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Bounds of Solutions of a Kind of Hyper-Chaotic Systems and Application

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Abstract This paper investigates the boundedness of a kind of hyper-chaotic systems that have wide applications in the secure communications. In particular, an accurate bound estimation is attained for this kind of hyper-chaotic systems. Then, the result is applied to the chaos synchronization. Some numerical simulations are also given to verify the corresponding theoretical results.

Keywords hyper-chaotic system; boundedness; numerical simulations.

MR(2010) Subject Classification 37B25; 45M10; 11D41; 26A18; 37A60

1. Introduction

Chaos has been studied extensively since the Lorenz chaotic system was established in 1963 (see [1]). A hyper-chaotic system is a chaotic system with more than one positive Lyapunov exponents. Thus, hyper-chaotic systems have more complex dynamical behaviors than the ordinary chaotic systems. At the same time, due to its theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear circuits, control, synchronization, hyperchaos has recently become a central topic in the research of nonlinear sciences.

In particular, the boundedness of a chaotic system is important for the study of the qualitative behavior of a chaotic system. If we can show that a chaotic system has a globally attractive set, then we know that the system cannot have equilibrium points, periodic solutions, quasi-periodic solutions, or other chaotic attractors outside the globally attractive set. This simplifies the analysis of the dynamical properties for the chaotic system. The boundedness of a chaotic system also plays an important role in chaos control and chaos synchronization [2, 3] and it is also important for estimating the hausdorff dimension [4]. However, it is a very difficult task to get the boundedness of a chaotic system. Ever since the Lorenz system was put forward, researchers have been investigating its bound. From 1985 to 1987, Leonov gave the original results of globally ultimate bound of the Lorenz system [5, 6]. Then, Swinnerton-Dyer showed that Lyapunov

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functions can be used to study the bounds for trajectories of the Lorenz equations [7]. This idea was further developed by Liao et al. to get the new globally attractive set and positively invariant set of the Lorenz system by constructing a family of generalized Lyapunov functions, and this result was applied to study chaos control and chaos synchronization [2]. In the recent paper, Yu and Liao made a new survey of the global attractive and positively invariant set of the Lorenz system [8]. Sun got an ultimate bound of a generalized Lorenz system [9]. And Zhang et al. got the bounds for a synchronous motor system [10]. The estimation of the ultimate bound of the smooth Chua's system was given in [11]. And recently an ultimate bound and positively invariant set for the Lorenz-Haken system was obtained in [12]. However, the ultimate bounds of many other chaotic systems remain to be solved. Qin and Chen investigated the ultimate bound of the Chen system, but the parameter values considered do not cover the most interesting case of the Chen's chaotic attractor [13]. And the boundedness of the Lü system was studied in [14] only in the case of $2a > b > 2c > 0$. Due to the complexity of hyper-chaotic systems, it is more difficult to study the boundedness of hyper-chaotic systems. As far as the authors know, there is only one paper [15] that discusses the ultimate bound of the hyper-chaotic L-S system. Moreover, there is no unified approach for the bound estimation of the chaotic systems. Therefore, it is necessary to study the boundedness of the new hyper-chaotic systems.

The structure of this paper is organized as follows. In Section 2, we study the boundedness of a kind of hyper-chaotic systems (1.1). In Section 3, we study synchronization between the driver system and the response system according to Theorem 1. Some numerical simulations are given in Section 4. Section 5 contains conclusions.

Recently, Li et al. introduced a kind of hyper-chaotic systems in [16] as follows,

$$\begin{cases} \dot{x} = -ax + ay + \omega, \\ \dot{y} = -y + xz, \\ \dot{z} = b - xy - cz, \\ \dot{\omega} = -d\omega - yz, \end{cases} \tag{1.1}$$

where a, b, c, d are positive parameters. When $a = 5, b = 16, c = 1, d = 0.5$, the system (1.1) has the following four Lyapunov exponents, $LE(1) = 1.4002400 > 0, LE(2) = 0.3132080 > 0, LE(3) = -0.968585, LE(4) = -3.327758$ (see [16]). Since there are two positive Lapunov exponents for the system (1.1), the system (1.1) is a hyper-chaotic system [16]. When $a = 5, b = 16, c = 1, d = 0.5$, the phase portrait of the hyper-chaotic system (1.1) in x-y-z space is illustrated in Figure 1.

Some basic dynamical properties of the hyper-chaotic systems (1.1) were studied in [16], but many properties of the systems (1.1) remain to be uncovered. In the following, we will discuss the boundedness of the hyper-chaotic systems (1.1).

