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New Generalized L-KKM Type Theorems in Topological Spaces with Applications

Min FANG
Department of Economic Mathematics, South Western University of Finance and Economics, Sichuan 610074*, P. R. China*

Abstract In this paper, some new generalized L-KKM type theorems with finitely open values and with finitely closed values are established without any convexity structure in topological spaces. As applications, some new matching theorem, fixed point theorem and existence theorem of equilibrium problem with lower and upper bounds are also given under some suitable conditions. These theorems presented in this paper unify and generalize some corresponding known results in recent literatures.

Keywords generalized L-KKM mapping; finitely closed valued; finitely open valued; fixed point; equilibrium problem.

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1. Introduction

KKM theory and its applications are of fundamental importance in modern nonlinear analysis. The famous KKM theorem was first established in finite dimensional spaces by Knaster, Kureatowski and Mazurkiewiez [1]. Fan [2] extended the KKM theorem to infinite dimensional topological vector spaces and gave some applications in several directions. Since then, lots of generalizations and applications have been obtained. In early forms of the fundamental result, convexity assumption played a crucial role and restricted the ranges of applicable areas (see, for example [3–20] and the references therein). In [11] Horvath, replacing convex hulls by contract subset, gave a purely topological version of the KKM theorem. Tian [14] proved the F-KKM theorems, Park and Kim [12, 13] introduced G-convex space and developed KKM-type theorems. Deng and Xia [4] and Ding [8, 9] proved some generalized R-KKM type theorems in general topological spaces without any convexity structure. Recently, Fang and Huang [15] introduced L-KKM mapping in topological spaces without any convexity, which unified and generalized some known results.

The main purpose of this paper is to establish some new generalized L-KKM type theorems with finitely open valued and with finitely closed values in topological spaces without any convexity structure under much weaker assumptions. As applications, some new matching theorem,

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E-mail address: fangmingracie@163.com

fixed point theorem and existence theorem of equilibrium problem with lower and upper bounds are also given in topological spaces. These results presented in this paper unify and generalize some corresponding known results of Deng and Xia [4], Ding [8, 9], Li [16], Chadli, Chiang and Yao [17].

2. Preliminaries

Let X and Y be two nonempty sets. We denote by 2^Y and $\langle X\rangle$ the families of all subsets of Y and the family of all nonempty finite subsets of X, respectively. For each $A\in\langle X\rangle$, $|A|$ denotes the cardinality of A. Let Δ_n denote the standard n-dimensional simplex with vertices $\{e_0,e_1,\dots,e_n\}$. If J is a nonempty subset of $\{0,1,\dots,n\}$, we shall denote by Δ_J the convex hull of vertices $\{e_j: j\in J\}$. Let X be a topological space. A subset A of X is said to be compactly open (resp. compactly closed) if for each nonempty compact subset K of X, $A\cap K$ is open (resp. closed) in K.

Let X and Y be two topological spaces. A set-valued mapping $T: X\to 2^Y$ is said to be lower (resp. upper) semicontinuous on X if, for each open set $U\subseteq Y$, the set $\{x\in X: T(x)\cap U\neq\emptyset\}$ (resp. $\{x\in X: T(x)\subseteq U\}$) is open in X.

Definition 2.1 ([15]) *Let X be a nonempty set and Y be a topological space. A set-valued mapping $G: X\to 2^Y$ is said to be a generalized L-KKM mapping if, for any $N=\{x_0,\dots,x_n\}\in\langle X\rangle$ (where some elements in N may be same), there exists a lower semicontinuous mapping $\varphi_N:\Delta_n\to 2^Y$ such that for each $\{e_{i_0},\dots,e_{i_k}\}\subset\{e_0,\dots,e_n\}$,*

$$\varphi_N(\Delta_k)\subseteq\bigcup_{j=0}^{k}G(x_{i_j}),$$

where $\Delta_k=\mathrm{co}(\{e_{i_0},\dots,e_{i_k}\})$.

Remark 2.1 Generalized L-KKM mapping extends the generalized R-KKM mapping of Deng and Xia [4]. We also know that the generalized L-KKM mapping defined by Definition 2.1 unifies the generalized R-KKM mapping of Verma [5], the generalized G-KKM mapping of Ding [6], and the generalized L-KKM mapping of Ding [7].

