Journal of Mathematical Research with Applications
May, 2013, Vol. 33, No. 3, pp. 353–360
DOI:10.3770/j.issn:2095-2651.2013.03.010
Http://jmre.dlut.edu.cn

A Note on Monotonically Metacompact Spaces

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Abstract In the first part of this note, we mainly prove that monotone metacompactness is hereditary with respect to closed subspaces and open F_σ-subspaces. For a generalized ordered (GO)-space X, we also show that X is monotonically metacompact if and only if its closed linearly ordered extension X^* is monotonically metacompact. We also point out that every non-Archimedean space X is monotonically ultraparacompact. In the second part of this note, we give an alternate proof of the result that McAuley space is paracompact and metacompact.

Keywords GO-space; paracompact; monotonically metacompact; monotonically ultraparacompact.

MR(2010) Subject Classification 54E30; 54E35; 54F05

1. Introduction

Recently, the monotone covering properties are investigated. The monotone Lindelöf property was discussed firstly. In [2], it was proved that any separable generalized ordered (GO)-space is hereditarily monotonically Lindelöf. In 2009, Popvassilev introduced the concept of monotonically countable metacompact. In 2010, H. R. Bennett introduced the concept of monotonically metacompact spaces and posed a question that whether McAuley space is monotonically metacompact. In the first part of this note, we mainly prove that monotonically metacompact spaces are hereditary with respect to closed subspaces and open F_σ-subspaces. For a GO-space X, we also show that X is monotonically metacompact if and only if its closed linearly ordered extension X^* is monotonically metacompact. Futhermore, we give an example of a monotonically metacompact space. By analysing the properties of this example, we introduce the concept of monotonically ultraparacompact spaces, and prove that monotonically ultraparacompact spaces are hereditary with respect to closed subspaces and any non-Archimedean space X is monotonically ultraparacompact.

In [1], M. Amono and T. Mizokami showed that McAuley space is an M_1-space. Thus we can get that McAuley space is paracompact and metacompact. In the second part of this note, we give an alternate proof of the result that McAuley space is paracompact and metacompact

Received November 25, 2011; Accepted March 27, 2012
Supported by the National Natural Science Foundation of China (Grant No. 11271036), Beijing Natural Science Foundation (Grant No. 1102002) and Doctoral Fund of Innovation of Beijing University of Technology.
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and get a conclusion on paracompact spaces. We hope it can be useful to solve the question pointed above.

For two collections $\mathcal{U}$ and $\mathcal{V}$ of subsets of a space X, we say that $\mathcal{U}$ is a refinement of $\mathcal{V}$, if for each $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subset V$ and $\bigcup\mathcal{U} = \bigcup\mathcal{V}$. If $\bigcup\mathcal{U} = \bigcup\mathcal{V}$ is not required in above definition, we say that $\mathcal{V}$ is a weak refinement of $\mathcal{U}$. We use $\mathcal{U} \prec \mathcal{V}$ to represent that $\mathcal{U}$ is a refinement of $\mathcal{V}$ or $\mathcal{U}$ is a weak refinement of $\mathcal{V}$. If $\mathcal{U}$ is a collection of subsets of a space X and $x \in X$, then we define $\mathrm{ord}(x,\mathcal{U}) = |\{U : x \in U, U \in \mathcal{U}\}|$. Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Z}$ denote the set of all positive integers, the set of all real numbers and the set of all integers, respectively. We will follow [5] for other notations and terminology.

2. Monotonically metacompact spaces

Definition 2.1 ([3]) *A space $(X,\mathcal{T})$ is monotonically metacompact if each open cover $\mathcal{U}$ of the space X has a point-finite open refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ and $\mathcal{V}$ are open covers of the space X and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$. In this case, the operator r is called a monotone metacompactness operator for the space X.*

Lemma 2.2 *Monotonically metacompact spaces are hereditary with respect to closed subspaces.*

