Journal of Mathematical Research with Applications
May, 2013, Vol. 33, No. 3, pp. 312–320
DOI:10.3770/j.issn:2095-2651.2013.03.005
Http://jmre.dlut.edu.cn

Abundant Semigroups Which Are Disjoint Unions of Multiplicative Adequate Transversals

Haijun LIU†, **Xiaojiang GUO***
Department of Mathematics, Jiangxi Normal University, Jiangxi 330022*, P. R. China*

Abstract The aim of this paper is to study abundant semigroups which are disjoint unions of multiplicative adequate tranversals. After obtaining some properties of such semigroup, we prove that a semigroup is a disjoint union of multiplicative adequate transversals if and only if it is isomorphic to the direct product of a rectangular band and an adequate semigroup.

Keywords abundant semigroup; adequate transversal; multiplicative adequate transversal; [left; right] normal band.

MR(2010) Subject Classification 20M10

1. Introduction

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are generalizations of Green's relations $\mathcal{L}$ and $\mathcal{R}$: elements a, b of a semigroup S are related by $\mathcal{L}^*$ [resp., $\mathcal{R}^*$] in S if and only if they are related by $\mathcal{L}$ [resp., $\mathcal{R}$] in some oversemigroup of S. A semigroup is abundant if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains at least one idempotent. An abundant semigroup in which the idempotents commute is named adequate. Regular semigroups are abundant semigroups, and inverse semigroups are adequate semigroups. In these cases, we have $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. In [3], El-Qallali and Fountain defined idempotent-connected abundant semigroups, which intermediate the class of abundant semigroups and the class of regular semigroups.

As a generalization of inverse transversals in the range of abundant semigroups, El-Qallali [4] defined adequate transversals. In the same reference, he established the structure of abundant semigroups satisfying the regularity condition with a multiplicative type-A transversal. Adequate transversals have been attracting due attentions. The second author in [8] completely determined the structure of abundant semigroups with a multiplicative adequate transversal. Chen in [2] considered abundant semigroups with a quasi-ideal adequate transversal. He obtained a structure of such abundant semigroups. Guo-Shum [10] also probed abundant semigroups with a quasi-ideal adequate transversal. Guo-Wang in [11] investigated idempotent-connected abundant semigroups

Received December 20, 2011; Accepted March 1, 2012
Supported by the National Natural Science Foundation of China (Grant No. 10961014), the Natural Science Foundation of Jiangxi Province (Grant No. 2014BAB201009) and the Science Foundation of Education Department of Jiangxi Province (Grant No. GJJ11388).
†Present address: The Second Technician Institute of Shaoguan, Guangdong 512031, P. R. China
* Corresponding author
E-mail address: haijunliu123@163.com (Haijun LIU); xjguo@jxnu.edu.cn (Xiaojinag GUO)

which are disjoint unions of quasi-ideal adequate transversals. This kind of abundant semigroups can be regarded as generalizations of completely simple semigroups in the range of abundant semigroups.

Inspired by Guo-Wang [11], in this paper, we will investigate abundant semigroups which are disjoint unions of multiplicative adequate transversals. We proceed as follows: after providing some basic concepts and results, we will give some properties of such semigroups.

2. Preliminaries

Throughout this paper we shall use the notions and notations of [3] and [6]. Other terms can be found in Howie [13]. Here we provide some known results used repeatedly in the sequel without mentions. Firstly, we recall some basic facts about the relations $\mathcal{L}^*$ and $\mathcal{R}^*$.

Lemma 2.1 ([6]) *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*$ $[\mathcal{R}^*]$ b.

(2) For all $x, y \in S^1$, $ax = ay$ $[xa = ya]$ if and only if $bx = by$ $[xb = yb]$.

As an easy but useful consequence of Lemma 2.1, we have

Corollary 2.2 ([6]) *Let S be a semigroup and $e^2 = e, a \in S$. Then the following statements are equivalent:*

(1) $a\mathcal{L}^*e$ $[a\mathcal{R}^*e]$.

(2) $ae = a$ $[ea = a]$ and for all $x, y \in S^1$, $ax = ay$ $[xa = ya]$ implies that $ex = ey$ $[xe = ye]$.

