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Multiple Positive Solutions of Boundary Value Problems for Systems of Nonlinear Third-Order Differential Equations

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Abstract In this paper, we consider boundary value problems for systems of nonlinear third-order differential equations. By applying the fixed point theorems of cone expansion and compression of norm type and Leggett-Williams fixed point theorem, the existence of multiple positive solutions is obtained. As application, we give some examples to demonstrate our results.

Keywords boundary value problem; multiple positive solutions; fixed-point theorem.

MR(2010) Subject Classification 34B15; 34B18

1. Introduction

In this paper, we study the existence of multiple positive solutions of boundary value problems for systems of nonlinear third-order differential equations:

$$\begin{cases} -u'''(t) = a_1(t)f_1(t, v(t)),\ t \in (0,1), \\ -v'''(t) = a_2(t)f_2(t, u(t)),\ t \in (0,1), \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha_1 u'(\eta_1), \\ v(0) = v'(0) = 0, \quad v'(1) = \alpha_2 v'(\eta_2), \end{cases} \tag{1.1}$$

where $f_i \in C([0,1] \times [0,+\infty), [0,+\infty)),\ 0 < \eta_i < 1,\ 1 < \alpha_i < \frac{1}{\eta_i},\ a_i(t) \in C([0,1],[0,+\infty))\ (i = 1,2)$.

In recent years, the existence of positive solutions for the third-order nonlinear boundary value problems received a special attention (see [1–4] and references therein). By using a Krasnosel'skii fixed point theorem, Guo et al. [5] studied the existence of at least one positive solutions for the following boundary value problem:

$$\begin{cases} u'''(t) + a(t)f(u(t)) = 0,\ t \in (0,1), \\ u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\eta). \end{cases} \tag{1.2}$$

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In [6], Hu et al. considered the existence of at least one and two positive solutions for systems of nonlinear second-order differential equations:

$$\begin{cases} -u''(t) = a_1(t)f_1(t, v(t)),\ t \in (0,1), \\ -v''(t) = a_2(t)f_2(t, u(t)),\ t \in (0,1), \\ \alpha u(0) + \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \\ \alpha v(0) + \beta v'(0) = 0, \quad \gamma v(1) + \delta v'(1) = 0. \end{cases} \tag{1.3}$$

Motivated by the works of [5] and [6], in this paper we aim at investigating the existence of at least two positive solutions associated with BVP (1.1) by applying the fixed point theorems of cone expansion and compression of norm type, and investigating the existence of at least three positive solutions for BVP(1.1) by using Leggett-Williams fixed point theorem. The results obtained in this paper are different from those in [5] and [6].

2. Preliminaries and lemmas

Lemma 2.1 ([5]) *Suppose that $\alpha_i\eta_i \neq 1$ $(i = 1, 2)$. Then for any $y \in C[0, 1]$, the problem*

$$\begin{cases} w_i'''(t) + y(t) = 0,\ t \in (0,1), \\ w_i(0) = w_i'(0) = 0, \quad w_i'(1) = \alpha_i w'(\eta_i), \end{cases} \tag{2.1}$$

has a unique solution

$$w_i(t) = \int_0^1 K_i(t,s)y(s)\mathrm{d}s, \quad i = 1, 2, \tag{2.2}$$

where

$$K_i(t,s) = \frac{1}{2(1-\alpha_i\eta_i)} \begin{cases} (2ts - s^2)(1-\alpha_i\eta_i) + t^2 s(\alpha_i - 1),\ s \le \min\{\eta_i, t\}, \\ t^2(1-\alpha_i\eta_i) + t^2 s(\alpha_i - 1),\ t \le s \le \eta_i, \\ (2ts - s^2)(1-\alpha_i\eta_i) + t^2(\alpha_i\eta_i - s),\ \eta_i \le s \le t, \\ t^2(1-s),\ \max\{\eta_i, t\} \le s. \end{cases} \tag{2.3}$$

Lemma 2.2 ([5]) *Let $0 < \eta_i < 1$ and $1 < \alpha_i < \frac{1}{\eta_i}$ $(i = 1, 2)$. Green's function $K_i(t, s)$ $(i = 1, 2)$ defined by (2.3) satisfies $0 \le K_i(t, s) \le K_i(s)$, $\forall (t, s) \in [0, 1] \times [0, 1]$, $i = 1, 2$, and*

$$\min_{t\in[\frac{\eta_i}{\alpha_i},1]} K_i(t,s) \ge \gamma_i K_i(s),\ \forall\ s \in [0,1],\ i = 1, 2, \tag{2.4}$$

where $K_i(s) = \frac{1+\alpha_i}{1-\alpha_i\eta_i} s(1-s)$, $s \in [0, 1]$, $0 < \gamma_i = \frac{\eta_i^2}{2\alpha_i^2(1+\alpha_i)} \min\{\alpha_i - 1, 1\} < 1$.