In the paper [15], the ultimate bound and positively invariant set were discussed for the hyper-chaotic L-S system. Compared with the hyper-chaotic system in the paper [15], it is more difficult to construct Lyapunov functions for the hyper-chaotic systems (1.1) in our paper. Synchronization of hyper-chaotic systems has been studied with increasing interest in the last few years due to its potential technological applications. And many methods have been successfully applied to study chaos synchronization such as PC method, linear feedback control, adaptive control, backstepping design, active control, and nonlinear control, etc. Among them, the linear feedback control is especially attractive. On the one hand, it can be easily applied to practical implementation due to its simplicity in configuration. On the other hand, linear state error feedback control is robust and easily implemented. The difference between our paper and other papers is to show that two linear control inputs are enough to force synchronization for two identical hyper-chaotic systems. Firstly, we have obtained $\lim_{t\to+\infty} e_1 = 0$, $\lim_{t\to+\infty} e_2 = 0$, $\lim_{t\to+\infty} e_3 = 0$ according to Lyapunov function theory. Secondly, we have proved $\lim_{t\to+\infty} |e_4| = 0$.

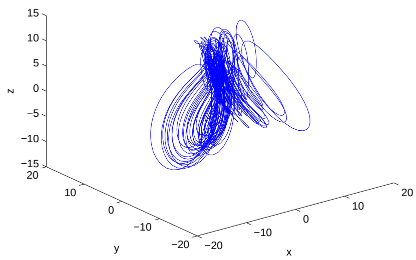

Figure 1 Phase portrait of the system (1.1) in the x-y-z space with parameters $a = 5, b = 16, c = 1, d = 0.5$ and the initial value $(x_0, y_0, z_0, w_0) = (3.2; 8.5; 3.5; 2.0)$

2. Main results

In this section, we will discuss the boundedness of the hyper-chaotic systems (1.1). Before going into details, let us introduce the following Lemma 1.

Lemma 1 *Define a set* $\Gamma = \{(y,z)|\frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1, b > 0, c > 0\}$, *and* $G = y^2 + z^2$, $H = y^2 + (z-2c)^2$, $(y,z) \in \Gamma$, *then we have*

$$G1 = \max_{(y,z)\in\Gamma} G = H1 = \max_{(y,z)\in\Gamma} H = \begin{cases} \frac{b^4}{b^2-c^2}, & b \geq \sqrt{2}c, \\ 4c^2, & b < \sqrt{2}c. \end{cases}$$

Proof It can be easily calculated by the Lagrange multiplier method. □

Theorem 1 *When* $a > 0$, $b > 0$, $c > 0$, $d > 0$, *the following set defined by*

$$\Omega = \left\{(x,y,z,w)|y^2 + z^2 \leq R_1^2,\ w^2 \leq \frac{R_1^4}{d^2},\ x^2 \leq \frac{R_1^2(ad+R_1)^2}{a^2d^2}\right\},$$

is the bound for systems (1.1), where

$$R_1^2 = \begin{cases} \frac{b^2}{4(c-1)}, & c \geq 2, \\ \frac{b^2}{c^2}, & c < 2. \end{cases}$$

Proof Define the following generalized positively definite and radially unbounded Lyapunov function $V_1(y,z) = y^2 + z^2$. Computing the derivative of $V_1(y,z)$ along the trajectory of systems

(1.1), we have

$$\dot{V}_1 = 2y\dot{y} + 2z\dot{z} = 2y\left(xz - y\right) + 2z\left(b - xy - cz\right) = -2y^2 - 2cz^2 + 2bz.$$

Let $\dot{V}_1 = 0$. We can get the following two-dimensional surface Γ,

$$\Gamma = \Big\{ (y,z)| \, \frac{y^2}{\frac{b^2}{4c}} + \frac{\left(z - \frac{b}{2c}\right)^2}{\frac{b^2}{4c^2}} = 1, b > 0, c > 0 \Big\}.$$

Outside $\Gamma, \dot{V}_1 < 0$, while inside $\Gamma, \dot{V}_1 > 0$. Since the function $V_1(y,z) = y^2 + z^2$ is continuous on the closed set Γ, $V_1(y,z) = y^2 + z^2$ can reach its maximum on the surface Γ. In the following, we will compute the maximum for $V_1(y,z) = y^2 + z^2$ on the surface Γ. By Lemma 1, we can easily get

$$V_1\left(X\right) \le \max_{X\in\Gamma} V_1\left(X\right) = R_1^2 = \begin{cases} \frac{b^2}{4(c-1)}, & c \ge 2, \\ \frac{b^2}{c^2}, & c < 2. \end{cases} \tag{2.1}$$

