 \rightarrow C \otimes_R F^1 \rightarrow \cdots,$$

and furthermore, this sequence stays exact when we apply the functor $\text{Hom}_R(C, E) \otimes_R -$ to it for any injective R -module E . We set

$$\mathcal{GI}_C(R) = \text{the subcategory of } C\text{-Gorenstein injective } R\text{-modules,}$$

$\mathcal{GP}_C(R)$ = the subcategory of C -Gorenstein projective R -modules,

$\mathcal{GF}_C(R)$ = the subcategory of C -Gorenstein flat R -modules.

Fact 2.5 (1) By Holm-Jørgensen [12, Ex. 2.8], if I is an injective R -module, then $\text{Hom}_R(C, I)$ and I are C -Gorenstein injective, and if P is a projective R -module, then $C \otimes_R P$ and P are C -Gorenstein projective. Similarly, if F is a flat R -module, then $C \otimes_R F$ and F are C -Gorenstein flat.

(2) By [17], we know that an R -module M is C -Gorenstein injective if and only if there exists an exact sequence of R -modules

$$X = \cdots \xrightarrow{\partial_2^X} \text{Hom}_R(C, I_1) \xrightarrow{\partial_1^X} \text{Hom}_R(C, I_0) \xrightarrow{\partial_0^X} I_{-1} \xrightarrow{\partial_{-1}^X} I_{-2} \xrightarrow{\partial_{-2}^X} \cdots$$

such that $M \cong \text{Coker} \partial_1^X$, each I_i is injective and the complex $\text{Hom}_R(\mathcal{I}_C(R), X)$ is exact. In which case, X is called a complete $\mathcal{I}_C\mathcal{I}$ -resolution of M .

Similarly, M is C -Gorenstein flat if and only if there exists an exact sequence of R -modules

$$X = \cdots \xrightarrow{\partial_2^X} F_1 \xrightarrow{\partial_1^X} F_0 \xrightarrow{\partial_0^X} C \otimes_R F_{-1} \xrightarrow{\partial_{-1}^X} C \otimes_R F_{-2} \xrightarrow{\partial_{-2}^X} \cdots$$

such that $M \cong \text{Coker} \partial_1^X$, each F_i is flat and the complex $\mathcal{I}_C(R) \otimes_R X$ is exact. In which case, X is called a complete \mathcal{FF}_C -resolution of M .

It was shown in [13] that the Auslander categories $A_C(R)$ and $B_C(R)$ satisfy the two-of-three property. Here, we can prove the similar results hold for $A'_C(R)$ and $B'_C(R)$. Recall an R -module M is cotorsion if $\text{Ext}_R^1(F, M) = 0$ for all flat R -modules F .

Lemma 2.6 *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of R -modules over a local noetherian ring R . The following assertions hold.*

- (1) *If any two of M', M, M'' are in $A'_C(R)$, then so is the third.*
- (2) *Assume that M' is a cotorsion R -module. If any two of M', M, M'' belong to $B'_C(R)$, then so does the third.*

Proof (1) The exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ yields the exact sequence $0 \rightarrow \hat{R} \otimes_R M' \rightarrow \hat{R} \otimes_R M \rightarrow \hat{R} \otimes_R M'' \rightarrow 0$. Now, the conclusion follows from [13, Cor. 6.3].

(2) The exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ yields the exact sequence $0 \rightarrow \text{Hom}_R(\hat{R}, M') \rightarrow \text{Hom}_R(\hat{R}, M) \rightarrow \text{Hom}_R(\hat{R}, M'') \rightarrow 0$. Now, using [13, Cor. 6.3] again. \square

Lemma 2.7 *Let (R, \mathfrak{M}, k) be a local noetherian ring and M a cotorsion R -module. Then for all $i > 0$, the following conditions are equivalent:*

- (1) $\text{Ext}_R^i(\text{Hom}_R(C, E), M) = 0$ for all injective R -modules E .
- (2) $\text{Ext}_{\hat{R}}^i(\text{Hom}_{\hat{R}}(\hat{C}, \bar{I}), \text{Hom}_R(\hat{R}, M)) = 0$ for all injective \hat{R} -modules \bar{I} .

Proof Assume that L is an \hat{R} -module and $\mathbf{F} \cdot \rightarrow \text{Hom}_{\hat{R}}(\hat{C}, L)$ is a free resolution of $\text{Hom}_{\hat{R}}(\hat{C}, L)$. By [8, Prop. 1.4.16] and the fact that M is cotorsion and every flat \hat{R} -module is flat as an R -module, then for every $i > 0$, we have

$$\text{Ext}_{\hat{R}}^i(\text{Hom}_{\hat{R}}(\hat{C}, L), \text{Hom}_R(\hat{R}, M))$$

$$\begin{aligned}
 &= H^i(\text{Hom}_{\hat{R}}(\mathbf{F}., \text{Hom}_R(\hat{R}, M))) \cong H^i(\text{Hom}_R(\mathbf{F} \otimes_{\hat{R}} \hat{R}, M)) \\
 &\cong H^i(\text{Hom}_R(\mathbf{F}., M)) \cong \text{Ext}_R^i(\text{Hom}_R(C, L), M).
 \end{aligned}$$

(1) \Rightarrow (2). By Lemma 2.2, every \hat{C} -injective \hat{R} -module is C -injective as an R -module. So, the assertion holds from the above isomorphism.