Corollary 2.1 *Let $0 < \eta_i < 1$ and $1 < \alpha_i < \frac{1}{\eta_i}$ $(i = 1, 2)$. Green's function $K_i(t, s) (i = 1, 2)$ defined by (2.3) satisfies $\min\limits_{t\in[\theta,1]} K_i(t, s) \ge \gamma K_i(s)$, where $\theta = \max\{\frac{\eta_1}{\alpha_1}, \frac{\eta_2}{\alpha_2}\}$, $\gamma = \min\{\gamma_1, \gamma_2\}$.*

It is easy to prove by Lemma 2.1 that $(u(t), v(t)) \in C^3([0, 1], (0, +\infty)) \times C^3([0, 1], (0, +\infty))$

is a positive solution of BVP(1.1) if and only if $(u(t), v(t))$ is a positive solution of system (2.5)

$$\begin{cases} u(t) = \int_0^1 K_1(t,s)a_1(s)f_1(s,v(s))\mathrm{d}s, \\ v(t) = \int_0^1 K_2(t,s)a_2(s)f_2(s,u(s))\mathrm{d}s, \end{cases} \tag{2.5}$$

where $K_i(t,s)$ $(i = 1, 2)$ is the Green's function defined by Lemma 2.1.

In real Banach space $C[0,1]$, the norm is defined by $\|u\| = \max_{t\in[0,1]} |u(t)|$. Set

$$P = \{u \in C[0,1]|, u(t) \geq 0 \text{ for } t \in [0,1], \min_{t\in[\theta,1]} u(t) \geq \gamma \parallel u \parallel\}. \tag{2.6}$$

Obviously, P is a positive cone in $C[0,1]$, where θ, γ are defined by Corollary 2.1.

For convenience, we make the following assumptions:

(A_1) $a_i(t) \in C([0,1], [0,+\infty))$ and $a_i(t)$ do not vanish identially for $t \in [\frac{\eta_i}{\alpha_i}, 1]$ $(i = 1, 2)$;

(A_2) $f_i \in C([0,1] \times [0,+\infty), [0,+\infty))$ $(i = 1, 2)$;

(A_3) $\alpha_i\eta_i < 1$ $(i = 1, 2)$.

Define the operators $T_1, T_2 : P \to E$ by

$$T_1u(t) = \int_0^1 K_1(t,s)a_1(s)f_1(s,v(s))\mathrm{d}s, \quad \forall t \in [0,1], \tag{2.7}$$

$$T_2v(t) = \int_0^1 K_2(t,s)a_2(s)f_2(s,u(s))\mathrm{d}s, \quad \forall\, t \in [0,1]. \tag{2.8}$$

Lemma 2.3 $T_1, T_2 : P \to P$ are completely continuous.

Proof For $u \in P$, consider (2.7), by Lemma 2.2, we have

$$0 \leq \|T_1u\| = \max_{0\leq t\leq 1} |T_1u(t)| \leq \int_0^1 K_1(s)a_1(s)f_1(s,v(s))\mathrm{d}s. \tag{2.9}$$

It follows from Corollary 2.1 and (2.9) that

$$\min_{t\in[\theta,1]} T_1u(t) \geq \gamma \int_0^1 K_1(s)a_1(s)f_1(s,v(s))\mathrm{d}s \geq \gamma\|T_1u\|. \tag{2.10}$$

Therefore $T_1 : P \to P$. It is easy to prove that $T_1 : P \to P$ is continuous since $K_1(t,s), f_1(t,v(s))$, $a_1(s)$ are continuous. Standard applications of the Arzela-Ascoli theorem imply that T_1 is a completely continuous operator. Similarly, it can be proven that $T_2 : P \to P$ is completely continuous.

In order to obtain our main results, we need the following fixed point theorems, which are useful methods to prove the existence of positive solutions for differential equations, for example [6–8] and [9, 10].

Lemma 2.4 ([11]) *Suppose E is a real Banach space and P is cone in E, and let Ω_1, Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega}_2\backslash\Omega_1) \to P$ be completely continuous. Suppose that one of the following two conditions holds*

(i) $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$; $\|Tx\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2$;

(ii) $\|Tx\| \geq \|x\|, \forall u \in P \cap \partial\Omega_1$; $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$,
then T *has at least one fixed point in* $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Lemma 2.5 ([12]) *Let* $A : \overline{P}_c \to \overline{P}_c$ *be completely continuous operator and* β *be a nonnegative continuous concave functional on* P *such that* $\beta(x) \leq \|x\|$ *for* $x \in \overline{P}_c$. *Suppose there exist* $0 < a < b < d \leq c$ *such that*

(i) $\{x \in P(\beta, b, d) : \beta(x) > b\} \neq \varnothing$ *and* $\beta(Ax) > b$ *for* $x \in P(\beta, b, d)$;

(ii) $\| Ax \| < a$ *for* $\| x \| \leq a$;

(iii) $\beta(Ax) > b$ *for* $x \in P(\beta, b, c)$ *with* $\| Ax \| > d$.