Evidently, $\mathcal{L}^*$ is a right congruence while $\mathcal{R}^*$ is a left congruence. In general, we have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. But for regular elements a and b, $a\mathcal{L}^*b$ $[a\mathcal{R}^*b]$ if and only if $a\mathcal{L}b$ $[a\mathcal{R}b]$. For convenience, we use a^* to denote the typical idempotents $\mathcal{L}^*$-related to a while $a^\dagger$ those $\mathcal{R}^*$ -related to a. It is easy to check that in an adequate semigroup, each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains exactly one idempotent. Also, we can define $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$ and $\mathcal{D}^* = \mathcal{L}^* \vee \mathcal{R}^*$. If $\mathcal{K}^*$ is one of Green's $*$-relations $\mathcal{L}^*, \mathcal{R}^*, \mathcal{H}^*$ and $\mathcal{D}^*$, we denote by K_a^* the $\mathcal{K}^*$-class of S containing a.

Following [3], an abundant semigroup S is idempotent-connected, for short, IC, provided for each $a \in S$ and for some $a^\dagger, a^*$, there exists a bijection $\theta : \langle a^\dagger \rangle \to \langle a^* \rangle$ such that $xa = a(x\theta)$ for all $x \in \langle a^\dagger \rangle$, where $\langle e \rangle$ $(e \in E(S))$ is the subsemigroup of S generated by the idempotents of eSe. In this case, θ is indeed an isomorphism.

Recall that ω is the natural partial order on the idempotents of semigroup T defined by: for $e, f \in E(T)$,

$$e\omega f \text{ if and only if } e = ef = fe.$$

In what follows, we denote the set $\{f \in E(T) : f\omega e\}$ by $\omega(e)$.

As in [14], define on an abundant semigroup S: for $x, y \in S$,

$$x \leq_r y \Longleftrightarrow R^*(x) \subseteq R^*(y) \text{ and } x = ey \text{ for some } e \in E(R_x^*); \text{and}$$

$$x \leq_\ell y \Longleftrightarrow L^*(x) \subseteq L^*(y) \text{ and } x = yf \text{ for some } f \in E(L_x^*),$$

and $\leq = \leq_r \cap \leq_\ell$, where $R^*(x)$ [resp., $L^*(x)$] is the smallest right [resp., left] *-ideal containing x (for $*$-ideals, see [6]). In particular, the restriction of $[\leq_r, \leq_\ell]$ $\leq$ to $E(S)$ coincides with ω. Lawson noticed that if $x \leq_r y$ [resp., $x \leq_\ell y$] for each (some) idempotent $y^\dagger \in R_y^*$ [resp., $y^* \in L_y^*$], then there exists an idempotent $x^\dagger \in R_x^*$ [resp., $x^* \in L_x^*$] such that $x^\dagger \omega y^\dagger$ [resp., $x^* \omega y^*$] and $x = x^\dagger y$ [resp., $x = yx^*$] (for natural partial orders, also see [9]).

Let S be an abundant semigroup with set of idempotents E, and U be an abundant subsemigroup of S. U is a left [right] $*$-subsemigroup if for all $a \in U$, there exists $e \in U \cap E$ such that $a\mathcal{L}^*(S)e$ [$a\mathcal{R}^*(S)e$]. Furthermore, if U is both a left and a right $*$-subsemigroup, we call U a $*$-subsemigroup of S.

Lemma 2.3 ([4]) *Let S be an abundant semigroup and let U be an abundant subsemigroup of S. Then U is a left [right] $*$-subsemigroup if and only if*

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)\ [\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)].$$

Lemma 2.4 ([14]) *Let S be an abundant semigroup. If $e \in E(S)$, then eSe is a $*$-subsemigroup of S.*

Lemma 2.5 ([5]) *Let S be an abundant semigroup and T a $*$-subsemigroup of S. If $x \in S$ and $a \in T$ such that $x = eaf$ with $e, f \in E(S)$, $e\mathcal{L}a^\dagger$ and $f\mathcal{R}a^*$ for $a^\dagger, a^* \in T$, then $e\mathcal{R}^*x\mathcal{L}^*f$.*

Let S be an abundant semigroup and S° be an adequate $*$-subsemigroup of S. Let E° be the idempotent semilattice of S°. S° is an adequate transversal for S, if for any element $x \in S$, there exists a unique element $x^\circ \in S^\circ$ and idempotents $e, f \in E$ such that $x = ex^\circ f$ where $e\mathcal{L}^*x^{\circ\dagger}, f\mathcal{R}^*x^{\circ *}$ for $x^{\circ\dagger}, x^{\circ *} \in E^\circ$. In this case, e and f are uniquely determined by x. We shall denote by e_x such unique idempotent e and by f_x such unique idempotent f. Also, write $I_{S^\circ} = \{e_x : x \in S\}$ and $\Lambda_{S^\circ} = \{f_x : x \in S\}$. Moreover, if for all $x, y \in S, f_y e_x \in E^\circ$, we call the adequate transversal S° a multiplicative adequate transversal for S (for details, see [4]).