(2) \Rightarrow (1). Let E be an injective R -module and let $\text{Hom}_R(-, E(k))$ be denoted by $(-)^{\vee}$, where $E(k)$ is the injective envelope of k . By [8, Thm. 3.4.1], $E(k)$ is an injective cogenerator for R and $E(k) \cong E_{\hat{R}}(\hat{R}/\mathfrak{M})$ as an \hat{R} -module. The R -module $(\text{Hom}_R(C, E)^{\vee})^{\vee}$ considered with the \hat{R} -module structure coming from $E(k)$, that is, $(\hat{r}f)(x) = \hat{r}(f(x))$, for all $\hat{r} \in \hat{R}$, $f \in \text{Hom}_R(\text{Hom}_R(C, E)^{\vee}, E(k))$ and $x \in \text{Hom}_R(C, E)^{\vee}$. Since E is injective, $\text{Hom}_R(C, E)^{\vee}$ is a C -flat R -module from [18, Lem. 3.3], and so $\text{Hom}_R(C, E)^{\vee} \cong C \otimes_R F$ for some flat R -module F . Then we have $\text{Hom}_R(C, E)^{\vee} \otimes_R \hat{R} \cong (C \otimes_R F) \otimes_R \hat{R} \cong \hat{C} \otimes_{\hat{R}} (F \otimes_R \hat{R})$, since $F \otimes_R \hat{R}$ is a flat \hat{R} -module by [8, p43, Ex. 9]. Consequently, $\text{Hom}_R(C, E)^{\vee} \otimes_R \hat{R}$ is a \hat{C} -flat \hat{R} -module. By the adjoint isomorphism, we have

$$\text{Hom}_{\hat{R}}(\text{Hom}_R(C, E)^{\vee} \otimes_R \hat{R}, E(k)) \cong \text{Hom}_R(\text{Hom}_R(C, E)^{\vee}, E(k))$$

as \hat{R} -module. Hence $(\text{Hom}_R(C, E)^{\vee})^{\vee}$ is a \hat{C} -injective \hat{R} -module by [18, Lem. 3.3].

Consider the natural monomorphism $\text{Hom}_R(C, E) \xrightarrow{f} (\text{Hom}_R(C, E)^{\vee})^{\vee}$. Note that $\text{Hom}_R(C, E)$ is a pure submodule of $(\text{Hom}_R(C, E)^{\vee})^{\vee}$, see [21, Prop. 2.3.5]. Also, [7, Lem. 2.1] implies that $\text{Hom}_R(C, E)$ is pure injective, so $\text{Hom}_R(C, E)$ is a direct summand of $(\text{Hom}_R(C, E)^{\vee})^{\vee}$. Thus, it suffices to show that $\text{Ext}_R^i((\text{Hom}_R(C, E)^{\vee})^{\vee}, M) = 0$ for all $i > 0$. To this end, by the assumption and the above isomorphism, for all $i > 0$, we have

$$\text{Ext}_R^i((\text{Hom}_R(C, E)^{\vee})^{\vee}, M) \cong \text{Ext}_{\hat{R}}^i((\text{Hom}_R(C, E)^{\vee})^{\vee}, \text{Hom}_R(\hat{R}, M)) = 0. \quad \square$$

Lemma 2.8 *Let R be a noetherian ring. If $\text{Tor}_i^R(\text{Hom}_R(C, I), M) = 0$ for all $i > 0$ and all injective R -modules I , then $\text{Ext}_R^i(M, C \otimes_R K) = 0$ for all $i > 0$ and all cotorsion R -modules K with finite flat dimension.*

Proof We use induction on the finite number $\text{fd}_R K = n$. First, assume that $n = 0$. Then K is flat, and hence K is a summand of an R -module $\text{Hom}_R(E, E')$ by [7, Lem. 2.3], where E, E' are injective. It suffices to show that $\text{Ext}_R^i(M, C \otimes_R \text{Hom}_R(E, E')) = 0$ for all $i > 0$. By [8, Thms. 3.2.11, 3.2.1], we have

$$\begin{aligned}
 \text{Ext}_R^i(M, C \otimes_R \text{Hom}_R(E, E')) &\cong \text{Ext}_R^i(M, \text{Hom}_R(\text{Hom}_R(C, E), E')) \\
 &\cong \text{Hom}_R(\text{Tor}_i^R(\text{Hom}_R(C, E), M), E') \\
 &= 0
 \end{aligned}$$

for all $i > 0$. So $\text{Ext}_R^i(M, C \otimes_R K) = 0$ for all $i > 0$. Now assume that $\text{fd}_R K = n > 0$. Let $F \rightarrow K$ be a flat cover of K with Kernel L . Then L is cotorsion and $\text{fd}_R L = n - 1$. Since $\text{fd}_R K = n < \infty$, by [13, Cor. 6.2], we have $K \in A_C(R)$, so $\text{Tor}_1^R(C, K) = 0$, then we obtain the following exact sequence

$$0 \rightarrow C \otimes_R L \rightarrow C \otimes_R F \rightarrow C \otimes_R K \rightarrow 0.$$

Now, applying the induction hypothesis and the long exact sequence

$$\cdots \rightarrow 0 = \text{Ext}_R^i(M, C \otimes_R F) \rightarrow \text{Ext}_R^i(M, C \otimes_R K) \rightarrow \text{Ext}_R^{i+1}(M, C \otimes_R L) = 0 \rightarrow \cdots,$$

we have the desired conclusion. \square

3. C-Gorenstein homological dimensions and Auslander categories

This section focuses on the connections between C-Gorenstein homological dimensions and the Auslander categories of \hat{R} .

In the following result, the conclusion of (1) was proved in [22, Prop. 4.3(1)] under an extra assumption that \hat{R} is a projective R -module.

Proposition 3.1 *Let R be a local noetherian ring and M an R -module. Then the following conclusions hold:*

- (1) *If M is a C-Gorenstein injective R -module, then $\text{Hom}_R(\hat{R}, M)$ is \hat{C} -Gorenstein injective as an \hat{R} -module.*
- (2) *If M is a C-Gorenstein injective R -module, then $M \in B'_C(R)$.*
- (3) *If M is a C-Gorenstein injective R -module, then M is cotorsion.*