Definition 2.2 *Let X be a nonempty set and Y be a topological space. A generalized L-KKM mapping $G: X\to 2^Y$ is said to be finitely closed-valued (resp. finitely open-valued) if, for each $N=\{x_0,\dots,x_n\}\in\langle X\rangle$ and each $x\in N$, $\varphi_N(\Delta_n)\cap G(x)$ is closed (resp. open) in $\varphi_N(\Delta_n)$, where $\varphi_N:\Delta_n\to 2^Y$ is the lower semicontinuous mapping in touch with N in Definition 2.1.*

Remark 2.2 Definition 2.2 extends the definition 2.3 of Ding [9]. It is easy to obtain that each generalized L-KKM mapping with compactly closed (resp. compactly open) values must be finitely closed (resp. finitely open) valued since for each $N=\{x_0,\dots,x_n\}\in\langle X\rangle$, $\varphi_N(\Delta_n)$ is compact in Y. The converse is not true in general.

Lemma 2.1 ([10]) *Let X and Y be two topological spaces, and $F: X\to 2^Y$ be a set-*

valued mapping. Then F is lower semicontinuous if and only if for each closed set S of Y, $F^{-1}(S) = \{x \in X : F(x) \subset S\}$ is closed set of X.

The following Lemma is the equal case of Theorem 3.1 in [15]

Lemma 2.2 ([15]) *Let X be a nonempty set, Y be a topological space, and $G : X \to 2^Y$ be a generalized L-KKM mapping with nonempty finite closed values. Then*

$$\varphi_N(\Delta_n) \bigcap \big(\bigcap_{i=0}^{n} G(x_i)\big) \neq \emptyset.$$

3. Generalized L-KKM Type Theorems

Theorem 3.1 *Let X be a nonempty set, Y be a topological space, and $G : X \to 2^Y$ be a generalized L-KKM mapping with nonempty finitely closed values. If for some $M \in \langle X \rangle$, $\bigcap_{x\in M} G(x)$ is compact, then $\bigcap_{x\in X} G(x) \neq \emptyset$.*

Proof By Lemma 2.2, $\bigcap_{i=0}^{n} G(x_i) \neq \emptyset$. Since for some $M \in \langle X \rangle$, $\cap_{x\in M} G(x)$ is compact. Therefore, $\cap_{x\in X} G(x) \neq \emptyset$. $\square$

Theorem 3.2 *Let X be a nonempty set, Y be a topological space, $G : X \to 2^Y$ be a generalized L-KKM mapping with nonempty finitely open values. Then for each $N = \{x_0, \dots, x_n\} \in \langle X \rangle$,*

$$\varphi_N(\Delta_n) \bigcup \big(\bigcap_{i=0}^{n} G(x_i)\big) \neq \emptyset.$$

Proof Suppose the conclusion is not true. Then there exists $N = \{x_0, \dots, x_n\} \in \langle X \rangle$ such that

$$\varphi_N(\Delta_n) \bigcap \big(\bigcap_{i=0}^{n} G(x_i)\big) = \emptyset,$$

where $\varphi_N : \Delta_n \to 2^Y$ is the lower semicontinuous mapping in touch with N in Definition 2.1. Then

$$\begin{aligned} \varphi_N(\Delta_n) &= \varphi_N(\Delta_n) \backslash \big(\bigcap_{i=0}^{n} (\varphi_N(\Delta_n) \cap G(x_i))\big) \\ &= \bigcup_{i=0}^{n} (\varphi_N(\Delta_n) \backslash (\varphi_N(\Delta_n) \cap G(x_i))). \end{aligned}$$

Since G is finitely open-valued, $\varphi_N(\Delta_n) \cap G(x_i)$ is open in $\varphi_N(\Delta_n)$ for each $x \in N$. For each $z \in \Delta_n$, let

$$I(z) = \{i \in \{0, 1, \dots, n\} : \varphi_N(z) \notin G(x_i)\}$$

and

$$S(z) = \mathrm{co}(\{e_i : i \in I(z)\}).$$

If for some $z \in \Delta_n$, $I(z) = \emptyset$, then we have

$$\varphi_N(z) \in G(x_i), \text{ for all } i \in \{0, 1, \dots, n\},$$

which contradicts the assumption

$$\varphi_N(\Delta_n)\bigcap\big(\bigcap_{i=0}^{n}G(x_i)\big)=\emptyset.$$

Therefore we can assume that $I(z)\neq\emptyset$, for each $z\in\Delta_n$. And hence $S(z)$ is nonempty compact convex subset of Δ_n for each $z\in\Delta_n$.