A subset M of a semigroup S is a quasi-ideal of S if $MSM \subseteq M$. In this paper, by a quasi-ideal adequate transversal, we mean an adequate transversal which is also a quasi-ideal of S. The papers [2], [8], [10] and [11] are all works on this aspect. Recently, Guo-Xie [12] pointed out naturally ordered abundant semigroups in which each idempotent has a greatest inverse have also some kind of quasi-ideal transversals. Our objective is to explore abundant semigroups which are disjoint unions of multiplicative adequate transversals, in notation, DUMT-semigroups.

The following lemma collects some results of Chen [2], Guo [8] and Guo-Shum [10].

Lemma 2.6 *Assume S° is an adequate transversal of an abundant semigroup S.*

(1) For any $x \in S$, $e_x\mathcal{R}^*x\mathcal{L}^*f_x$.

(2) If $x \in S^\circ$, then $e_x = x^\dagger, f_x = x^*$ and $x^\circ = x$.

(3) If $x \in E^\circ$, then $e_x = x^\circ = x = f_x$.

(4) If S° is a multiplicative adequate transversal of S, then S° is a quasi-ideal adequate transversal of S.

(5) S° is a quasi-ideal adequate transversal if and only if $\Lambda_{S^\circ} I_{S^\circ} \subseteq S^\circ$.

Let U and V be index sets and let us assume that a semigroup S is the disjoint union of nonempty sets $S_{i\lambda}$, where $(i,\lambda) \in U \times V$. We call S the rectangular band $U \times V$ of the semigroups $S_{i\lambda}$, with $(i,\lambda) \in U \times V$, if for all $(i,\lambda),(j,\mu) \in U \times V$,

$$S_{i\lambda}S_{j\mu} \subseteq S_{i\mu}. \tag{1}$$

Condition (1) is equivalent to saying that

$$S \to U \times V, \quad x \to (i,\lambda), \text{ if } x \in S_{i\lambda}$$

is a homomorphism of S onto the rectangular band $U \times V$. A semigroup S which is a rectangular band $U \times V$ of the semigroup $S_{i\lambda}$, with $(i,\lambda) \in U \times V$, is said to be a perfect rectangular band $U \times V$ of the semigroups $S_{i\lambda}$, with $(i,\lambda) \in U \times V$, if for all $(i,\lambda),(j,\mu) \in U \times V$, $S_{i\lambda}S_{j\mu} = S_{i\mu}$.

For brevity, in what follows, by the phrase "Let $S = \bigcup_{i \in U} S_i$ be a DUMT-semigroup", we mean that S is an abundant semigroup which is the disjoint union of multiplicative adequate transversals S_i with $i \in U$.

3. Main results

The objective of this section is to give some properties for DUMT-semigroups.

Proposition 3.1 *Let $S = \bigcup_{i \in U} S_i$ be a DUMT-semigroup. Then*

(1) For any $x \in S_i, y \in S_j$ with $i,j \in U$, if $x \le y$, then $S_i = S_j$.

(2) For each $i \in U, S_i$ is a maximum adequate $*$-subsemigroup of S.

(3) S is a locally adequate semigroup (that is, a semigroup satisfying that eSe is an adequate subsemigroup of S for every $e \in E(S)$).

Proof (1) If $x \in S_i, y \in S_j$ and $x \le y$, then for $y^\dagger, y^* \in S_j$, there exist $x^\dagger$ and x^* such that $x^\dagger \omega y^\dagger$, $x^* \omega y^*$ and $x = x^\dagger y = yx^*$. By $x^* \omega y^*$, we have $x^* = x^* y^* = y^* x^*$ and $x^* = y^* x^* y^*$, which imply that $x^* \in S_j$ by noticing that multiplicative adequate transversals are quasi-ideal adequate transversals, so that S_j is a quasi-ideal; and similarly, $x^\dagger \in S_j$. Because $x = x^\dagger y x^*$ and since S_j is a quasi-ideal, we have $x \in S_j$. Thus $S_i \cap S_j \ne \emptyset$ and $S_i = S_j$ since S is a disjoint union of S_k, with $k \in U$.