Then A *has at least three fixed points* x_1, x_2, x_3 *in* $\overline{P}_c$ *such that* $\|x_1\| < a, b < \beta(x_2)$ *and* $\|x_3\| > a$ *with* $\beta(x_3) < b$.

3. The existence of two positive solutions

For convenience, we introduce the following notations. Let

$$R_1 = \max\Big\{ \int_0^1 K_1(s)a_1(s)\mathrm{d}s, \int_0^1 K_2(s)a_2(s)\mathrm{d}s\Big\},$$
$$R_2 = \int_\theta^1 K_1(s)a_1(s)\mathrm{d}s \int_\theta^1 K_2(\tau)a_2(\tau)\mathrm{d}\tau,$$
$$R_3 = \gamma \min\Big\{ \int_\theta^1 K_1(s)a_1(s)\mathrm{d}s, \int_\theta^1 K_2(s)a_2(s)\mathrm{d}s\Big\}.$$

Theorem 3.1 *Suppose that the conditions* (A_1)–(A_3) *and the following assumptions hold*

(B_1) $\lim\limits_{v\to 0^+} \inf\limits_{t\in[0,1]} \frac{f_1(t,v)}{v} = \infty, \lim\limits_{u\to 0^+} \inf\limits_{t\in[0,1]} \frac{f_2(t,u)}{u} = \infty$;

(B_2) $\lim\limits_{v\to \infty} \inf\limits_{t\in[0,1]} \frac{f_1(t,v)}{v} = \infty, \lim\limits_{u\to \infty} \inf\limits_{t\in[0,1]} \frac{f_2(t,u)}{u} = \infty$,

(B_3) *There exists a constant* $\rho^* > 0$ *such that*

$$f_1(t,v) \leq R_1^{-1}\rho^*,\ f_2(t,u) \leq R_1^{-1}\rho^*,\ \text{for } (t,v),(t,u) \in [0,1]\times[0,\rho^*],$$

then BVP(1.1) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t)) \in C^3[0,1]\times C^3[0,1]$ *satisfying* $0 <\| u_1 \|< \rho^* <\| u_2 \|$ *and* $0 <\| v_1 \|< \rho^* <\| v_2 \|$.

Proof At first, it follows from the assumption (B_1) that we may choose $0 < \rho_1 < \rho^*$ such that $f_1(t,v) \geq \lambda_1 v, f_2(t,u) \geq \lambda_1 u$, for each $(t,v),(t,u) \in [0,1]\times[0,\rho_1]$, where $\lambda_1^2\gamma^3R_2 \geq 1$. Set $\Omega_1 = \{u \in C[0,1] :\| u \|< \rho_1\}$ and for $u, v \in P \cap \partial\Omega_1$, by Corollary 2.1 and (2.10), we have

$$\begin{aligned} T_1u(t) &\geq \lambda_1 \int_\theta^1 K_1(t,s)a_1(s)v(s)\mathrm{d}s \geq (\lambda_1\gamma)^2 \int_\theta^1 K_1(s)a_1(s)\mathrm{d}s \int_\theta^1 K_2(\tau)a_2(\tau)u(\tau)\mathrm{d}\tau \\ &\geq (\lambda_1\gamma)^2\gamma R_2\|u\|. \end{aligned}$$

Therefore

$$\|T_1u\| \geq \|u\|,\quad u \in P \cap \partial\Omega_1. \tag{3.1}$$

Further, it follows from the condition (B_2) that there exists $\rho_* > \rho^* > 0$ such that $f_1(t,v) \geq \lambda_2 v, f_2(t,u) \geq \lambda_2 u$, for each $(t,v),(t,u) \in [0,1]\times[\rho_*,+\infty)$, where $\lambda_2^2\gamma^3R_2 \geq 1$. Let $\rho_2 =$