Proof Let X be a monotonically metacompact space and let r be a monotone metacompactness operator for X. Suppose that $Y \subset X$ is closed. Let $\mathcal{U}$ be any open cover of Y. For any $U \in \mathcal{U}$, there exists an open set V_U of X such that $V_U \cap Y = U$. The set Y is a closed subset of X, so $V_U \cup (X \setminus Y)$ is open in X and $(V_U \cup (X \setminus Y)) \cap Y = U$. Let $O_U = V_U \cup (X \setminus Y)$ and $\mathcal{U}^* = \{O_U : U \in \mathcal{U}\}$. Then $X = \bigcup\{O_U : U \in \mathcal{U}\}$. So $X = \bigcup r(\mathcal{U}^*)$ and $r(\mathcal{U}^*)$ is a point-finite open refinement of $\mathcal{U}^*$. Let $r_Y(\mathcal{U}) = \{V \cap Y : V \in r(\mathcal{U}^*)\}$. Then $r_Y(\mathcal{U})$ is a point-finite open refinement of $\mathcal{U}$ (in Y). Obviously, if $\mathcal{U}$ and $\mathcal{V}$ are open covers of Y and $\mathcal{U} \prec \mathcal{V}$, then $r_Y(\mathcal{U}) \prec r_Y(\mathcal{V})$. Thus Y is monotonically metacompact. $\Box$

Lemma 2.3 *Let F be any F_σ-subspace of a monotonically metacompact space X. If there exist open families $\mathcal{U}$ and $\mathcal{V}$ of X such that $F \subset \bigcup\mathcal{U} = \bigcup\mathcal{V}$, $\mathcal{U} \prec \mathcal{V}$, and for any $U \in \mathcal{U}$, $V \in \mathcal{V}$, $U \cap F \neq \emptyset$, $V \cap F \neq \emptyset$, then there exists an operator r of F such that $r(\mathcal{U}) \prec \mathcal{U}$, $r(\mathcal{V}) \prec \mathcal{V}$, $r(\mathcal{U})$ and $r(\mathcal{V})$ are point-finite open covers of F and $r(\mathcal{U}) \prec r(\mathcal{V})$.*

Proof Let r' be a monotone metacompactness operator of X and $F = \bigcup\{F_n : n \in \mathbb{N}\}$, where F_n is closed for each $n \in \mathbb{N}$. If $\mathcal{U}$ is a family of open sets of X and $F \subset \bigcup\mathcal{U}$, then let $\mathcal{U}_n = \{U : U \in \mathcal{U}, U \cap F_n \neq \emptyset\}\bigcup\{X\backslash F_n\}$ for each $n \in \mathbb{N}$, thus $\mathcal{U}_n$ covers X. So $\mathcal{U}_n$ has a point-finite open refinement $r'(\mathcal{U}_n)$. If $\mathcal{V}$ is a family of open sets of X, $F \subset \bigcup\mathcal{U} = \bigcup\mathcal{V}$ and $\mathcal{U} \prec \mathcal{V}$, then $\mathcal{U}_n \prec \mathcal{V}_n$, where $\mathcal{V}_n = \{V : V \in \mathcal{V}, V \cap F_n \neq \emptyset\}\bigcup\{X \setminus F_n\}$. Thus $r'(\mathcal{U}_n) \prec r'(\mathcal{V}_n)$. Let $r''(\mathcal{U}_1) = \{V \cap F : V \in r'(\mathcal{U}_1), V \cap F_1 \neq \emptyset\}$ and $l(\mathcal{U}_1) = r''(\mathcal{U}_1)$. If $n > 1$, let $r''(\mathcal{U}_n) = \{V \cap F : V \in r'(\mathcal{U}_n), V \cap F_n \neq \emptyset\}$ and $l(\mathcal{U}_n) = \{W\backslash\bigcup_{i<n} F_i : W \in r''(\mathcal{U}_n)\}$.

Denote $r(\mathcal{U}) = \bigcup\{l(\mathcal{U}_n) : n \in \mathbb{N}\}$. We prove that $r(\mathcal{U})$ is a point-finite open cover of F and $r(\mathcal{U})$ is a weak refinement of $\mathcal{U}$.

(1) For any $A \in r(\mathcal{U})$, there exists some $n \in \mathbb{N}$ such that $A \in l(\mathcal{U}_n)$. Thus there exists some $V \in r'(\mathcal{U}_n)$ such that $A = (V \cap F)\backslash\bigcup_{i<n} F_i$. Since $r'(\mathcal{U}_n) \prec \mathcal{U}_n$, there exists some $U \in \mathcal{U}_n$ such that $V \subset U$. Then $A \subset V \subset U$. So $r(\mathcal{U})$ is a weak refinement of $\mathcal{U}$.