(2) For $i \in U$, it is clear that S_i is an adequate $*$-subsemigroup of S. Thus we need still to verify that S_i is maximal. For this, we let M be an adequate $*$-subsemigroup of S such that $S_i \subseteq M$. If $x \in M$ and $x^\dagger, x^* \in E(M)$, by hypothesis, we have that S_i is a multiplicative adequate transversal of S, so that there exists $x^\circ \in S_i$ such that $x = ex^\circ f$, where $e,f \in E(S)$ with $e\mathcal{L}x^{\circ\dagger}$ and $f\mathcal{R}x^{\circ *}$, for $x^{\circ\dagger}, x^{\circ *} \in E(S_i)$ $(\subseteq E(M))$. By Lemma 2.5 (1), $e\mathcal{R}^* x$ and $f\mathcal{L}^* x$, hence $e\mathcal{R}x^\dagger$ and $f\mathcal{L}x^*$. On the other hand, since $E(S_i) \subseteq E(M)$, we have $x^{\circ *}x^* \le x^{\circ *}$ and $x^{\circ *}x^* \le x^*$, thus since $x^{\circ *}x^* = x^* x^{\circ *}$, giving $x^{\circ *}x^* \in S_i$, and by (1), we can obtain $x^* \in S_i$; similarly, $x^\dagger \in S_i$. But $x = x^\dagger x x^*$, now $x \in S_i$ since S_i is a quasi-ideal, whence $M \subseteq S_i$. Therefore S_i is a maximum adequate $*$-subsemigroup of S.

(3) By Lemma 2.4, eSe is an abundant subsemigroup of S. Suppose $e \in S_i$ and notice that multiplicative adequate transversals are quasi-ideals, so $eSe \subseteq S_i$ and $E(eSe)$ is a semilattice

(since S_i is an adequate $*$-subsemigroup). This means that eSe is an adequate subsemigroup of S, thus S is a locally adequate semigroup. $\square$

Guo [8] pointed out that an abundant semigroup with a multiplicative adequate transversal must satisfy the regularity condition (that is, the set of all regular elements forms a regular subsemigroup). Clearly, a DUMT-semigroup satisfies the regularity condition. Moreover, we can prove

Proposition 3.2 *Let S be a DUMT-semigroup. Then* $\mathrm{Reg}\, S$ *(the set of regular elements of S) is a regular semigroup which is a disjoint union of multiplicative inverse transversals.*

Proof Assume S is the disjoint union of multiplicative adequate transversals S_i with $i \in U$. Let $x \in \mathrm{Reg}\, S \cap S_i$ and $x^\dagger, x^* \in S_i$. If x' is an inverse of x, then $x = xx'x = x(x^*x'x^\dagger)x$. Since S_i is a quasi-ideal of S, $x^*x'x^\dagger \in S_i$. This, together with $x = xx'x = x(x^*x'x^\dagger)x$, implies that x is regular in the semigroup S_i. Therefore $\mathrm{Reg}\, S \subseteq \bigcup_{i\in U} \mathrm{Reg}\, S_i$ and whence $\mathrm{Reg}\, S = \bigcup_{i\in U} \mathrm{Reg}\, S_i$.

Obviously, $\mathrm{Reg}\, S_i$ is an inverse subsemigroup of $\mathrm{Reg}\, S$, for all $i \in U$. By [8, Theorem 6.5], multiplicative adequate transversals of a regular semigroup must be its multiplicative inverse transversals. Now, we need only to verify that $\mathrm{Reg}\, S_i$ is a multiplicative adequate transversal for $\mathrm{Reg}\, S$. Since S_i is a multiplicative adequate transversal for S and by the definition of multiplicative adequate transversals, it remains to show that for any $x \in \mathrm{Reg}\, S$, if $x = e_x x^\circ f_x$ with $x^\circ \in S_i$, then x° is regular. Indeed, $x^\circ = x^{\circ\dagger}\, x^\circ\, x^{\circ *} = x^{\circ\dagger} e_x x^\circ f_x x^{\circ *} = x^{\circ\dagger}\, x x^{\circ *}$. Since S_i is a quasi-ideal of S , and S satisfies the regularity condition, we have $x^\circ \in \mathrm{Reg}\, S_i$. The proof is completed. $\square$

A band B is called [left; right] normal band, if [$xyz = xzy$; $xyz = yxz$] $xyzw = xzyw$, for any $x, y, z, w \in B$. By a strong normal band, we mean a normal band B which is the disjoint union of some semilattices each of which is isomorphic to the structure semilattice of B. For brevity, we always assume that the phrase "let $S = \bigcup_{i\in I} E_i$ be a strong normal band" means that S is a strong normal band and S has the decomposition $\bigcup_{i\in I} E_i$ in which each E_i is a semilattice isomorphic to the structure semilattice of S.