Proof (1) If M is a C-Gorenstein injective R -module, then by Fact 2.5 (2), M admits a complete $\mathcal{IC}\mathcal{I}$ -resolution:

$$X = \cdots \xrightarrow{\partial_2^X} \text{Hom}_R(C, I_1) \xrightarrow{\partial_1^X} \text{Hom}_R(C, I_0) \xrightarrow{\partial_0^X} I_{-1} \xrightarrow{\partial_{-1}^X} I_{-2} \xrightarrow{\partial_{-2}^X} \cdots$$

such that $M \cong \text{Coker} \partial_1^X$. As an R -module, \hat{R} has projective dimension at most $\dim R < \infty$ (see [8, Cor. 2.4.31] and [21, Thm. 4.2.8]). We use induction on the finite number $\text{pd}_R \hat{R}$. If $\text{pd}_R \hat{R} = 0$, then \hat{R} is a projective R -module, and so $\text{Hom}_R(\hat{R}, X)$ is exact. Assume $\text{pd}_R \hat{R} \geq 1$. Let $0 \rightarrow K \rightarrow P \rightarrow \hat{R} \rightarrow 0$ be a projective resolution of \hat{R} with $\text{pd}_R K = \text{pd}_R \hat{R} - 1$. Then it follows from [8, Thm. 3.2.1] that $\text{Ext}_R^1(\hat{R}, \text{Hom}_R(C, E)) \cong \text{Hom}_R(\text{Tor}_1^R(C, \hat{R}), E) = 0$ for any injective R -module E since $\hat{R} \in A_C(R)$ (see [13, Cor. 6.2]), and so the following sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_R(\hat{R}, \text{Hom}_R(C, I_i)) \rightarrow \text{Hom}_R(P, \text{Hom}_R(C, I_i)) \\ \rightarrow \text{Hom}_R(K, \text{Hom}_R(C, I_i)) \rightarrow 0, \end{aligned} \tag{*}_i$$

and

$$0 \rightarrow \text{Hom}_R(\hat{R}, I_{-j}) \rightarrow \text{Hom}_R(P, I_{-j}) \rightarrow \text{Hom}_R(K, I_{-j}) \rightarrow 0 \tag{*}_{-j}$$

are exact for $i = 0, 1, \dots$ and $j = 1, 2, \dots$. Thus $0 \rightarrow \text{Hom}_R(\hat{R}, X) \rightarrow \text{Hom}_R(P, X) \rightarrow \text{Hom}_R(K, X) \rightarrow 0$ is exact, which gives that

$$\text{Hom}_R(\hat{R}, X) = \cdots \rightarrow \text{Hom}_{\hat{R}}(\hat{C}, \text{Hom}_R(\hat{R}, I_0)) \rightarrow \text{Hom}_R(\hat{R}, I_{-1}) \rightarrow \cdots$$

is exact by induction and since $\text{Hom}_R(\hat{R}, \text{Hom}_R(C, I_i)) \cong \text{Hom}_{\hat{R}}(\hat{C}, \text{Hom}_R(\hat{R}, I_i))$, where every $\text{Hom}_R(\hat{R}, I_i)$ is an injective \hat{R} -module. Note that

$$\text{Hom}_R(\hat{R}, M) \cong \text{Hom}_R(\hat{R}, C_1^X) \cong \text{Hom}_R(\hat{R}, Z_{-1}^X) \cong Z_{-1}^{\text{Hom}_R(\hat{R}, X)} \cong C_1^{\text{Hom}_R(\hat{R}, X)}.$$

Let \bar{I} be any injective \hat{R} -module. Then \bar{I} is an injective R -module, and so

$$\text{Hom}_{\hat{R}}(\text{Hom}_{\hat{R}}(\hat{C}, \bar{I}), \text{Hom}_R(\hat{R}, X)) \cong \text{Hom}_R(\text{Hom}_R(C, \bar{I}), X)$$

is exact. Hence $\text{Hom}_R(\hat{R}, M)$ is a \hat{C} -Gorenstein injective \hat{R} -module.

(2) Let M be a C -Gorenstein injective R -module. By (1), $\text{Hom}_R(\hat{R}, M)$ is \hat{C} -Gorenstein injective as an \hat{R} -module. Hence, by [12, Thm. 4.6], $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$, and so $M \in B'_C(R)$, by the definition.

(3) Let F be any flat R -module. By [21, Thm. 4.2.8], $\text{pd}_R F = n < \infty$. Since M is a C -Gorenstein injective R -module, we have an exact sequence

$$0 \rightarrow K_n \rightarrow \text{Hom}_R(C, I_{n-1}) \rightarrow \cdots \rightarrow \text{Hom}_R(C, I_1) \rightarrow \text{Hom}_R(C, I_0) \rightarrow M \rightarrow 0$$

with I_i injective. Breaking this sequence into short exact ones and noting that

$$\text{Ext}_R^i(F, \text{Hom}_R(C, I_i)) \cong \text{Hom}_R(\text{Tor}_i^R(C, F), I_i) = 0$$

for all $i > 0$ by [8, Thm. 3.2.1], by dimension shifting, we have

$$\text{Ext}_R^m(F, M) \cong \text{Ext}_R^{m+n}(F, K_n) = 0$$

for all $m > 0$. \square

Dually, we get the next result.

Proposition 3.2 *Let R be a local noetherian ring and M an R -module. Then the following conclusions hold:*

- (1) *If M is a C -Gorenstein flat R -module, then $\hat{R} \otimes_R M$ is \hat{C} -Gorenstein flat as an \hat{R} -module.*
- (2) *If M is a C -Gorenstein flat R -module, then $M \in A'_C(R)$.*

Let \mathcal{X} be any class of R -modules and M an R -module. An \mathcal{X} -precover of M is an R -homomorphism $\varphi : X \rightarrow M$, where $X \in \mathcal{X}$ and such that the sequence

$$\text{Hom}_R(X', X) \xrightarrow{\text{Hom}_R(X', \varphi)} \text{Hom}_R(X', M) \longrightarrow 0$$

is exact for every $X' \in \mathcal{X}$. If, moreover, $\varphi f = \varphi$ for $f \in \text{Hom}_R(X, X)$ implies f is an automorphism of X , then φ is called an \mathcal{X} -cover of M . Also, an \mathcal{X} -preenvelope and \mathcal{X} -envelope of M are defined “dually”.