(2) For any $x \in F$, there exists a minimal number $m \in \mathbb{N}$ such that $x \in F_m$. If $n > m$, then $x \notin V\backslash \bigcup_{i<n} F_i$ for any $V \in r'(\mathcal{U}_n)$. Thus $x \notin \bigcup l(\mathcal{U}_n)$, if $n > m$. For any $n \leq m$, $l(\mathcal{U}_n)$ is point-finite. Thus $|\{O : x \in O, O \in r(\mathcal{U})\}| < \omega$, hence $r(\mathcal{U})$ is point-finite.

(3) Since $F \subset \bigcup\mathcal{U} = \bigcup\mathcal{V}$, and $\mathcal{U} \prec \mathcal{V}$, we have $r'(\mathcal{U}_n) \prec r'(\mathcal{V}_n)$ for each $n \in \mathbb{N}$. Thus $r''(\mathcal{U}_n) \prec r''(\mathcal{V}_n)$. So $l(\mathcal{U}_n) \prec l(\mathcal{V}_n)$ and $\bigcup r(\mathcal{U}) = \bigcup r(\mathcal{V}) = F$, hence $r(\mathcal{U}) \prec r(\mathcal{V})$. □

By Lemma 2.3, it is easy to get the following theorem.

Theorem 2.4 *Monotone metacompactness is hereditary with respect to open F_σ-subspaces.*

For a GO-space X and $Y \subset X$, if for any $a, b \in Y$ and $a < b$, $(a, b) = \{x \in X : a < x < b\} \subset Y$, then Y is called a convex subset of X. For any non-empty open subset G of a GO-space X, if we let $\mathcal{U}_G = \{P : P \subset G, P \text{ is a convex open subset of } X\}$, then $G = \bigcup\mathcal{U}_G$. For each $P \in \mathcal{U}_G$, let $st(P,\mathcal{U}_G) = \bigcup\{A : A \cap P \neq \emptyset, A \in \mathcal{U}_G\}$, $st^{i+1}(P,\mathcal{U}_G) = st(st^i(P,\mathcal{U}_G),\mathcal{U}_G), i \in \mathbb{N}$. For each $P \in \mathcal{U}_G$, let $S_P = \bigcup\{st^n(P,\mathcal{U}_G) : n \in \mathbb{N}\}$. Obviously, S_P is a convex open set for each $P \in \mathcal{U}_G$, and for any P_1, $P_2 \in \mathcal{U}_G$, $S_{P_1} = S_{P_2}$ or $S_{P_1} \cap S_{P_2} = \emptyset$. The set $G = \bigcup\mathcal{U}$, where $\mathcal{U} = \{S_P : P \subset G, P \text{ is a convex set of } X\}$. The set S_P is called a maximal convex open set of G. Thus any non-empty open subset G of a GO-space X can be uniquely represented as the union of some maximal convex open sets. For a GO-space X, a maximal convex set of an open set is also called a maximal convex component of this open set.

If a GO-space X can be topologically embedded in a linearly ordered topological space (LOTS) X^*, then the LOTS X^* is called an ordered extension of X. If the embedding is order-preserving, then the LOTS X^* is called a linearly ordered extension of X. Let X be a GO-space with the topology τ and let λ be the usual open interval topology on X. Put $R = \{x \in X : [x, \rightarrow) \in \tau\backslash\lambda\}$, $L = \{x \in X : (\leftarrow, x] \in \tau\backslash\lambda\}$. Define $X^* \subset X \times \mathbb{Z}$ as follows: $X^* = (X \times \{0\}) \cup (R \times \{k : k \in \mathbb{Z}, k < 0\}) \cup (L \times \{k : k \in \mathbb{Z}, k > 0\})$. Let X^* have the open interval topology generated by the lexicographical order. Then $e(x) = \langle x, 0\rangle$ defined by $e : X \to X^*$ is an order-preserving homeomorphism from X onto the closed subspace $X \times \{0\}$ of X^*. So the space X^* is called a closed linearly ordered extension of X (see [8]).