Now let $S = \bigcup_{j\in J} S_j$ be a DUMT-semigroup. Then by Proposition 3.2, $\mathrm{Reg}\, S$ is a regular subsemigroup of S which is a disjoint union of multiplicative inverse transversals $\mathrm{Reg}\, S_j$ with $j \in J$, and further by [11, Theorem 2.4], $\mathrm{Reg}\, S$ is a perfect rectangular band of inverse subsemigroups $T_{i\lambda}$ with $(i, \lambda) \in I \times \Lambda$, where each $T_{i\lambda}$ is just some $\mathrm{Reg}\, S_j$. For convenience, we index S_j as $S_{i\lambda}$ if $\mathrm{Reg}\, S_j = T_{i\lambda}$. Now, we can assume S is a disjoint union of multiplicative adequate transversals $S_{i\lambda}$ with $(i, \lambda) \in I \times \Lambda$. Denote $E_{i\lambda} = E(S_{i\lambda})$. Let $E_i = \bigcup_{\lambda\in\Lambda} E_{i\lambda}$ and $E_\lambda = \bigcup_{i\in I} E_{i\lambda}$. By [11, Lemma 3.5], we have

- $E_{i\lambda}E_{i\mu} = E_{i\mu}$ ($E_{i\lambda} = E_{i\lambda}E_{j\lambda}$) and E_i (E_λ) is a strong right (left) normal band.

Notice that a regular semigroup is an IC abundant semigroup. Applying [11, Lemma 2.3] to $\mathrm{Reg}\, S$, we have

- For any $(i, \lambda) \in I \times \Lambda$,

(1) For each $(i, \mu) \in I \times \Lambda$, there exists $e_{i\mu} \in E_{i\mu}$ such that $e_{i\mu}\mathcal{R}e_{i\lambda}$.

(2) For each $(j, \lambda) \in I \times \Lambda$, there exists $e_{j\lambda} \in E_{j\lambda}$ such that $e_{j\lambda}\mathcal{L}e_{i\lambda}$.

Based on the above arguments, we may prove

Corollary 3.3 *Let $S = \bigcup_{(i,\lambda)\in I\times\Lambda} S_{i\lambda}$ be a DUMT-semigroup. Then*

(1) $I_{S_{i\lambda}} = E_\lambda$.

(2) $\Lambda_{S_{i\lambda}} = E_i$.

Proof We only prove (1) because (2) is dual to (1). If $x = e_{k\lambda} \in E_{k\lambda}$, then there exists $e_{i\lambda} \in E_{i\lambda}$ such that $e_{i\lambda}\mathcal{L}e_{k\lambda}$. Now, $e_{k\lambda} = e_{k\lambda}e_{i\lambda}e_{i\lambda}$ and so $e_x = e_{k\lambda}$ since $S_{i\lambda}$ is an adequate transversal for S. This means $E_{k\lambda} \subseteq I_{S_{i\lambda}}$, and so $E_\lambda \subseteq I_{S_{i\lambda}}$. Conversely, if $y \in I_{S_{i\lambda}}$, then for some $(k,\mu) \in I \times \Lambda$, $y \in E_{k\mu}$. This shows that there is $f_{i\lambda} \in E_{i\lambda}$ such that $y\mathcal{L}f_{i\lambda}$, since $S_{i\lambda}$ is an adequate transversal for S. So, $y = yf_{i\lambda} \in E_{k\lambda}$ and whence $I_{S_{i\lambda}} \subseteq E_\lambda$. This completes the proof. □

Lemma 3.4 *Let $S = \bigcup_{(i,\lambda)\in I\times\Lambda} S_{i\lambda}$ be a DUMT-semigroup. Then for any $(i,\lambda) \in I\times\Lambda$, $S_{i\lambda} = \Lambda_{S_{i\lambda}} S I_{S_{i\lambda}}$. Moreover, $S_{i\lambda} = E_{i\lambda}SE_{i\lambda}$.*

Proof Since $S_{i\lambda}$ is a multiplicative adequate transversal for S, we observe that $S_{i\lambda}$ is a quasi-ideal adequate transversal for S, and by Corollary 3.3, $S_{i\lambda} = E(S_{i\lambda})S_{i\lambda}E(S_{i\lambda}) \subseteq \Lambda_{S_{i\lambda}}SI_{S_{i\lambda}}$. On the other hand, by Corollary 3.3 and since $S_{i\lambda}$ is a quasi-ideal adequate transversal, we have

$$\begin{aligned}\Lambda_{S_{i\lambda}}SI_{S_{i\lambda}} &= \cup_{(\mu,k)\in\Lambda\times I}E_{i\mu}SE_{k\lambda} = \cup_{(\mu,k)\in\Lambda\times I}E_{i\lambda}E_{j\mu}SE_{k\lambda}E_{i\lambda}\\ &\subseteq E_{i\lambda}SE_{i\lambda} \subseteq S_{i\lambda}SS_{i\lambda} \subseteq S_{i\lambda}.\end{aligned}$$

Consequently, $S_{i\lambda} = \Lambda_{S_{i\lambda}}SI_{S_{i\lambda}}$.