$\max\{2\rho_1, \gamma^{-1}\rho_*\}$. Set $\Omega_2 = \{u \in C[0,1] :\| u \|< \rho_2\}$. For $u, v \in P \cap \partial\Omega_2$, by (2.10) we have $\min_{t\in[\theta,1]} u(t) \geq \gamma\|u\| = \gamma\rho_2 \geq \rho_*$, and

$$\begin{aligned} T_1u(t) &\geq \lambda_2\gamma \int_\theta^1 K_1(s)a_1(s)v(s)\mathrm{d}s \geq (\lambda_2\gamma)^2 \int_\theta^1 K_1(s)a_1(s)\mathrm{d}s \int_\theta^1 K_2(\tau)a_2(\tau)u(\tau)\mathrm{d}\tau \\ &\geq (\lambda_2\gamma)^2\gamma R_2\|u\|. \end{aligned}$$

Therefore

$$\|T_1u\| \geq \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.2}$$

Finally, set $\Omega_3 = \{u \in C[0,1] : \|u\| < \rho^*\}$ and for $u, v \in P \cap \partial\Omega_3$, by Lemma 2.2 and the condition (B$_3$), we have

$$T_1u(t) \leq \int_0^1 K_1(s)a_1(s)f_1(t, v(s))\mathrm{d}s \leq R_1^{-1}\rho^* \int_0^1 K_1(s)a_1(s)\mathrm{d}s \leq \rho^* = \|u\|$$

which implies

$$\| T_1u \|\leq \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{3.3}$$

Thus by (3.1)–(3.3), Lemmas 2.3 and 2.4, T_1 has a fixed point u_1 in $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$ and a fixed point u_2 in $P\cap(\overline{\Omega}_2\setminus\Omega_3)$. Similarly, it can be proven that T_2 has a fixed point v_1 in $P\cap(\overline{\Omega}_3\setminus\Omega_1)$ and a fixed point v_2 in $P \cap (\overline{\Omega}_2 \setminus \Omega_3)$. This means that BVP(1.1) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t)) \in C^3[0,1] \times C^3[0,1]$ satisfying $0 < u_1(t) < \rho^* \leq u_2(t), 0 < v_1(t) < \rho^* \leq v_2(t)$.

Theorem 3.2 *Suppose that the conditions (A$_1$)–(A$_3$) and the following assumptions hold*

(B$_4$) $\lim\limits_{v\to0^+} \sup\limits_{t\in[0,1]} \frac{f_1(t,v)}{v} = 0, \lim\limits_{u\to0^+} \sup\limits_{t\in[0,1]} \frac{f_2(t,u)}{u} = 0;$

(B$_5$) $\lim\limits_{v\to\infty} \sup\limits_{t\in[0,1]} \frac{f_1(t,v)}{v} = 0, \lim\limits_{u\to\infty} \sup\limits_{t\in[0,1]} \frac{f_2(t,u)}{u} = 0,$

(B$_6$) There exists a constant $\rho' > 0$ such that

$$f_1(t,u) \geq R_3^{-1}\rho', f_2(t,u) \geq R_3^{-1}\rho', \text{ for } (t,v), (t,u) \in [0,1] \times [\gamma\rho', \rho'].$$

Then BVP(1.1) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t)) \in C^3[0,1]\times C^3[0,1]$ satisfying $0 < \|u_1\| < \rho' < \|u_2\|$ and $0 < \|v_1\| < \rho' < \|v_2\|$.

Proof At first, it follows from the assumption (B$_4$) that there exists $0 < \rho_3 < \rho'$ such that $f_1(t,v) \leq \lambda_3 v, f_2(t,u) \leq \lambda_3 u$, for each $(t,v), (t,u) \in [0,1] \times [0, \rho_3]$, where $\lambda_3 R_1 \leq 1$. Set $\Omega_4 = \{u \in C[0,1] : \|u\| < \rho_3\}$. For $u, v \in P \cap \partial\Omega_4$, by Lemma 2.2, we have

$$\begin{aligned} T_1u(t) &\leq \int_0^1 K_1(s)a_1(s)\lambda_3 v(s)\mathrm{d}s \leq \lambda_3^2 \int_0^1 K_1(s)a_1(s)\mathrm{d}s \int_0^1 K_2(\tau)a_2(\tau)u(\tau)\mathrm{d}\tau \\ &\leq (\lambda_3R_1)^2\|u\|. \end{aligned}$$

Therefore

$$\|T_1u\| \leq \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{3.4}$$

Further, by the condition (B$_5$) we consider four cases:

Case (i) If f_1 and f_2 are bounded, then there exists $N > 0$ such that $f_1(t, v(t)) \le N$, $f_2(t, u(t)) \le N$, for $(t, u), (t, v) \in [0, 1] \times [0, +\infty)$. In this case, we may choose $\rho_4 = \max\{2\rho_3, NR_1\}$, so that for any $u \in P$ with $\|u\| = \rho_4$, we have

$$T_1 u(t) \le N \int_0^1 K_1(s) a_1(s) \mathrm{d}s \le \rho_4, \quad t \in [0, 1].$$

Therefore, $\|T_1 u\| \le \|u\|$. Similarly, we may obtain $\|T_2 v\| \le \|v\|$ for any $v \in P$ with $\|v\| = \rho_4$.