For a convex set S of a GO-space X, put $I(S) = \{x \in S : \exists a, b \in S \text{ with } a < x < b\}$, and define the subset $S^\sim$ of X^* as follows: $S^\sim = \{\langle x, k\rangle \in X^* : x \in I(S)\} \cup \{\langle x, 0\rangle : x \in S\backslash I(S)\}$ (see [8]).

For a GO-space X, it was proved that X is monotonically meta-Lindelöf if and only if its closed linearly ordered extension X^* is monotonically meta-Lindelöf in [7]. By a proof which is similar to that of Proposition 5 in [7], we can get Theorem 2.7. Firstly, we introduce some lemmas.

Lemma 2.5 ([8]) *Let X be a GO space.*

(a) If $S \subset T$ is convex in X, then $S^\sim \subset T^\sim$.

(b) If S is convex in X, then $S^\sim$ is open in X^* iff S is open in X.

(c) If J is convex in X^* and if $S \subset J$, where S is convex in X, then $S^\sim \subset J$.

Lemma 2.6 *For a GO-space X, the following are equivalent:*

(1) X is monotonically metacompact;

(2) Each open cover $\mathcal{U}$ of X by convex sets has a point-finite open refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ and $\mathcal{V}$ are open covers of X by convex sets and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$;

(3) Each open cover $\mathcal{U}$ of X by convex sets has a point-finite open refinement $r(\mathcal{U})$, where each member of $r(\mathcal{U})$ is a convex set such that if $\mathcal{U}$ and $\mathcal{V}$ are open covers of X by convex sets and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$.

Proof (1)$\Rightarrow$(2). Obviously.

(2)$\Rightarrow$(3). For any open sets U and V of X, if $U \subset V$, then a maximal convex component of U is a subset of some maximal convex component of V. Let r' be an operator which satisfies the condition of (2). For any open cover $\mathcal{U}$ of X by convex sets, we only need to let every element of $r(\mathcal{U})$ be a maximal convex component of some element of $r'(\mathcal{U})$. Thus we complete the proof.

(3)$\Rightarrow$(1). Any non-empty open subset G of the GO-space X can be uniquely represented as the union of some maximal convex open sets, that is $G = \bigcup\{S_i : i \in I\}$, where S_i is the maximal convex component of G. The set G is an open set, so each S_i is open. If $G \subset G^{'}$, $G^{'} = \bigcup\{S_i' : i \in I'\}$, where $\{S_i' : i \in I'\}$ is a set consisting of all the maximal convex component of G', then $\{S_i : i \in I\} \prec \{S_i' : i \in I'\}$. For any open cover $\mathcal{U}$ of X, let $\mathcal{U}^*$ be a family consisting of all the maximal convex component of some members of $\mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are open covers of X and $\mathcal{U} \prec \mathcal{V}$, then $\mathcal{U}^* \prec \mathcal{V}^*$ by the former analysis. If r is an operator which satisfies the condition of (3), then $r(\mathcal{U}^*) \prec \mathcal{U}$, $r(\mathcal{V}^*) \prec \mathcal{V}$, and $r(\mathcal{U}^*) \prec r(\mathcal{V}^*)$. If we denote $r'(\mathcal{U}) = r(\mathcal{U}^*)$, then r' is a monotone metacompactness operator for the space X. □

Theorem 2.7 *For a GO-space X, the following are equivalent:*

(1) X is monotonically metacompact;

(2) The closed linearly ordered extension X^* of X is monotonically metacompact.

Proof (2)$\Rightarrow$(1). Suppose X^* is monotonically metacompact. By Lemma 2.2, the closed subspace $X \times \{0\}$ of X^* is monotonically metacompact. $X \times \{0\}$ is homeomorphic to X, so X is monotonically metacompact.