The rest follows from $S_{i\lambda} = E_{i\lambda}S_{i\lambda}E_{i\lambda} \subseteq E_{i\lambda}SE_{i\lambda} \subseteq \Lambda_{S_{i\lambda}}SI_{S_{i\lambda}} = S_{i\lambda}$. □

Lemma 3.5 *With the notations of Corollary 3.3, S is a perfect rectangular band of the adequate semigroups $S_{i\lambda}$ with $(i,\lambda) \in I\times\Lambda$.*

Proof By Lemma 3.4, we have

$$\begin{aligned}S_{i\lambda}S_{j\mu} &= E_{i\lambda}SE_{i\lambda}E_{j\mu}SE_{j\mu} \subseteq E_{i\lambda}SE_{j\mu}\\ &= E_{i\mu}E_{i\lambda}SE_{j\mu}E_{i\mu} \subseteq E_{i\mu}SE_{i\mu} = S_{i\mu}\end{aligned}$$

and S is a rectangular band of $S_{i\lambda}$ with $(i,\lambda) \in I\times\Lambda$. Notice that the proof of [11, Lemma 2.3] does not need the assumption that S should be IC, so by [11, Lemma 2.3], S is a perfect rectangular band of $S_{i\lambda}$ with $(i,\lambda) \in I\times\Lambda$. □

Lemma 3.6 *Let $S = R\times T$ be the direct product of the rectangular band R and the adequate semigroup T. Then S is a disjoint union of multiplicative adequate transversals $\{r\}\times T$ with $r \in R$.*

Proof Notice that any rectangular band R is direct product of a left zero band I and a right zero band Λ, so we can let $S = I\times T\times\Lambda$, where T is an adequate semigroup. Now let $x = (j,t,\mu) \in I\times T\times\Lambda$. By the multiplication of direct products, it is a routine calculation to show $(j,t,\mu)\mathcal{L}^*$ $[\mathcal{R}^*]$ (k,s,γ) if and only if $\mu = \gamma$ $[j = k]$ and $t\mathcal{L}^*$ $[\mathcal{R}^*]$ s. This shows that

$(j, t^*, \mu)\mathcal{L}^*(j, t, \mu)\mathcal{R}^*(j, t^\dagger, \mu)$, so S is an abundant semigroup. Also, it is easy to check that the set $S_{j\mu} = \{j\} \times T \times \{\mu\}$ is a subsemigroup of S. Since the mapping $(j, t, \mu) \mapsto t$ is an isomorphism of $S_{j\mu}$ onto T, $S_{j\mu}$ is an adequate $*$-subsemigroup of S.

If $(i, t, \lambda) \in S_{i\lambda}$, it is obvious that $(i, t, \lambda) = (i, t^\dagger, \mu)(j, t, \mu)(j, t^*, \lambda)$, and $(i, t^\dagger, \mu)\mathcal{L}(j, t^\dagger, \mu)$ and $(j, t^*, \lambda)\mathcal{R}(j, t^*, \mu)$. On the other hand, if there is another element $(j, s, \mu) \in I \times T \times \Lambda$ such that

$$(*) \quad (i, t, \lambda) = (p, u, \tau)(j, s, \mu)(q, v, \xi),$$

with $(p, u, \tau), (q, v, \xi) \in E(S)$ and

$$(\dagger) \quad (p, u, \tau)\mathcal{L}(j, s^\dagger, \mu), (q, v, \xi)\mathcal{R}(j, s^*, \mu),$$

then by Lemma 2.5, $(p, u, \tau)\mathcal{R}^*(i, t, \lambda)\mathcal{L}^*(q, v, \xi)$ and so by the above proof, $i = p,\ \lambda = \xi$ and $u\mathcal{R}^*t\mathcal{L}^*v$, so that $u = t^\dagger, v = t^*$ since T is an adequate semigroup. Now, by $(*)$ and comparing the components, $t = t^\dagger s t^*$ while by $(\dagger)$,