Lemma 3.3 *Let R be a noetherian ring and M an R -module.*

(1) *If R is a local ring and M is a cotorsion R -module such that $M \in B'_C(R)$, then there exists an epimorphism $L \rightarrow M$ with $\mathcal{I}_C\text{-id}_R(L) < \infty$.*

(2) *Assume $\varphi : L \rightarrow M$ is an epimorphism such that $\mathcal{I}_C\text{-id}_R(L) < \infty$ and that*

$$\text{Ext}_R^i(\text{Hom}_R(C, I), M) = 0$$

for all injective R -modules I and all $i > 0$. Then M has an epic $\tilde{\mathcal{I}}_C(R)$ -precover $E \rightarrow M$, in which E is C -injective.

Proof (1) Note that $M \in B'_C(R)$, then $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$. By [12, Thm. 4.6],

$$\mathcal{GI}_{\hat{C}}\text{-id}_{\hat{R}}(\text{Hom}_R(\hat{R}, M)) < \infty.$$

It follows from [19, Lem. 1.9], there are an \hat{R} -module L and an \hat{R} -epimorphism $L \rightarrow \text{Hom}_R(\hat{R}, M)$ such that $\mathcal{I}_C\text{-id}_{\hat{R}}(L) = \mathcal{GI}_C\text{-id}_{\hat{R}}(\text{Hom}_R(\hat{R}, M)) < \infty$. Note that every \hat{C} -injective \hat{R} -module is C -injective as an R -module by Lemma 2.2, and hence $\mathcal{I}_C\text{-id}_R(L) < \infty$. So, it suffices to show that the natural R -homomorphism $\text{Hom}_R(\hat{R}, M) \rightarrow M$ is epic.

The natural exact sequence $0 \rightarrow R \rightarrow \hat{R} \rightarrow \hat{R}/R \rightarrow 0$ yields an exact sequence

$$\text{Hom}_R(\hat{R}, M) \rightarrow \text{Hom}_R(R, M) \rightarrow \text{Ext}_R^1(\hat{R}/R, M).$$

The last module is zero because \hat{R}/R is a flat R -module and M is a cotorsion R -module. These considerations prove that $\text{Hom}_R(\hat{R}, M) \rightarrow M$ is epic.

(2) By [13, Prop. 5.3 (e)], there exists an $\mathcal{I}_C(R)$ -precover $f : \text{Hom}_R(C, E) \rightarrow M$. At first, we show that f is an $\tilde{\mathcal{I}}_C(R)$ -precover. To this end, let $\varphi' : L' \rightarrow M$ be an R -homomorphism such that $\mathcal{I}_C\text{-id}_R(L') < \infty$. By [15, Cor. 2.9], we have $L' \in A_C(R)$, and so $L' \cong \text{Hom}_R(C, C \otimes_R L')$ and $\text{Ext}_R^1(C, C \otimes_R L') = 0$. On the other hand, by [15, Thm. 2.11], $\text{id}_R(C \otimes_R L') = \mathcal{I}_C\text{-id}_R(L') < \infty$, there in particular exists an exact sequence $0 \rightarrow C \otimes_R L' \rightarrow E' \rightarrow K \rightarrow 0$ with E' injective and $\text{id}_R K < \infty$. Applying $\text{Hom}_R(C, -)$, we get an exact sequence

$$0 \longrightarrow L' \xrightarrow{g} \text{Hom}_R(C, E') \longrightarrow \text{Hom}_R(C, K) \longrightarrow 0$$

with $\mathcal{I}_C\text{-id}_R(\text{Hom}_R(C, K)) = \text{id}_R K < \infty$ (see [15, Thm. 2.11]). From the assumption and [16, Lem. 1.7], we have $\text{Ext}_R^1(\text{Hom}_R(C, K), M) = 0$. Now, applying the functor $\text{Hom}_R(-, M)$ to the above exact sequence, we obtain the following exact sequence

$$\text{Hom}_R(\text{Hom}_R(C, E'), M) \rightarrow \text{Hom}_R(L', M) \rightarrow \text{Ext}_R^1(\text{Hom}_R(C, K), M) = 0.$$

So, there exists an R -homomorphism $\phi : \text{Hom}_R(C, E') \rightarrow M$ such that $\varphi' = \phi g$. Also as f is an $\mathcal{I}_C(R)$ -precover, there exists an R -homomorphism $h : \text{Hom}_R(C, E') \rightarrow \text{Hom}_R(C, E)$ such that $\phi = fh$. Then there exists an R -homomorphism $hg : L' \rightarrow \text{Hom}_R(C, E)$ such that $f(hg) = \varphi'$, which means f is an $\tilde{\mathcal{I}}_C(R)$ -precover.

Since f is an $\tilde{\mathcal{I}}_C(R)$ -precover, then there is an R -homomorphism $\psi : L \rightarrow \text{Hom}_R(C, E)$ such that $\varphi = f\psi$. Since φ is epic, f is also epic. \square

Dually, we can prove the conclusions (1) and (2) of the next lemma using Lemma 2.8 and [18, Cor. 5.10].

Lemma 3.4 *Let R be a noetherian ring and M an R -module.*

(1) *If R is a local ring and $M \in A'_C(R)$, then there exists a monomorphism $M \rightarrow L$ with $\mathcal{F}_C\text{-pd}_R(L) < \infty$.*

(2) *Assume $\varphi : M \rightarrow L$ is a monomorphism such that $\mathcal{F}_C\text{-pd}_R(L) < \infty$ and that $\text{Tor}_i^R(\text{Hom}_R(C, I), M) = 0$ for all injective R -modules I and all $i > 0$. Then M has a monic $\tilde{\mathcal{F}}_C(R)$ -preenvelope $M \rightarrow F$, in which F is C -flat.*

(3) *Let R -homomorphism $f : M \rightarrow L'$ be a $\tilde{\mathcal{P}}_C(R)$ -preenvelope. Assume $\psi : M \rightarrow L$ is a monomorphism such that $\mathcal{P}_C\text{-pd}_R(L) < \infty$ and that $\text{Ext}_R^i(M, C \otimes_R Q) = 0$ for all projective R -modules Q and all $i > 0$. Then M has a monic $\tilde{\mathcal{P}}_C(R)$ -preenvelope $M \rightarrow P$, in which P is C -projective.*