(1)$\Rightarrow$(2). We will identify X with the subspace $X \times \{0\}$ of X^*. If $\mathcal{U}$ is an open cover of X^* by convex sets, then $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$ is an open cover of X by convex sets. Since X is monotonically metacompact, $\mathcal{U}_X$ has a point-finite open refinement $r_X(\mathcal{U}_X)$ by Lemma 2.6, where r_X is a monotone metacompactness operator for X and $r_X(\mathcal{U}_X)$ consists of convex sets of X. For a convex set S of X, put

$$I(S) = \{x \in S : \exists a, b \in S \text{ with } a < x < b\},$$

$$S^\sim = \{\langle x, k\rangle \in X^* : x \in I(S)\} \cup \{\langle x, 0\rangle : x \in S \backslash I(S)\},$$

$$\varphi^\sim = \{S^\sim : S \in r_X(\mathcal{U}_X)\}.$$

For any $S^{\sim} \in \varphi^{\sim}$ with $S \in r_X(\mathcal{U}_X)$, there exists a $U \in \mathcal{U}$ such that $S \subset U$. Since S is an open convex set and $U \subset X^*$ is convex, $S^{\sim}$ is open and $S^{\sim} \subset U$ by Lemma 2.5.

Let $r(\mathcal{U}) = \varphi^{\sim} \cup \{\{\langle x, k\rangle : \langle x, k\rangle \in X^* \backslash X\}\}$. For any $x \in X$, if $S \in r_X(\mathcal{U}_X)$ and $x \notin S$, then $\langle x, 0\rangle \notin S^{\sim}$. Since $r_X(\mathcal{U}_X)$ is a point-finite open cover of X^*, $r(\mathcal{U})$ is point-finite at $\langle x, 0\rangle$. For any $\langle x, k\rangle \in X^* \setminus X$, $x \in X$, if $x \notin S$ and $S \in r_X(\mathcal{U}_X)$, then $\langle x, k\rangle \notin S^{\sim}$, so $r(\mathcal{U})$ is point-finite at $\langle x, k\rangle$. Each $\{\langle x, k\rangle\}$ with $(k \neq 0)$ is an open set, then $r(\mathcal{U})$ is a point-finite open cover of X^* and $r(\mathcal{U}) \prec \mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are open covers of X^* by convex sets and $\mathcal{U} \prec \mathcal{V}$, then $r_X(\mathcal{U}) \prec r_X(\mathcal{V})$. For any $S \in r_X(\mathcal{U}_X)$, there exists $T \in r_X(\mathcal{V}_X)$ such that $S \subset T$. By Lemma 2.5 we have $S^{\sim} \subseteq T^{\sim}$, so $r(\mathcal{U}) \prec r(\mathcal{V})$. Hence X^* is monotonically metacompact by Lemma 2.6. □

Example 2.8 *Let $X = \omega_1 + 1$. We define a topology on the linearly ordered set X with a base as follows: if $x \in X \backslash \{\omega_1\}$, then $\mathcal{B}(x) = \{\{x\}\}$; $\mathcal{B}(\omega_1) = \{(\alpha, \omega_1] : \alpha < \omega_1\}$. Then the space X is a monotonically metacompact space.*

Proof For any open cover $\mathcal{U}$ of X, let $\alpha(\mathcal{U})$ =min$\{\alpha' : \alpha' \in [0, \omega_1), (\alpha', \omega_1] \subset U$ for some $U \in \mathcal{U}\}$ and $r(\mathcal{U}) = \{(\alpha(\mathcal{U}), \omega_1]\} \cup \{\{\beta\} : \beta \leq \alpha(\mathcal{U})\}$.

In what follows, we prove that r is a monotone metacompactness operator for the space X:

(1) For each $x \in X$, if $x \leq \alpha(\mathcal{U})$, then $|\{U : x \in U \in r(\mathcal{U})\}| = 1 < \omega$; if $\alpha(\mathcal{U}) < x \leq \omega_1$, then only the element $(\alpha(\mathcal{U}), \omega_1]$ of $r(\mathcal{U})$ contains the point x. So $r(\mathcal{U})$ is a point-finite open cover of X.

(2) For any $V \in r(\mathcal{U})$: if $V = (\alpha(\mathcal{U}), \omega_1]$, where $\alpha(\mathcal{U})$ =min$\{\alpha' : \alpha' \in [0, \omega_1), (\alpha', \omega_1] \subset U$ for some $U \in \mathcal{U}\}$, then there exists some $U \in \mathcal{U}$ such that $V \subset U$; if there is some $\beta \in \omega_1$ such that $V = \{\beta\}$, then there is some $U \in \mathcal{U}$ such that $V \subset U$. Thus $r(\mathcal{U}) \prec \mathcal{U}$.