Case (ii) If f_1 is bounded and f_2 is unbounded, then there exists $N > 0$ such that $f_1(t, v(t)) \le N$ for $(t, v) \in [0, 1] \times [0, +\infty)$, and by the assumption (B_4) there exists $H > 0$ such that $f_2(t, u(t)) \le \delta u(t)$ for all $(t, u) \in [0, 1] \times [H, +\infty)$, where $\delta \int_0^1 K_2(s) a_2(s) \mathrm{d}s \le 1$.
Therefore, we may choose $\rho_4 = \max\{2\rho_3, H, N \int_0^1 K_1(s) a_1(s) \mathrm{d}s\}$, such that $f_2(t, u) \le f_2(t, \rho_4)$ for $(t, u) \in [0, 1] \times [0, \rho_4]$. So, for any $u \in P$ with $\|u\| = \rho_4$, we have

$$T_1 u(t) \le N \int_0^1 K_1(s) a_1(s) \mathrm{d}s \le \rho_4, \quad t \in [0, 1].$$

Therefore, $\|T_1 u\| \le \|u\| = \rho_4$. For any $v \in P$ with $\|v\| = \rho_4$, we have

$$T_2 v(t) \le \int_0^1 K_2(s) a_2(s) f_2(t, \rho_4) \mathrm{d}s \le \delta \rho_4 \int_0^1 K_2(s) a_2(s) \mathrm{d}s \le \rho_4, \quad t \in [0, 1].$$

So $\|T_2 v\| \le \|v\|$.

Case (iii) If f_2 is bounded and f_1 is unbounded, then there exists $N > 0$ such that $f_2(t, u(t)) \le N$ for all $(t, u) \in [0, 1] \times [0, +\infty)$, and by the assumption (B_4) there exists $H > 0$ such that $f_1(t, v(t)) \le \delta v(t)$ for $(t, u) \in [0, 1] \times [H, +\infty)$, where $\delta \int_0^1 K_1(s) a_1(s) \mathrm{d}s \le 1$. Therefore, we may choose $\rho_4 = \max\{2\rho_3, H, N \int_0^1 K_2(s) a_2(s) \mathrm{d}s\}$. For any $u, v \in P$ with $\|u\| = \|v\| = \rho_4$, similarly to Case (ii), we can obtain $\|T_1 u\| \le \|u\|, \|T_2 v\| \le \|v\|$.

Case (iv) If f_2 is unbounded and f_1 is unbounded, by the assumption (B_4) there exists $H > 0$ such that $f_1(t, v(t)) \le \delta v(t), f_2(t, u(t)) \le \delta u(t)$ for $(t, v), (t, u) \in [0, 1] \times [H, +\infty)$,where $\delta R_1 \le 1$. Therefore, we may choose $\rho_4 = \max\{2\rho_3, H, NR_1\}$. For any $u, v \in P$ with $\|u\| = \|v\| = \rho_4$, we can obtain $\|T_1 u\| \le \|u\|, \|T_2 v\| \le \|v\|$.
Therefore, in either case we may set $\Omega_5 = \{u \in C[0, 1] : \|u\| < \rho_4\}$, for $u, v \in P \cap \partial\Omega_5$ and we have

$$\|T_1 u\| \le \|u\|, \quad u \in P \cap \partial\Omega_5. \tag{3.5}$$

Finally, set $\Omega_6 = \{u \in C[0, 1] :\| u \|< \rho'\}$, for $u \in P \cap \partial\Omega_6$. Lemma 2.2 implies $\min_{t \in [\theta, 1]} u(t) \ge \gamma \|u\| = \gamma \rho'$, and by the condition (B_6), Corollary 2.1, (2.7), we have

$$T_1 u(t) \ge \int_\theta^1 K_1(t, s) a_1(s) f_1(t, v(s)) \mathrm{d}s \ge \gamma R_3^{-1} \rho' \int_\theta^1 K_1(s) a_1(s) \mathrm{d}s \ge \rho' = \|u\|.$$

Hence

$$\|T_1 u\| \ge \|u\|, \quad u \in P \cap \partial\Omega_6. \tag{3.6}$$

By (3.4)–(3.6), Lemmas 2.3 and 2.4, T_1 has a fixed point u_1 in $P \cap (\overline{\Omega}_6 \setminus \Omega_4)$ and a fixed u_2 in $P \cap (\overline{\Omega}_5 \setminus \Omega_6)$. Similarly, it can be proven that T_2 has a fixed point v_1 in $P \cap (\overline{\Omega}_6 \setminus \Omega_4)$

and a fixed v_2 in $P \cap (\overline{\Omega}_5 \setminus \Omega_6)$. This means that BVP(1.1) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t)) \in C^3[0,1] \times C^3[0,1]$ satisfying $0 < u_1(t) < \rho' \leq u_2(t), 0 < v_1(t) < \rho' \leq v_2(t)$.