Proof We should only prove (3). By assumption, $\psi : M \rightarrow L$ is monic, then $f : M \rightarrow L'$ is also monic. It follows from [15, Cor. 2.9], $L' \in B_C(R)$. Now, there exists an exact sequence $0 \rightarrow K \rightarrow C \otimes_R P \xrightarrow{\pi} L' \rightarrow 0$ such that P is a projective R -module by [15, Cor. 2.4]. It is easy to see that $\mathcal{P}_C\text{-pd}_R(K) < \infty$. So from the assumption and [16, Lem. 1.7], we have $\text{Ext}_R^1(M, K) = 0$, and thus a lifting $\mu : M \rightarrow C \otimes_R P$ with $f = \pi\mu$. The injectivity of f gives also μ is monic. It is easy to see that μ is also a $\tilde{\mathcal{P}}_C(R)$ -preenvelope. \square

Theorem 3.5 *Let R be a local noetherian ring and M an R -module. Then the following conditions are equivalent:*

- (1) M is C -Gorenstein injective.
- (2) M is cotorsion and $\text{Hom}_R(\hat{R}, M)$ is \hat{C} -Gorenstein injective as an \hat{R} -module.
- (3) $M \in B'_C(R)$, M is cotorsion and $\text{Ext}_R^i(\text{Hom}_R(C, E), M) = 0$ for all injective R -modules E and all $i > 0$.

Proof (1) \Rightarrow (2). It follows from Proposition 3.1.

(2) \Rightarrow (3). By Definition 2.4 and Lemma 2.7, we have $\text{Ext}_R^i(\text{Hom}_R(C, E), M) = 0$ for all injective R -modules E and all $i > 0$. On the other hand, [12, Thm. 4.6] implies that $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$, so $M \in B'_C(R)$.

(3) \Rightarrow (1). By the definition of C -Gorenstein injective R -modules, it suffices to show that M admits a proper $\mathcal{I}_C(R)$ -resolution. By Lemma 3.3, there exists an exact sequence

$$0 \rightarrow N \rightarrow \text{Hom}_R(C, E) \xrightarrow{f} M \rightarrow 0 \tag{*}$$

with f an $\tilde{\mathcal{I}}_C(R)$ -precover and E injective. If we have proved that N satisfies the given assumption on M , then the result is obtained because one can obtain the sequence in Definition 2.4 (I2) by iterating (*).

Let I be an injective R -module. Applying the functor $\text{Hom}_R(\text{Hom}_R(C, I), -)$ to the exact sequence (*) yields an exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Ext}_R^i(\text{Hom}_R(C, I), M) &\rightarrow \text{Ext}_R^{i+1}(\text{Hom}_R(C, I), N) \\ &\rightarrow \text{Ext}_R^{i+1}(\text{Hom}_R(C, I), \text{Hom}_R(C, E)) \rightarrow \cdots \end{aligned}$$

Since $I \in B_C(R)$, we have

$$\text{Ext}_R^i(\text{Hom}_R(C, I), \text{Hom}_R(C, E)) \cong \text{Hom}_R(\text{Tor}_i^R(C, \text{Hom}_R(C, I)), E) = 0.$$

Also $\text{Ext}_R^i(\text{Hom}_R(C, I), M) = 0$ for all $i > 0$ by the assumption. Thus $\text{Ext}_R^j(\text{Hom}_R(C, I), N) = 0$ for all $j \geq 2$. From (*) we also have the following exact sequence

$$\text{Hom}_R(\text{Hom}_R(C, I), M) \rightarrow \text{Ext}_R^1(\text{Hom}_R(C, I), N) \rightarrow \text{Ext}_R^1(\text{Hom}_R(C, I), \text{Hom}_R(C, E)).$$

It is clear that $\text{Ext}_R^1(\text{Hom}_R(C, I), \text{Hom}_R(C, E)) = 0$. On the other hand, since f is an $\tilde{\mathcal{I}}_C(R)$ -precover, $\text{Hom}_R(\text{Hom}_R(C, I), f)$ is an epimorphism. So $\text{Ext}_R^1(\text{Hom}_R(C, I), N) = 0$ by ‘‘Five Lemma’’.

Next, we show that N is a cotorsion R -module. Since $M \in B'_C(R)$, $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$, and so $\mathcal{GI}_{\hat{C}\text{-id}_{\hat{R}}}(\text{Hom}_R(\hat{R}, M)) < \infty$. By [19, Lem. 1.9], there exists an exact sequence of \hat{R} -

modules

$$0 \longrightarrow K \longrightarrow E' \xrightarrow{\alpha} \text{Hom}_R(\hat{R}, M) \longrightarrow 0$$

such that $\mathcal{I}_{\hat{C}}\text{-id}_{\hat{R}}(E') = \mathcal{GI}_{\hat{C}}\text{-id}_{\hat{R}}(\text{Hom}_R(\hat{R}, M)) < \infty$ and K is a \hat{C} -Gorenstein injective \hat{R} -module. By Proposition 3.1, K is a cotorsion \hat{R} -module, which implies that K is cotorsion as an R -module by [10, Lem. 2.4]. Now, let $\varphi : \text{Hom}_R(\hat{R}, M) \rightarrow M$ be the natural R -homomorphism. Since $\mathcal{I}_{\hat{C}}\text{-id}_{\hat{R}}(E') < \infty$, by Lemma 2.2, $\mathcal{I}_C\text{-id}_R(E') < \infty$. Note that f is an $\tilde{\mathcal{I}}_C(R)$ -precover of M , then there exists an R -homomorphism $\psi : E' \rightarrow \text{Hom}_R(C, E)$ such that $\varphi\alpha = f\psi$. Thus there exists an R -homomorphism $\theta : K \rightarrow N$ such that the following diagram is commutative