(3) If $\mathcal{U}_1 \prec \mathcal{U}_2$, for open covers $\mathcal{U}_1$ and $\mathcal{U}_2$ of X, we have $\alpha(\mathcal{U}_1) \geq \alpha(\mathcal{U}_2)$ and $r(\mathcal{U}_1) = \{(\alpha(\mathcal{U}_1), \omega_1]\} \cup \{\{\beta\} : \beta \leq \alpha(\mathcal{U}_1)\}$, $r(\mathcal{U}_2) = \{(\alpha(\mathcal{U}_2), \omega_1]\} \cup \{\{\beta\} : \beta \leq \alpha(\mathcal{U}_2)\}$, it is easy to see that $r(\mathcal{U}_1) \prec r(\mathcal{U}_2)$. So X is a monotonically metacompact space. □

If X is the space which appears in Example 2.8, then we can see that $r(\mathcal{U})$ is a pairwise disjoint open refinement of $\mathcal{U}$. Thus we have the following definition.

Definition 2.9 *A space $(X, \mathcal{T})$ is monotonically ultra-paracompact if each open cover $\mathcal{U}$ of X has a pairwise disjoint open refinement $r(\mathcal{U})$ such that if $\mathcal{U}$ and $\mathcal{V}$ are open covers of the space X and $\mathcal{U} \prec \mathcal{V}$, then $r(\mathcal{U}) \prec r(\mathcal{V})$. In this case, the operator r is called a monotone ultra-paracompact operator for the space X.*

Remark In Example 2.8, any two elements of $r(\mathcal{U}) = \{(\alpha(\mathcal{U}), \omega_1]\} \cup \{\{\beta\} : \beta \leq \alpha(\mathcal{U})\}$ are pairwise disjoint. By Definition 2.9, the space X in Example 2.8 is also monotonically ultra-paracompact.

By the definition of monotonically ultra-paracompact, we can get that monotonically ultra-paracompact spaces are monotonically metacompact and monotonically meta-Lindelöf. By a proof which is similar to that of Lemma 2.2, we can get the following theorem.

Theorem 2.10 *Monotonically ultra-paracompact spaces are hereditary with respect to closed* *subspaces.*

Recall that a topological space X is termed non-Archimedean if it has a base $\mathcal{B}$ where any two members are either disjoint, or comparable (viz, one is contained inside the other). The base $\mathcal{B}$ is also called a rank-1 base of X (see [10, 11]).

Lemma 2.11 ([11]) *Every non-Archimedean space has a base which is a tree by reverse inclusion.*

Theorem 2.12 *Every non-Archimedean space is monotonically ultraparacompact.*

Proof Let X be a non-Archimedean space and let $\mathcal{U}$ be any open cover of the space X. By Lemma 2.11, there exists a base $\mathcal{B}$ which is a tree by reverse inclusion. Let $r(\mathcal{U})$ be the family of all members of the tree base $\mathcal{B}$ which are $\subset$-maximal with respect to being contained in members of $\mathcal{U}$.

(1) For any $x \in X$, there exist $U \in \mathcal{U}$ and $B \in \mathcal{B}$ such that $x \in B \subset U$. Then there exists $B_x \in r(\mathcal{U})$ such that $x \in B \subset B_x$. By the maximality of the members of $r(\mathcal{U})$ and any two members of $\mathcal{B}$ are either disjoint or comparable, any two members of $r(\mathcal{U})$ are obviously disjoint. Then it is easy to get that $r(\mathcal{U})$ is a pairwise disjoint open refinement of the cover $\mathcal{U}$.

(2) If $\mathcal{U}$ and $\mathcal{V}$ are open covers of the space X and $\mathcal{V} \prec \mathcal{U}$, then it is enough to show that $r(\mathcal{V}) \prec r(\mathcal{U})$. For any $W \in r(\mathcal{V})$, there is $V_W \in \mathcal{V}$ such that $W \subset V_W$. If $\mathcal{V} \prec \mathcal{U}$, then there exists $U_W \in \mathcal{U}$ such that $V_W \subset U_W$. By the maximality of the members of $r(\mathcal{U})$, there exists $B_W \in r(\mathcal{U})$ such that $W \subset B_W$. Then $r(\mathcal{V}) \prec r(\mathcal{U})$. So X is a monotonically ultraparacompact space.