4. The existence of three positive solutions

Let E be a real Banach space with cone P. A map $\beta : P \to [0, +\infty)$ is said to be a nonnegative continuous concave functional on P if β is continuous and

$$\beta(tx + (1-t)y) \geq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $t \in [0,1]$. Let a, b be two numbers such that $0 < a < b$ and β be a nonnegative continuous concave functional on P. We define the following convex sets:

$$P_a = \{x \in P : \|x\| < a\}, \quad \partial P_a = \{x \in P : \|x\| = a\}, \quad \overline{P}_a = \{x \in P : \|x\| \leq a\},$$

$$P(\beta, a, b) = \{x \in P : a \leq \beta(x), \|x\| \leq b\}.$$

Theorem 4.1 *Suppose that* (A_1)*–*(A_3) *hold. There exist nonnegative numbers* a, b, c *such that* $0 < a < b \leq \min\{\gamma, \frac{m_1}{M_1}, \frac{m_2}{M_2}\}c$ *and* $f_1(t,v), f_2(t,u)$ *satisfy the following growth conditions:*

(C_1) $f_1(t,v) \leq \frac{c}{M_1}$, $f_2(t,u) \leq \frac{c}{M_2}$, $(t,v),(t,u) \in [0,1] \times [0,c]$,

(C_2) $f_1(t,v) < \frac{a}{M_1}$, $f_2(t,u) < \frac{a}{M_2}$, $(t,v),(t,u) \in [0,1] \times [0,a]$,

(C_3) $f_1(t,v) > \frac{b}{m_1}$, $f_2(t,u) > \frac{b}{m_2}$, $(t,v),(t,u) \in [\theta,1] \times [b, \frac{b}{\gamma}]$,

where $m_i = \min_{t \in [\theta,1]} \int_\theta^1 K_i(t,s)a_i(s)\mathrm{d}s$, $M_i = \max_{t \in [0,1]} \int_0^1 K_i(t,s)a_i(s)\mathrm{d}s$, $i = 1,2$.
Then BVP(1.1) has at least three positive solutions $(u_{11}, u_{21}), (u_{12}, u_{22}), (u_{13}, u_{23}) \in C^3[0,1] \times C^3[0,1]$ *such that* $\|u_{i1}\| < a$, $b < \beta(u_{i2})$, *and* $\|u_{i3}\| > a$ *with* $\beta(u_{i3}) < b$, $i = 1,2$.

Proof Let P be defined by (2.6) and T_1, T_2 be defined by (2.7) (2.8). For $u \in P$, let $\beta(u) = \min_{t \in [\theta,1]} u(t)$. Then it is easy to check that β is a nonnegative continuous concave functional on P with $\beta(u) \leq \| u \|$ and by Lemma 2.3, $T_1, T_2 : P \to P$ are completely continuous operators.

First, we prove that if (C_1) holds, then $T_1 : \overline{P}_c \to \overline{P}_c$. In fact, if $u, v \in \overline{P}_c$, then $\|u\| \leq c$ and by condition (C_1), we have

$$\|T_1 u\| = \max_{t \in [0,1]} \left| \int_0^1 K_1(t,s)a_1(s)f_1(t,v(s))\mathrm{d}s \right| \leq \max_{t \in [0,1]} \frac{c}{M_1} \int_0^1 K_1(t,s)a_1(s)\mathrm{d}s = c. \tag{4.1}$$

Hence (4.1) shows that $T_1 : \overline{P}_c \to \overline{P}_c$.

In a completely analogous argument, the condition (C_2) implies that the condition (ii) of Lemma 2.5 is satisfied.

Now we show that the condition (i) of Lemma 2.5 is satisfied. Clearly, $\{u \in P(\beta, b, \frac{b}{\gamma}) : \beta(u) > b\} \neq \varnothing$. If $u \in P(\beta, b, \frac{b}{\gamma})$, then $b \leq u(s) \leq \frac{b}{\gamma}, s \in [\theta,1]$. Therefore, by ($C_3$) we obtain

$$\beta(T_1 u) = \min_{t \in [\theta,1]} \int_0^1 K_1(t,s)a_1(s)f_1(t,v(s))\mathrm{d}s > \frac{b}{m_1} \min_{t \in [\theta,1]} \int_\theta^1 K_1(t,s)a_1(s)\mathrm{d}s = b. \tag{4.2}$$

Therefore, the condition (i) of Lemma 2.5 is satisfied.