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & E' & \xrightarrow{\alpha} & \text{Hom}_R(\hat{R}, M) \longrightarrow 0 \\ & & \downarrow \theta & & \downarrow \psi & & \downarrow \varphi \\ 0 & \longrightarrow & N & \longrightarrow & \text{Hom}_R(C, E) & \xrightarrow{f} & M \longrightarrow 0. \end{array}$$

Let F be any flat R -module. Since $\text{Ext}_R^1(F, \text{Hom}_R(C, E)) \cong \text{Hom}_R(\text{Tor}_1^R(C, F), E) = 0$, we obtain the following commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \text{Hom}_R(F, \text{Hom}_R(\hat{R}, M)) & \xrightarrow{\beta} & \text{Ext}_R^1(F, K) = 0 & \longrightarrow & \cdots & (**) \\ & & \downarrow \text{Hom}_R(F, \varphi) & & \downarrow \theta_1 & & \\ \cdots & \longrightarrow & \text{Hom}_R(F, M) & \xrightarrow{\gamma} & \text{Ext}_R^1(F, N) & \longrightarrow & 0. \end{array}$$

Note that \hat{R}/R is a flat R -module and M is a cotorsion R -module, then we have $\text{Ext}_R^1(\hat{R}/R, M) = 0$. Thus the natural exact sequence $0 \rightarrow R \rightarrow \hat{R} \rightarrow \hat{R}/R \rightarrow 0$ yields the following exact sequence

$$0 \longrightarrow \text{Hom}_R(\hat{R}/R, M) \longrightarrow \text{Hom}_R(\hat{R}, M) \xrightarrow{\varphi} M \longrightarrow 0.$$

Now, applying the functor $\text{Hom}_R(F, -)$ to it yields an exact sequence

$$\begin{array}{ccccccc} & & \text{Hom}_R(F, \text{Hom}_R(\hat{R}, M)) & \xrightarrow{\text{Hom}_R(F, \varphi)} & \text{Hom}_R(F, M) & & \\ & \longrightarrow & \text{Ext}_R^1(F, \text{Hom}_R(\hat{R}/R, M)) & \longrightarrow & \text{Ext}_R^1(F, \text{Hom}_R(\hat{R}, M)) & \longrightarrow & 0 & (\#) \end{array}$$

since M is cotorsion. Also, [14, Lem. 2.16] implies that $\text{Hom}_R(\hat{R}, M)$ is a cotorsion R -module, so $\text{Ext}_R^1(F, \text{Hom}_R(\hat{R}, M)) = 0$. On the other hand, since F is a flat R -module, we get that the sequence $0 \rightarrow F \rightarrow F \otimes_R \hat{R} \rightarrow F \otimes_R \hat{R}/R \rightarrow 0$ is exact. Thus the sequence

$$\text{Hom}_R(F \otimes_R \hat{R}, M) \rightarrow \text{Hom}_R(F, M) \rightarrow \text{Ext}_R^1(F \otimes_R \hat{R}/R, M) \rightarrow \text{Ext}_R^1(F \otimes_R \hat{R}, M) \rightarrow 0 \quad (\#\#)$$

is exact and since $F \otimes_R \hat{R}$ is a flat R -module and M is cotorsion, $\text{Ext}_R^1(F \otimes_R \hat{R}, M) = 0$.

Now, from (#) and (##) we obtain the following commutative diagram

$$\begin{array}{ccc}
 \text{Hom}_R(F, \text{Hom}_R(\hat{R}, M)) & \xrightarrow{\cong} & \text{Hom}_R(F \otimes_R \hat{R}, M) \\
 \downarrow & & \downarrow \\
 \text{Hom}_R(F, M) & \xrightarrow{=} & \text{Hom}_R(F, M) \\
 \downarrow & & \downarrow \\
 \text{Ext}_R^1(F, \text{Hom}_R(\hat{R}/R, M)) & \longrightarrow & \text{Ext}_R^1(F \otimes_R \hat{R}/R, M) \\
 \downarrow & & \downarrow \\
 0 & \xrightarrow{=} & 0.
 \end{array}$$

By ‘‘Five Lemma’’, $\text{Ext}_R^1(F, \text{Hom}_R(\hat{R}/R, M)) \cong \text{Ext}_R^1(F \otimes_R \hat{R}/R, M)$. Since $F \otimes_R \hat{R}/R$ is a flat R -module and M is cotorsion, $\text{Ext}_R^1(F, \text{Hom}_R(\hat{R}/R, M)) = 0$. Thus, from (#), $\text{Hom}_R(F, \varphi)$ is an epimorphism. Then by (**), $\theta_1\beta$ is epic and so θ_1 is epic. Thus $\text{Ext}_R^1(F, N) = 0$. It follows that N is cotorsion.

Finally, by Fact 2.5 and Proposition 3.1 (2), we have $\text{Hom}_R(C, E) \in B'_C(R)$. So $N \in B'_C(R)$ by Lemma 2.6. \square

Corollary 3.6 *Let R be a local noetherian ring of Krull dimension d and assume that $\text{Ext}_R^i(\hat{R}, M) = 0$ for all $i > 0$. Then the following conditions are equivalent: (1) $M \in B'_C(R)$. (2) $\mathcal{GI}_{C\text{-id}_R}(M) < \infty$. (3) $\mathcal{GI}_{C\text{-id}_R}(M) \leq d$.*

Proof (1) \Rightarrow (3). Since $M \in B'_C(R)$, $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$. So [12, Thm. 4.6] implies that $\mathcal{GI}_{\hat{C}\text{-id}_{\hat{R}}}(\text{Hom}_R(\hat{R}, M)) < \infty$. On the other hand, [12, Thm. 2.16] gives that

$$\mathcal{GI}_{\hat{C}\text{-id}_{\hat{R}}}(\text{Hom}_R(\hat{R}, M)) = \text{Gid}_{\hat{R} \times \hat{C}}(\text{Hom}_R(\hat{R}, M)).$$

By [11, Thm. 2.29], $\text{Gid}_{\hat{R} \times \hat{C}}(\text{Hom}_R(\hat{R}, M)) \leq \mathbf{FID}(\hat{R} \times \hat{C})$, where

$$\mathbf{FID}(R) = \sup\{\text{id}_R N \mid N \text{ is an } R\text{-module with finite injective dimension}\}.$$

Also, [4, Cor. 5.5] implies that $\mathbf{FID}(\hat{R} \times \hat{C}) \leq \dim(\hat{R} \times \hat{C})$, and \hat{R} and $\hat{R} \times \hat{C}$ are finitely generated as modules over each other, so $\dim(\hat{R} \times \hat{C}) = \dim(\hat{R}) = d$. This shows that $\mathcal{GI}_{\hat{C}\text{-id}_{\hat{R}}}(\text{Hom}_R(\hat{R}, M)) \leq d$.