$$(p, t^\dagger, \tau)(j, s^\dagger, \mu) = (p, t^\dagger s^\dagger, \mu) = (p, t^\dagger, \tau)$$

and

$$(j, s^\dagger, \mu)(p, t^\dagger, \tau) = (j, s^\dagger t^\dagger, \tau) = (j, s^\dagger, \mu),$$

so $t^\dagger = t^\dagger s^\dagger = s^\dagger t^\dagger = s^\dagger$ since T is an adequate semigroup. Similarly, $s^* = t^*$. Thus $t = t^\dagger s t^* = s^\dagger s s^* = s$, so that $(j, t, \mu) = (j, s, \mu)$. We have now proved that $S_{j\mu}$ is an adequate transversal for S with $e_{(i,t,\lambda)} = (i, t^\dagger, \mu),\ f_{(i,t,\lambda)} = (j, t^*, \lambda)$. Furthermore, since $(j, t^*, \lambda)(i, t^\dagger, \mu) = (j, t^* t^\dagger, \mu) \in E(S_{j\mu})$, $S_{j\mu}$ is indeed a multiplicative adequate transversal for S. This completes the proof. □

As for any non-empty sets A and B, $AB = \{ab : a \in A, b \in B\}$ is the complex product of A and B. Recall from pastijn-petrich that a non-empty subset H of an inverse semigroup T is called permissible if $E(T)H, HE(T) \subseteq H$ and $HH^{-1}, H^{-1}H \subseteq E(T)$, where $H^{-1} = \{h^{-1} : h \in H\}$ and h^{-1} is the inverse element of h in T. By [16, Lemma 2.6, p213], the set $C(T)$ of all permissible subsets of T forms an inverse monoid with identity $E(T)$ under complex multiplication; moreover, the set of idempotents of $C(T)$ is equal to $\{H \in C(T) :\ H \subseteq E(T)\}$.

Lemma 3.7 *Let $S = \bigcup_{(i,\lambda) \in I \times \Lambda} S_{i\lambda}$ be a DUMT-semigroup and $e_{i\lambda} \in E_{i\lambda}$, $f_{j\mu} \in E_{j\mu}$, $e_{i\mu}, f_{i\mu} \in E_{i\mu}$ for $i, j \in I, \lambda, \mu \in \Lambda$. If $e_{i\lambda}\mathcal{R}e_{i\mu}$ and $f_{j\mu}\mathcal{L}f_{i\mu}$, then $e_{i\lambda}f_{j\mu} = e_{i\mu}f_{i\mu}$.*

Proof By Proposition 3.2, $\operatorname{Reg} S$ is a disjoint union of multiplicative inverse transversal $\operatorname{Reg} S_{i\lambda}$ with $(i, \lambda) \in I \times \Lambda$, and so by [11, Lemma 4.4], $E_{i\lambda}E_{j\mu}$ is a unit of $C(\operatorname{Reg} S_{i\mu})$. Note that $S_{i\mu}$ is a multiplicative adequate transversal for S, thus $\Lambda_{S_{i\mu}} I_{S_{i\mu}} \subseteq E_{i\mu}$ and further by Corollary 3.3, $E_{i\lambda}E_{j\mu} \subseteq E_{i\mu}$, which implies that $E_{i\lambda}E_{j\mu}$ is an idempotent of $C(\operatorname{Reg} S_{i\mu})$. Now, since $E_{i\lambda}E_{j\mu}$ is a unit of $C(\operatorname{Reg} S_{i\mu})$, it is easy to see that $E_{i\lambda}E_{j\mu}$ is the identity of $C(\operatorname{Reg} S_{i\mu})$, that is, $E_{i\lambda}E_{j\mu} = E_{i\mu}$.

On the other hand, if denote $[a] = \{x \in S : x \leq a\}$, then by Proposition 3.1 (3), $\operatorname{Reg} S$ is a locally inverse semigroup and further by [11, Lemma 4.2], with respect to complex multiplication, the set $S_1 = \{[a] : a \in \operatorname{Reg} S\}$ is a semigroup and the mapping defined by

$$\theta :\ \operatorname{Reg} S \to S_1;\ a \mapsto [a]$$

is a semigroup isomorphism. By hypothesis, $e_{i\lambda} = e_{i\mu}e_{i\lambda}$ and $f_{j\mu} = f_{j\mu}f_{i\mu}$, hence by [11, Lemma 4.3], we can obtain $[e_{i\lambda}] = [e_{i\mu}]E_{i\lambda}$ and $[f_{j\mu}] = E_{j\mu}[f_{i\mu}]$. Thus

$$\begin{aligned}[e_{i\lambda}f_{j\mu}] &= [e_{i\lambda}][f_{j\mu}] = [e_{i\mu}]E_{i\lambda}E_{j\mu}[f_{i\mu}] = [e_{i\mu}]E_{i\mu}[f_{i\mu}] = [e_{i\mu}][f_{i\mu}] \\ &= [e_{i\mu}f_{i\mu}]\end{aligned}$$

and whence $e_{i\lambda}f_{j\mu} = e_{i\mu}f_{i\mu}$. □

We arrive now at the structure theorem of DUMT-semigroups.