From the formula (2.1), we obtain

$$|y| \le R_1, \quad |z| \le R_1. \tag{2.2}$$

At the same time, the fourth equation of formula (1.1) and (2.2) yield

$$\dot{w} = -dw - yz \le -dw + |y|\,|z| \le -dw + R_1^2. \tag{2.3}$$

Integrating both sides of formula (2.3), we have

$$w\left(t\right) \le w\left(t_0\right) e^{-d(t-t_0)} + \int_{t_0}^{t} e^{-d(t-\tau)} R_1^2 \mathrm{d}\tau = w\left(t_0\right) e^{-d(t-t_0)} + \frac{R_1^2}{d}(1 - e^{-d(t-t_0)}),$$

$$w\left(t\right) \le \frac{R_1^2}{d} + (w\left(t_0\right) - \frac{R_1^2}{d})e^{-d(t-t_0)}. \tag{2.4}$$

So we get

$$\lim_{t\to+\infty} w\left(t\right) \le \frac{R_1^2}{d}. \tag{2.5}$$

That is to say, the inequality $\omega^2 \le \frac{R_1^4}{d^2}$ holds as $t \longrightarrow +\infty$. Similarly, according to the first equation of formula (1.1), (2.2) and (2.5), we obtain

$$\dot{x} = -ax + ay + w \le -ax + aR_1 + \frac{R_1^2}{d} = -ax + \frac{adR_1 + R_1^2}{d}. \tag{2.6}$$

Integrating both sides of formula (2.6) gives

$$\begin{aligned} x\left(t\right) &\le x\left(t_0\right) e^{-a(t-t_0)} + \int_{t_0}^{t} e^{-a(t-\tau)} \frac{adR_1 + R_1^2}{d} \mathrm{d}\tau \\ &= x\left(t_0\right) e^{-a(t-t_0)} + \frac{adR_1 + R_1^2}{ad}(1 - e^{-a(t-t_0)}), \end{aligned}$$

$$x\left(t\right) \le \frac{adR_1 + R_1^2}{ad} + (x\left(t_0\right) - \frac{adR_1 + R_1^2}{ad})e^{-a(t-t_0)}.$$

So we get

$$\lim_{t\to+\infty} x\left(t\right) \le \frac{adR_1 + R_1^2}{ad}. \tag{2.7}$$

According to the formulae (2.2), (2.5) and (2.7), we have the conclusion that

$$\Omega = \left\{ (x,y,z,w) |\, y^2 + z^2 \le R_1^2, w^2 \le \frac{R_1^4}{d^2}, x^2 \le \frac{R_1^2 \left(ad + R_1\right)^2}{a^2 d^2} \right\}$$

is the bound for the hyper-chaotic systems (1.1). This completes the proof. □

3. Synchronization of the hyper-chaotic system

In this section, the boundedness of (1.1) will be applied to study chaos synchronization. Let the system (1.1) be the driver system, and the response system be

$$\begin{cases} \dot{x}_1 = -ax_1 + ay_1 + w_1 - k_1 \left(w_1 - w\right), \\ \dot{y}_1 = -y_1 + x_1 z_1 - k_2 \left(y_1 - y\right), \\ \dot{z}_1 = b - x_1 y_1 - cz_1, \\ \dot{w}_1 = -dw_1 - y_1 z_1, \end{cases} \tag{3.1}$$

where $k_1 > 0$ and $k_2 > 0$ are control coefficients. System (1.1) can synchronize the system (3.1) by adjusting parameters $k_1 > 0$ and $k_2 > 0$. We have the following theorem for (1.1) and (3.1).

Theorem 2 *Systems (1.1) and (3.1) are globally complete synchronization when* $k_2 > \frac{(a\rho+M_3)^2}{2a\rho} - 1$, $k_1 = 1$ ($\rho = \frac{M_2^2}{2ac} > 0$).