Finally, we show that the condition (iii) of Lemma 2.5 is satisfied. If $u \in P(\beta, b, c)$ and $\|T_1 u\| > \frac{b}{\gamma}$, then we have from Corollary 2.1 and (2.10) that

$$\beta(T_1 u) = \min_{t\in[\theta,1]} T_1 u(t) \geq \gamma \|T_1 u\| > \gamma \cdot \frac{b}{\gamma} = b. \tag{4.3}$$

Therefore, the condition (iii) of Lemma 2.5 is satisfied.

To sum up (4.1)–(4.3), all the conditions of Lemma 2.5 are satisfied. Hence, T_1 has at least three fixed points u_{11}, u_{12}, u_{13} such that $\|u_{11}\| < a$, $b < \beta(u_{12})$, and $\|u_{13}\| > a$ with $\beta(u_{13}) < b$. Similarly, it can be proven that T_2 has at least three fixed points u_{21}, u_{22}, u_{23} such that $\|u_{21}\| < a$, $b < \beta(u_{22})$, and $\|u_{23}\| > a$ with $\beta(u_{23}) < b$. This means that BVP(1.1) has at least three positive solutions $(u_{11}(t), u_{21}(t)), (u_{12}(t), u_{22}(t)), (u_{13}(t), u_{23}(t)) \in C^3[0,1] \times C^3[0,1]$ such that $\|u_{i1}\| < a$, $b < \beta(u_{i2})$, and $\|u_{i3}\| > a$ with $\beta(u_{i3}) < b$, $i = 1, 2$. □

In order to illustrate our results, we consider the following examples.

Example 4.1 In BVP(1.1), let $\alpha_1 = 2$, $\alpha_2 = \frac{3}{2}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{1}{2}$, $\alpha_1\eta_1 = \frac{2}{3} < 1$, $\alpha_2\eta_2 = \frac{3}{4} < 1$, $a_1(t) = (1-t)t$, $a_2(t) = \frac{1}{6}$, $f_1(t, v) = t + v^2 + v^{\frac{1}{3}}$, $f_2(t, u) = t + u^3 + u^{\frac{1}{2}}$. Clearly, the conditions (A_1)–(A_3) are satisfied. Then

$$\lim_{v\to 0^+} \inf_{t\in[0,1]} \frac{f_1(t,v)}{v} = \infty, \quad \lim_{u\to 0^+} \inf_{t\in[0,1]} \frac{f_2(t,u)}{u} = \infty;$$

$$\lim_{v\to \infty} \inf_{t\in[0,1]} \frac{f_1(t,v)}{v} = \infty, \quad \lim_{u\to \infty} \inf_{t\in[0,1]} \frac{f_2(t,u)}{u} = \infty.$$

Thus, the conditions (B_1)–(B_2) hold. Again

$$R_1 = \max\Big\{ \int_0^1 K_1(s) a_1(s) \mathrm{d}s, \int_0^1 K_2(s) a_2(s) \mathrm{d}s \Big\} \leq \frac{3}{10}.$$

Since $f_1(t, v), f_2(t, u)$ are monotone increasing functions for $(t, v), (t, u) \in [0,1] \times [0, +\infty)$, taking $\rho^* = 1$, and for $(t, v), (t, u) \in [0,1] \times [0, \rho^*]$, we have

$$f_1(t, v) \leq f_1(1,1) = 2 \leq R_1^{-1}\rho^*, \ \ f_2(t, u) \leq f_2(1,1) = 2 \leq R_1^{-1}\rho^*,$$

which implies that the condition (B_3) holds. Hence, by Theorem 3.1, BVP(1.1) has at least two positive solutions $(u_1(t), v_1(t)), (u_2(t), v_2(t)) \in C^3[0,1] \times C^3[0,1]$ satisfying $0 < u_1(t) < 1 < u_2(t), 0 < v_1(t) < 1 < v_2(t)$.