Theorem 3.8 *A semigroup S is a DUMT-semigroup if and only if S is isomorphic to the direct product of a rectangular band and an adequate semigroup.*

Proof By Lemma 3.6, it suffices to prove S is isomorphic to the direct product of a rectangular band and an adequate semigroup. By Lemma 3.5 we assume S is an abundant semigroup which is a perfect rectangular band $I \times \Lambda$ of adequate semigroups $S_{i\lambda}$ with $(i, \lambda) \in I \times \Lambda$. Without loss of generality, we may let $0 \in I \cap \Lambda$ and denote $T = S_{00}$. Clearly, T is a multiplicative adequate transversal of S. By the definition of adequate transversal, for any $a_{i\lambda} \in S$, there are unique elements $a \in T$, $e_{i0} \in E_{i0} \cap L_{a^\dagger}$, $f_{0\lambda} \in E_{0\lambda} \cap R_{a^*}$ such that $a_{i\lambda} = e_{i0}af_{0\lambda}$, so the mapping

$$\theta : S \to I \times T \times \Lambda;\ a_{i\lambda} \mapsto (i, a, \lambda),$$

where I is identified as a left zero band and Λ is identified as a right zero band, is well defined and further a bijection.

It remains to prove that θ is a homomorphism. For this, let $a_{i\lambda} \in S_{i\lambda}$ and $b_{j\mu} \in S_{j\mu}$. By the definition of adequate transversal, $a_{i\lambda} = e_{i0}af_{0\lambda}$ and $b_{j\mu} = e_{j0}bf_{0\mu}$, where $a, b \in T$. This means that $(i, a, \lambda) = (a_{i\lambda})\theta$ and $(j, b, \mu) = (b_{j\mu})\theta$. By Lemma 3.7, $f_{0\lambda}e_{j0} = f_{00}e_{00}$ and so

$$\begin{aligned}a_{i\lambda}b_{j\mu} &= e_{i0}af_{0\lambda}e_{j0}bf_{0\mu} = e_{i0}af_{00}e_{00}bf_{0\mu} = e_{i0}aa^*f_{00}e_{00}b^\dagger bf_{0\mu} \\ &= e_{i0}af_{00}a^*b^\dagger e_{00}bf_{0\mu} = e_{i0}aa^*b^\dagger bf_{0\mu} \\ &= e_{i0}abf_{0\mu},\end{aligned}$$

whence $(a_{i\lambda}b_{j\mu})\theta = (i, ab, \mu)$. Thus

$$(a_{i\lambda})\theta(b_{j\mu})\theta = (i, a, \lambda)(j, b, \mu) = (ij, ab, \lambda\mu) = (i, ab, \mu) = (a_{i\lambda}b_{j\mu})\theta.$$

This shows that θ is a semigroup homomorphism, as required. □

By [10, Lemma 2.1] and [8, Theorem 6.5], we have that if S is a regular semigroup with an adequate transversal S°, then S° is multiplicative if and only if S° is a multiplicative inverse transversal for S. Now, by Theorem 3.7, the following corollary is immediate.

Corollary 3.9 *A regular semigroup is a disjoint union of multiplicative inverse transversals if and only if it is isomorphic to the direct product of a rectangular band and an inverse semigroup.*

Reference [8, Lemma 6.11] told us that if S° is an adequate transversal for an abundant semigroup S, then S is IC if and only if S° is a type-A semigroup. This and Theorem 3.8 imply immediately the following corollary.

Corollary 3.10 *An abundant semigroup is a disjoint union of multiplicative type-A transversals if and only if it is isomorphic to the direct product of a rectangular band and a type-A semigroup.*

Remark 3.11 By applying Theorem 3.8, it is easy to check that a DUMT-semigroup is quasi-adequate. Again by Proposition 3.1 (3), any DUMT-semigroup is a locally adequate semigroup. This shows that the idempotent band of a DUMT-semigroup is a normal band. For this kind of quasi-adequate semigroups, the reader can refer to [7].

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