Proof Let the state errors be $e_1 = x_1 - x$, $e_2 = y_1 - y$, $e_3 = z_1 - z$, $e_4 = \omega_1 - \omega$. Then the error dynamics of systems (1.1) and (3.1) is described as

$$\begin{cases} \dot{e}_1 = -ae_1 + ae_2, \\ \dot{e}_2 = ze_1 + xe_3 + e_1 e_3 - \left(1 + k_2\right) e_2, \\ \dot{e}_3 = -ye_1 - xe_2 - e_1 e_2 - ce_3, \\ \dot{e}_4 = -de_4 - ye_3 - ze_2 - e_2 e_3. \end{cases} \tag{3.2}$$

Let us take the Lyapunov function

$$V(e_1, e_2, e_3) = \rho e_1^2 + e_2^2 + e_3^2, \quad \rho = \frac{M_2^2}{2ac} > 0.$$

Then, its derivative with respect to time t along the trajectory of the system (3.2) is

$$\begin{aligned} \frac{1}{2}\dot{V} =& \rho e_1 \dot{e}_1 + e_2 \dot{e}_2 + e_3 \dot{e}_3 \\ =& \rho e_1 \left(-ae_1 + ae_2\right) + e_2 \left(ze_1 + xe_3 + e_1 e_3 - \left(1 + k_2\right) e_2\right) + e_3 \left(-ye_1 - xe_2 - e_1 e_2 - ce_3\right) \\ =& - a\rho e_1^2 - \left(k_2 + 1\right) e_2^2 - ce_3^2 + \left(a\rho + z\right) e_1 e_2 - ye_1 e_3 \\ \le& - a\rho e_1^2 - \left(k_2 + 1\right) e_2^2 - ce_3^2 + \left(a\rho + M_3\right) |e_1| \, |e_2| + M_2 \, |e_1| \, |e_3| \\ =& - E^T P E, \end{aligned}$$

where,

$$E = \left[|e_1|, |e_2|, |e_3|\right]^{\mathrm{T}}, \quad P = \begin{bmatrix} a\rho & -\frac{a\rho+M_3}{2} & -\frac{M_2}{2} \\ -\frac{a\rho+M_3}{2} & 1+k_2 & 0 \\ -\frac{M_2}{2} & 0 & c \end{bmatrix}.$$

By some elementary calculation, we know that the matrix P is positively definite if $k_2 > \frac{(a\rho+M_3)^2}{2a\rho} - 1$. According to Lyapunov function theory, we get

$$\lim_{t\to+\infty} |e_1| = 0, \quad \lim_{t\to+\infty} |e_2| = 0, \quad \lim_{t\to+\infty} |e_3| = 0. \tag{3.3}$$

Next, we will prove $\lim_{t\to+\infty} |e_4| = 0$. From formula (3.3), we can get $\lim_{t\to+\infty} e_1 = 0$, $\lim_{t\to+\infty} e_2 = 0$, $\lim_{t\to+\infty} e_3 = 0$. Hence, for any $\varepsilon > 0$, there is a sufficiently large $T > t_0$ such that when $t \geq T$, we have $\left|\frac{ye_3+ze_2+e_2e_3}{d}\right| < \varepsilon$. At the same time, we can assume that $|y| \leq M_2 = R_1$, $|z| \leq M_3 = R_1$ according to Theorem 1. For any $\varepsilon > 0$, when $t \geq T$, from the fourth equation of formula (3.2) with the variational technique to estimate $e_4(t)$, we can get

$$\begin{aligned} e_4(t) &= e_4(t_0)\, e^{-d(t-t_0)} + e^{-dt}\int_{t_0}^{t} [-y(\tau) - z(\tau)\, e_2(\tau) - e_2(\tau)\, e_3(\tau)] e^{d\tau} \mathrm{d}\tau \\ &\leq e_4(t_0)\, e^{-d(t-t_0)} + e^{-dt}\int_{t_0}^{t} d\varepsilon e^{d\tau} \mathrm{d}\tau, \\ &= (e_4(t_0) - \varepsilon)\, e^{-d(t-t_0)} + \varepsilon. \end{aligned}$$

Thus, if the initial value $e_4(t_0) > \varepsilon$ and $t \longrightarrow +\infty$, we can get

$$e_4(t) - \varepsilon \leq (e_4(t_0) - \varepsilon) e^{-d(t-t_0)} \to 0.$$

Similarly,

$$\begin{aligned} e_4(t) &= e_4(t_0)\, e^{-d(t-t_0)} + e^{-dt}\int_{t_0}^{t} [-y(\tau) - z(\tau)\, e_2(\tau) - e_2(\tau)\, e_3(\tau)] e^{d\tau} \mathrm{d}\tau \\ &\geq e_4(t_0)\, e^{-d(t-t_0)} - e^{-dt}\int_{t_0}^{t} d\varepsilon e^{d\tau} \mathrm{d}\tau, \\ &= (e_4(t_0) + \varepsilon)\, e^{-d(t-t_0)} - \varepsilon. \end{aligned}$$