Consider the following exact sequence of R -modules

$$0 \rightarrow M \rightarrow E^0 \rightarrow E^1 \rightarrow \dots \rightarrow E^{d-1} \rightarrow L \rightarrow 0$$

with E^i injective for $0 \leq i \leq d - 1$. By Fact 2.5, it suffices to prove that L is a C -Gorenstein injective R -module. From the assumption, we have the following exact sequence

$$0 \rightarrow \text{Hom}_R(\hat{R}, M) \rightarrow \text{Hom}_R(\hat{R}, E^0) \rightarrow \dots \rightarrow \text{Hom}_R(\hat{R}, E^{d-1}) \rightarrow \text{Hom}_R(\hat{R}, L) \rightarrow 0.$$

Since $\mathcal{GI}_{\hat{C}\text{-id}_{\hat{R}}}(\text{Hom}_R(\hat{R}, M)) \leq d$, $\text{Hom}_R(\hat{R}, L)$ is a \hat{C} -Gorenstein injective \hat{R} -module.

Let F be any flat R -module. Then by [21, Thm. 4.2.8], $\text{pd}_R F \leq d$. Thus $\text{Ext}_R^i(F, L) \cong \text{Ext}_R^{i+d}(F, M) = 0$ for $i > 0$. This means that L is a cotorsion R -module. Now the result follows

from Theorem 3.5.

(3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1). Let $\mathcal{GI}_{C\text{-id}_R}(M) = s < \infty$. Then there exists an exact sequence of R -modules

$$0 \rightarrow M \rightarrow G^0 \rightarrow G^1 \rightarrow \dots \rightarrow G^s \rightarrow 0$$

such that G^0, G^1, \dots, G^s are C -Gorenstein injective R -modules. By [21, Thm. 4.2.8], $\text{pd}_R \hat{R}$ is finite. Using hypothesis and [17, Lem. 4.8], we get an exact sequence

$$0 \rightarrow \text{Hom}_R(\hat{R}, M) \rightarrow \text{Hom}_R(\hat{R}, G^0) \rightarrow \dots \rightarrow \text{Hom}_R(\hat{R}, G^s) \rightarrow 0.$$

This implies that $\mathcal{GI}_{\hat{C}\text{-id}_{\hat{R}}}(\text{Hom}_R(\hat{R}, M)) \leq s$. So [12, Thm. 4.6] gives that $\text{Hom}_R(\hat{R}, M) \in B_{\hat{C}^+}(\hat{R})$. Thus, $M \in B'_C(R)$. \square

The next result can be proved using a similar method as in Theorem 3.5.

Theorem 3.7 *Let R be a local noetherian ring, M an R -module, and n a non-negative integer. Then the following conditions are equivalent:*

- (1) $\mathcal{GF}_{C\text{-pd}_R}(M) \leq n$.
- (2) $M \in A'_C(R)$ and $\text{Tor}_i^R(\text{Hom}_R(C, I), M) = 0$ for all injective R -modules I and all $i > n$.
- (3) $M \in A'_C(R)$ and $\text{Ext}_R^i(M, C \otimes_R L) = 0$ for all cotorsion R -modules L with finite flat dimension and all $i > n$.
- (4) $M \in A'_C(R)$ and $\text{Ext}_R^i(M, C \otimes_R F) = 0$ for all cotorsion flat R -modules F and all $i > n$.

Corollary 3.8 *Let (R, \mathfrak{M}, k) be a local noetherian ring. If $M \in A'_C(R)$, then*

$$\mathcal{GF}_{\hat{C}\text{-pd}_{\hat{R}}}(\hat{R} \otimes_R M) = \mathcal{GF}_{C\text{-pd}_R}(M).$$

Proof The inequality “ \leq ” follows from Proposition 3.2 (1), then we should only prove the opposite inequality. Since $M \in A'_C(R)$, we have $\hat{R} \otimes_R M \in A_{\hat{C}^+}(\hat{R})$, which implies $\mathcal{GF}_{\hat{C}\text{-pd}_{\hat{R}}}(\hat{R} \otimes_R M) < \infty$ by [12, Thm. 4.6]. Set $\mathcal{GF}_{\hat{C}\text{-pd}_{\hat{R}}}(\hat{R} \otimes_R M) = s$. Then there exists an exact sequence of R -modules

$$0 \rightarrow K_s \rightarrow P_{s-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \tag{*}$$

with P_i projective for $0 \leq i \leq s - 1$. By Fact 2.5 (2), we need show that K_s is a C -Gorenstein flat R -module. From (*), we obtain the following exact sequence

$$0 \rightarrow K_s \otimes_R \hat{R} \rightarrow P_{s-1} \otimes_R \hat{R} \rightarrow \dots \rightarrow P_1 \otimes_R \hat{R} \rightarrow P_0 \otimes_R \hat{R} \rightarrow M \otimes_R \hat{R} \rightarrow 0.$$

By Fact 2.5 (2) and Proposition 3.2 (1), $K_s \otimes_R \hat{R}$ is a \hat{C} -Gorenstein flat \hat{R} -module. Then $K_s \otimes_R \hat{R} \in A_{\hat{C}^+}(\hat{R})$ by [12, Thm. 4.6]. Thus, $K_s \in A'_C(R)$.