3. Two conclusions on McAuley spaces

Definition 3.1 ([1]) *Let $X = X_0 \cup X_1$, where*

$$X_0 = \{(x, 0) : x \in \mathbb{R}\}, \quad X_1 = \{(x, y) \in \mathbb{R}^2 : y > 0\}.$$

We topologize X by defining neighborhood bases at each point $p \in X$ in the following two ways: (1) If $p = (x, y) \in X_1$, then p has a usual neighborhood base, and (2) If $p = (x, 0) \in X_0$, then p has a neighborhood base $\{M(p, \frac{1}{n}) : n \in \mathbb{N}\}$, where $M(p, \frac{1}{n}) = \{p\} \bigcup \{(x', y') \in X : y' < \frac{1}{n}|x' - x| < \frac{1}{n^2}\}$, $n \in \mathbb{N}\}$. Then the space X, thus defined, is called McAuley space.

In [1], it was proved that McAuley space is an M_1-space, then we can get that McAuley space is also paracompact and metacompact. In what follows we give an alternate proof of this result. In [3], H. R. Bennett posed a question whether stratifiable spaces are monotonically metacompact. We have known that McAuley space is a stratifiable space, then another question was posed whether McAuley space is monotonically metacompact. This is a very interesting and specific question. We ever considered this question, but we have not got good results. However, we can directly prove the result that McAuley space is paracompact and metacompact. We hope the method that we used can be useful to slove the problem above. So we give a direct proof for the following results.

Theorem 3.2 *McAuley space X is metacompact.*

Proof Let $\mathcal{U}$ be any open cover of X. Thus $X = \bigcup\mathcal{U}$ and $X_0 = \bigcup\{U \cap X_0 : U \in \mathcal{U}\}$. As the subspace of X, X_0 is homeomorphic to $\mathbb{R}$ (in the usual topology), so there exists a point-finite open refinement $\mathcal{V}$ of $\{U \cap X_0 : U \in \mathcal{U}\}$ in X_0. For any $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ such that $V \subset U$. For each $V \in \mathcal{V}$ and $x \in V$, the second component of the point x is 0. X_0 is homeomorphic to $\mathbb{R}$, for convenience, we take every element of $\mathcal{V}$ as a subspace of $\mathbb{R}$. Because any non-empty open subset of the space $\mathbb{R}$ can be uniquely represented as the union of some maximal convex open sets, then for any $V \in \mathcal{V}$, $V = \bigcup\{V_i : i \in \Lambda_V, V_i$ is a maximal convex component of the set $V\}$. Obviously, for any $i, j \in \Lambda_V$, if $i \neq j$, then $V_i \cap V_j = \emptyset$. If $\mathcal{V}' = \{I : I$ is a maximal convex component of some member of $\mathcal{V}\}$, then for any $I \in \mathcal{V}'$, there are $V_I \in \mathcal{V}$, $U_I \in \mathcal{U}$ such that $I \times \{0\} \subset V_I \times \{0\} \subset U_I$. As $I \in \mathcal{V}'$, I is an open set of $\mathbb{R}$, then we can easily get that $I \times [0, +\infty)$ is an open set of X. If $\mathcal{W}_0 = \{(I \times [0, +\infty)) \cap U_I : I \in \mathcal{V}'\}$, then $\mathcal{W}_0$ is a weak open refinement of $\mathcal{U}$. For any $p = (x, y) \in X$, as $\mathcal{V}$ is point-finite in $\mathbb{R}$, then $|\{V : x \in V, V \in \mathcal{V}\}| < \omega$. Therefore $|\{I : x \in I, I \in \mathcal{V}'\}| < \omega$, hence $\mathrm{ord}(p, \mathcal{W}_0) < \omega$. The topology of X_1 is the same as the usual topology of the upper half plane of $\mathbb{R}^2$ (without x-axis), so X_1 is a metric space, $\{U \setminus X_0 : U \in \mathcal{U}\}$ is an open cover of X_1, so there exists a point-finite open refinement $\mathcal{W}_1$ of $\{U \setminus X_0 : U \in \mathcal{U}\}$ in X_1. If $\mathcal{W} = \mathcal{W}_0 \bigcup \mathcal{W}_1$, then $\mathcal{W}$ is a point-finite open refinement of $\mathcal{U}$ and $X = \bigcup\mathcal{W}$, so X is metacompact. □

In what follows, we prove that McAuley space is paracompact.