Thus, if the initial value $e_4(t_0) < -\varepsilon$ and $t \longrightarrow +\infty$, we have

$$e_4(t) + \varepsilon \geq (e_4(t_0) + \varepsilon)\, e^{-d(t-t_0)} \to 0.$$

Therefore, when the initial value $|e_4(t_0)| > \varepsilon$ and $t \longrightarrow +\infty$, we have the distance $\rho((e_4(t), I) \to 0$, where $I = [-\varepsilon, \varepsilon]$. Hence, for any sufficiently small $\varepsilon > 0$, there is a sufficiently large $T > t_0$ such that when $t > T$, we have $|e_4(t)| < \varepsilon$. By the definition of limit, we get

$$\lim_{t\to+\infty} e_4(t) = 0. \tag{3.4}$$

To summarize (3.3)-(3.4), we can get

$$\lim_{t\to+\infty} |e_1| = 0, \ \lim_{t\to+\infty} |e_2| = 0, \ \lim_{t\to+\infty} |e_3| = 0, \ \lim_{t\to+\infty} |e_4| = 0. \tag{3.5}$$

This implies that the origin of the error system (3.2) is asymptotically stable, which is equivalent to saying that the system (1.1) can synchronize system (3.1) completely. This completes the proof. □

4. Simulation studies

The numerical simulations are carried out with the MATLAB 7.4. The initial conditions of the system (1.1) and the system (3.1) are chosen as $(x(0),(y(0),(z(0),(\omega(0)) = (0.1; 0.1; 0.1; 0.1)$, $(x_1(0),(y_1(0),(z_1(0),(\omega_1(0)) = (1; 2; 3; 4)$, respectively. When parameters $a = 5, b = 16, c = 1, d = 0.5$ (see [16]), it is easy to obtain $R_1 = \frac{b}{c} = 16$, $|y| < M_2 = R_1 = 16$, $|z| < M_3 = R_1 = 16$, $\rho = \frac{M_2^2}{2ac} = 25.6$, $k_2 > \frac{(a\rho+M_3)^2}{2a\rho} - 1 = 39.5$ according to Theorems 1 and 2. Choose $k_2 = 40$, the response system (1.1) synchronizes with the drive system (3.1) as shown in Figure 2. The trajectories of $y(t)$ and $z(t)$ of the system (1.1) are contained in the circle defined by $\Omega_1 = \{(y, z)|y^2 + z^2 = 16^2\}$ according to Theorem 1 as shown in Figure 3.

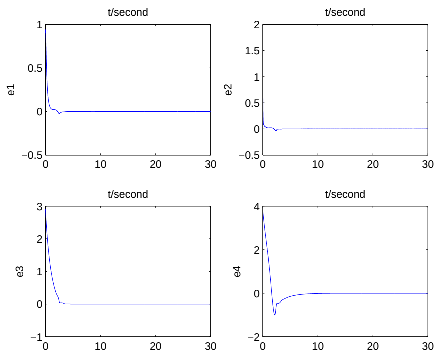

Figure 2 Effectiveness of the chaos synchronization between the system (1.1) and the system (3.1)

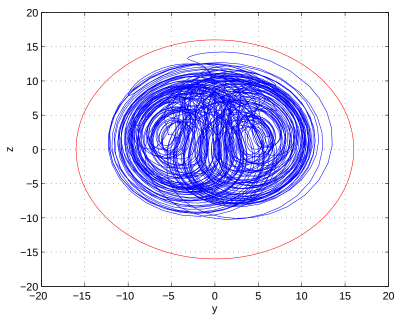

Figure 3 The trajectories of $y(t)$ and $z(t)$ of the system (1.1) are contained in the circle defined by $\Omega_1 = \{(y, z)|y^2 + z^2 = 16^2\}$

5. Conclusions

In this paper, we have studied the boundedness of a kind of hyper-chaotic systems (1.1) which have potential applications in secure communications. We have obtained the boundedness of systems (1.1). Finally, the boundedness with respect to y, z of the system (1.1) is applied to the chaos synchronization. Numerical simulations are presented to show the effectiveness of the proposed scheme. But for the boundedness of the Chen system in [13], the parameter values considered do not cover the most interesting case with the chaotic attractor of the Chen system.

The future research on the boundedness of the Chen system in [13] and the Lü system in [14] is still challenging and helpful.

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