Let E be any injective R -module and let $\text{Hom}_R(-, E(k))$ be denoted by $(-)^{\vee}$, where $E(k)$ is the injective envelope of k . By the similar arguments to the proof of Lemma 2.7, we know $\text{Hom}_R(C, E)$ is a direct summand of $(\text{Hom}_R(C, E)^{\vee})^{\vee}$. Thus, it suffices to show that

$$\text{Tor}_i^R((\text{Hom}_R(C, E)^{\vee})^{\vee}, K_s) = 0 \text{ for all } i > 0.$$

Consequently, we have $\text{Tor}_i^R(\text{Hom}_R(C, E), K_s) = 0$ for all $i > 0$ and the assertions hold. While

$(\text{Hom}_R(C, E)^\vee)^\vee$ is a \hat{C} -injective \hat{R} -module, we have $\text{Tor}_i^{\hat{R}}((\text{Hom}_R(C, E)^\vee)^\vee, K_s \otimes_R \hat{R}) = 0$ for all $i > 0$ by Theorem 3.7.

Now, suppose $\mathbf{F} \rightarrow K_s$ is a flat resolution of K_s . Then $\mathbf{F} \otimes_R \hat{R}$ is a flat resolution of $K_s \otimes_R \hat{R}$. So for every $i > 0$, we have

$$\begin{aligned} \text{Tor}_i^R((\text{Hom}_R(C, E)^\vee)^\vee, K_s) &= \text{H}_i((\text{Hom}_R(C, E)^\vee)^\vee \otimes_R \mathbf{F}) \\ &\cong \text{H}_i((\text{Hom}_R(C, E)^\vee)^\vee \otimes_{\hat{R}} (\mathbf{F} \otimes_R \hat{R})) \\ &= \text{Tor}_i^{\hat{R}}((\text{Hom}_R(C, E)^\vee)^\vee, K_s \otimes_R \hat{R}) \\ &= 0. \quad \square \end{aligned}$$

Corollary 3.9 *Let R be a local noetherian ring and M an R -module. Then the following conditions are equivalent:*

- (1) $M \in A'_C(R)$.
- (2) $\mathcal{GF}_C\text{-pd}_R(M) < \infty$.

The next result contains Theorem C from the introduction.

Theorem 3.10 *Let R be a local noetherian ring, M an R -module, and n a non-negative integer. Then the following conditions are equivalent:*

- (1) $\mathcal{GP}_C\text{-pd}_R(M) \leq n$.
- (2) $M \in A'_C(R)$ and $\text{Ext}_R^i(M, C \otimes_R P) = 0$ for all projective R -modules P and all $i > n$.
- (3) $\mathcal{GF}_C\text{-pd}_R(M) < \infty$ and $\text{Ext}_R^i(M, C \otimes_R P) = 0$ for all projective R -modules P and all $i > n$.

Proof (1) \Rightarrow (2). It follows from [17, Lem. (3.3)(a)] that $\mathcal{GF}_C\text{-pd}_R(M) \leq \mathcal{GP}_C\text{-pd}_R(M) \leq n$. Then by Theorem 3.7, $M \in A'_C(R)$. Also, [20, Prop. 2.12] implies that $\text{Ext}_R^i(M, C \otimes_R P) = 0$ for all projective R -modules P and all $i > n$.

(2) \Rightarrow (3). It follows from Corollary 3.9.

(3) \Rightarrow (1). There exists an exact sequence

$$0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

with P_i projective for $0 \leq i \leq n-1$. From Fact 2.5, we only need to show that K_n is C -Gorenstein projective.

Let P be any projective R -module. Then $\text{Ext}_R^i(K_n, C \otimes_R P) \cong \text{Ext}_R^{i+n}(M, C \otimes_R P) = 0$ for all $i > 0$. In view of Definition 2.4, it is enough to show that K_n admits a proper $\mathcal{P}_C(R)$ -coresolution.

[12, Thm. 2.16] and [11, Thm. 3.15] give that $\mathcal{GF}_C\text{-pd}_R(K_n) < \infty$, thus there exists a monomorphism $K_n \rightarrow L$ with $\mathcal{F}_C\text{-pd}_R(L) < \infty$ by [18, Cor. 5.10]. From [8, Thms. 3.2.1, 3.2.11], we have

$$\text{Ext}_R^i(K_n, C \otimes_R \text{Hom}_R(E, E')) \cong \text{Hom}_R(\text{Tor}_i^R(\text{Hom}_R(C, E), K_n), E')$$

for all $i > 0$ and all injective R -modules E and E' . Therefore, $\text{Tor}_i^R(\text{Hom}_R(C, E), K_n) = 0$ for all injective R -modules E . Now using Lemma 3.4 (2), (3), there exists a monic $\widehat{\mathcal{P}}_C(R)$ -preenvelope

$f : K_n \rightarrow C \otimes_R Q$ in which Q is projective. Let P be any projective R -module. Applying the functor $\text{Hom}_R(-, C \otimes_R P)$ to the exact sequence

$$0 \rightarrow K_n \xrightarrow{f} C \otimes_R Q \rightarrow B \rightarrow 0, \quad (*)$$

we have $\text{Ext}_R^i(B, C \otimes_R P) = 0$ for all $i > 0$ since $f : K_n \rightarrow C \otimes_R Q$ is a $\tilde{\mathcal{P}}_C(R)$ -preenvelope. Also, $\mathcal{G}\mathcal{F}_C\text{-pd}_R(B) < \infty$ by [12, Thm. 2.16] and [11, Thm. 3.15]. Now iterating the above procedure, we have the desired proper $\mathcal{P}_C(R)$ -coresolution of M . \square

Corollary 3.11 *Let R be a local noetherian ring and M an R -module. Then the following conditions are equivalent:*

- (1) $\mathcal{G}\mathcal{P}_C\text{-pd}_R(M) < \infty$.
- (2) $\mathcal{G}\mathcal{F}_C\text{-pd}_R(M) < \infty$.
- (3) $M \in A'_C(R)$.

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