Example 4.2 In BVP(1.1), let $\alpha_1 = 2$, $\alpha_2 = \frac{3}{2}$, $\eta_1 = \frac{1}{3}$, $\eta_2 = \frac{1}{2}$, $\alpha_1\eta_1 = \frac{2}{3} < 1$, $\alpha_2\eta_2 = \frac{3}{4} < 1$, $a_1(t) = 24$, $a_2(t) = 36$, $\theta = \max\{\frac{1}{3}, \frac{1}{2}\} = \frac{1}{2}$, $K_1(t) = 9t(1-t)$, $K_2(t) = 10t(1-t)$, $\gamma_1 = \frac{1}{216}$, $\gamma_2 = \frac{1}{45}$, $\gamma = \min\{\frac{1}{216}, \frac{1}{45}\} = \frac{1}{216}$ and

$$f_1(t, v) = \begin{cases} \frac{t}{1000} + 12v^9, v \leq 1, \\ \frac{t}{1000} + 12, v > 1. \end{cases} \quad f_2(t, u) = \begin{cases} \frac{t}{1000} + 9u^{11}, u \leq 1, \\ \frac{t}{1000} + 9, u > 1. \end{cases}$$

It is easy to check that (A_1)–(A_3) hold. By direct calculation, we can obtain that $\frac{1}{12} \leq m_1 \leq M_1 = 36, \frac{5}{36} \leq m_2 \leq M_2 = 60$. Set $a = \frac{1}{2}, b = 1, c = 600$, so the nonlinear terms f_1, f_2 satisfy

$$f_1(t, v) < \frac{1}{72} = \frac{a}{M_1}, f_2(t, u) < \frac{1}{120} = \frac{a}{M_2}, \ (t, v), (t, u) \in [0,1] \times [0, \frac{1}{2}],$$

$$f_1(t,v) > 12 > \frac{b}{m_1}, f_2(t,u) > 9 > \frac{b}{m_2},\ (t,v),(t,u) \in [\frac{1}{2},1] \times [1,216],$$

$$f_1(t,v) < 13 < \frac{c}{M_1}, f_2(t,u) < 10 = \frac{c}{M_2},\ (t,v),(t,u) \in [0,1] \times [0,600].$$

Then the conditions (C_1)–(C_3) in Theorem 4.1 are all satisfied, and BVP(1.1) has at least three positive solutions $(u_{11}(t),u_{21}(t)),(u_{12}(t),u_{22}(t)),(u_{13}(t),u_{23}(t)) \in C^3[0,1] \times C^3[0,1]$ such that

$$\max_{0\le t\le 1} u_{i1} < \frac{1}{2},\ 1 < \min_{\frac{1}{2}\le t\le 1} u_{i2},\ \text{and}\ \max_{0\le t\le 1} u_{i3} > \frac{1}{2}\ \text{with}\ \min_{\frac{1}{2}\le t\le 1} u_{i3} < 1,\ i=1,2.$$

References

[1] D. R. ANDERSON, C. C. Tisdell. *Third-order nonlocal problems with sign-changing nonlinearity on time scales*. Electron. J. Differential Equations, 2007, **19**: 1–12.

[2] Yuqing FENG. *Solution and positive solution of a semilinear third-order equation*. J. Appl. Math. Comput., 2009, **29**(1-2): 153–161.

[3] Qingliu YAO. *Positive solutions of singular third-order three-point boundary value problems*. J. Math. Anal. Appl., 2009, **354**(1): 207–212.

[4] Yongping SUN. *Positive solutions for third-order three-point nonhomogeneous boundary value problems*. Appl. Math. Lett., 2009, **22**(1): 45–51.

[5] Lijun GUO, Jianping SUN, Yahong ZHAO. *Existence of positive solutions for nonlinear third-order three-point boundary value problems*. Nonlinear Anal., 2008, **68**(10): 3151–3158.

[6] Ling HU, Lianglong WANG. *Multiple positive solutions of boundary value problems for systems of nonlinear second-order differential equations*. J. Math. Anal. Appl., 2007, **335**(2): 1052–1060.

[7] Bin LIU. *Positive solutions of a nonlinear three-point boundary value problem*. Appl. Math. Comput., 2002, **132**(1): 11–28.

[8] Yaohong LI, Zhongli WEI. *Multiple positive solutions for n th order multipoint boundary value problem*. Bound. Value Probl., 2010, Art. ID 708376, 13 pp.

[9] Jianli LI, Jianhua SHEN. *Multiple positive solutions for a second-order three-point boundary value problem*. Appl. Math. Comput., 2006, **182**(1): 258–268.

[10] Yanping GUO, Yude JI, Jiehua ZHANG. *Three positive solutions for a nonlinear nth-order m-point boundary value problem*. Nonlinear Anal., 2008, **68**(11): 3485–3492.

[11] Dajun GUO, V. LAKSHMIKANTHAN. *Nonlinear Problems in Abstract Cones*. Academic Press, Inc., Boston, MA, 1988.

[12] R. W. LEGGETT, L. R. WILLIAMS. *Multiple positive fixed points of nonlinear operator on ordered Banach spaces*. Indiana Univ. Math. J., 1979, **28**(4): 673–688.