Theorem 3.3 *McAuley space X is paracompact.*

Proof X_0 and X_1 as the subspaces of X are hereditary separable. So we can easily get that McAuley space is a hereditary separable regular space. By Lemma 3.2, McAuley space X is metacompact. Because every separable metacompact space is Lindelöf, McAuley space X is paracompact. □

In what follows, we get a conclusion on paracompact spaces.

Theorem 3.4 *Let F be a closed subspace of a normal space X and let $X \setminus F$ be hereditary paracompact. If for each open family $\mathcal{U}$ of X with $F \subset \bigcup\mathcal{U}$, there exists a weak refinement $\mathcal{V}$ of $\mathcal{U}$ and $\mathcal{V}$ is a locally finite open family of X such that $F \subset \bigcup\mathcal{V}$, then X is paracompact.*

Proof Let $\mathcal{U}$ be any open cover of X and $F \subset \bigcup\mathcal{U}$. There is a weak refinement $\mathcal{V}$ of $\mathcal{U}$, $F \subset \bigcup\mathcal{V}$, and $\mathcal{V}$ is locally finite open family of X. The set F is a closed subspace of a normal space X, so there is an open set O_1 of X such that $F \subset O_1 \subset \overline{O_1} \subset \bigcup\mathcal{V}$, and there is an open set O_2 of X such that $F \subset O_2 \subset \overline{O_2} \subset O_1$. $X \setminus F$ is hereditary paracompact, so $\bigcup\{U \setminus \overline{O_2} : U \in \mathcal{U}\} = X \setminus \overline{O_2}$ is paracompact. So there exists a locally finite open refinement $\mathcal{V}_1$ of $\{U \setminus \overline{O_2} : U \in \mathcal{U}\}$ in $X \setminus \overline{O_2}$ such that $\bigcup\mathcal{V}_1 = X \setminus \overline{O_2}$. For each $V_1 \in \mathcal{V}_1$, there exists some $U_1 \in \mathcal{U}$ such that $V_1 \subset U_1 \setminus \overline{O_2}$. For any $x \in X \setminus \overline{O_2}$, there exists a neighborhood $U_x \subset X \setminus \overline{O_2}$ of x such that $|\{V : V \cap U_x \neq \emptyset, V \in \mathcal{V}_1\}| < \omega$. If $\mathcal{V}_2 = \{V \setminus \overline{O_1} : V \in \mathcal{V}_1\}$, then $\bigcup\mathcal{V}_2 = X \setminus \overline{O_1}$ and $\overline{O_1} \subset \bigcup\mathcal{V}$. So $(\bigcup\mathcal{V}) \cup (\bigcup\mathcal{V}_2) = X$. $\mathcal{V}_1$ is locally finite in $X \setminus \overline{O_2}$, so $\mathcal{V}_2 = \{V \setminus \overline{O_1} : V \in \mathcal{V}_1\}$ is also locally finite in $X \setminus \overline{O_2}$. If $x \in O_1$, then for a neighborhood U_x of x, $U_x = O_1$, $U_x \cap V' = \emptyset$ for each $V' \in \mathcal{V}_2$. If $x \in X \setminus O_1$, then by $X \setminus O_1 \subset X \setminus \overline{O_2}$, we can get $x \in X \setminus \overline{O_2}$, then there exists a neighborhood U_x of x such that $|\{V' : V' \cap U_x \neq \emptyset, V' \in \mathcal{V}_2\}| \leq |\{V : V \cap U_x \neq \emptyset, V \in \mathcal{V}_1\}| < \omega$. So $\mathcal{V}_2$ is locally finite in X. $\mathcal{V}_1 \prec \{U \setminus \overline{O_2} : U \in \mathcal{U}\}$, so $\mathcal{V}_2 \prec \{U \setminus \overline{O_2} : U \in \mathcal{U}\} \prec \mathcal{U}$, $\mathcal{V} \bigcup \mathcal{V}_2$ is a locally finite open refinement of the open cover $\mathcal{U}$ of X, therefore X is paracompact. $\Box$

Acknowledgement The authors would like to thank the referee for his (or her) valuable remarks and suggestions which greatly improved the paper.

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