By the assumption, $\bigcup_{i\notin I(z)}(\varphi_N(\Delta_n)\backslash\varphi_N(\Delta_n)\cap G(x_i))$ is closed, and φ_N is lower semicontinuous, we have

$$U=\Delta_n\backslash\varphi_N^{-1}\big(\bigcup_{i\notin I(z)}(\varphi_N(\Delta_n)\backslash\varphi_N(\Delta_n)\cap G(x_i))\big)$$

is an open neighborhood of Z in Δ_n. For each $z'\in U$, we have

$$\varphi_N(z')\in G(x_i),\ \text{for all } i\notin I(z).$$

And hence $I(z')\subset I(z)$. It follows that

$$S(z')\subset S(z),\ \text{for all } z'\in U.$$

This shows that $S:\Delta_n\to 2^{\Delta_n}$ is an upper semicontinuous set-valued mapping with nonempty compact, convex values. By the Kakutain fixed point theorem, there exists a $z_0\in\Delta_n$ such that $z_0\in S(z_0)$.

Note that G is a generalized L-KKM mapping. It follows that

$$\varphi_N(z_0)\in\varphi_N(S(z_0))\subset\bigcup_{i\in I(z_0)}G(x_i).$$

Hence there exists an $i_0\in I(z_0)$ such that $\varphi_N(z_0)\in G(x_{i_0})$. By the definition of $I(z_0)$, we have

$$\varphi_N(z_0)\notin G(x_i),\quad\forall i\in I(z_0),$$

which is a contradiction. Therefore

$$\varphi_N(\Delta_n)\bigcap\big(\bigcap_{i=0}^{n}G(x_i)\big)\neq\emptyset.$$

Remark 3.1 Theorem 3.2 generalizes Theorem 3.2 of Ding [9] and Theorem 1 of Piao [18] from generalized R-KKM mapping to generalized L-KKM mapping. Theorem 3.2 also extends Theorem 2.2 of Ding [8] in the following aspects: (1) From generalized R-KKM mapping to generalized L-KKM mapping; (2) From compactly open-valued to finitely open-valued.

4. Applications

In this section, by applying the generalized L-KKM theorems established in the above section, some new matching theorem, fixed point theorem, and existence theorem of equilibrium problem with lower and upper bounds are established in topological spaces.

Theorem 4.1 *Let X be a nonempty set, Y be a topological space and $\{M_i\}_{i=0}^{n}$ be a family of finitely closed (or finitely open) subset of Y such that $\bigcup_{i=0}^{n}M_i=Y$. Suppose that for each*

$A = \{x_0, \dots, x_n\} \in \langle X \rangle$, *there exists at least a lower semicontinuous mapping* $\varphi_A : \Delta_n \to 2^Y$. *Let* $X_0 = \{x_0, \dots, x_n\} \in \langle X \rangle$ *be given. Then for any lower semicontinuous mapping* $\varphi_{X_0} : \Delta_n \to 2^Y$, *there exists* $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$ *such that*

$$\varphi_{X_0}(\Delta_k) \bigcap (\bigcap_{j=0}^{k} M_{i_j}) \neq \emptyset.$$

Proof Define a set-valued mapping $G : X_0 \to 2^Y$ as follows:

$$G(x_i) = Y \backslash M_i \text{ for each } i = 0, 1, \dots, n.$$

Since each M_i has finite closed (or finite open) values in Y, G is a set-valued mapping with finitely open (or finitely closed) values. Suppose, on the contrary to the conclusion, that there exists a lower semicontinuous mapping $\varphi_{X_0} : \Delta_n \to 2^Y$ such that for any $\{e_{i_j} : j = 0, 1, \dots, k\} \subset \{e_o, \dots, e_n\}$, we obtain

$$\varphi_{X_0}(\Delta_k) \bigcap (\bigcap_{j=0}^{k} M_{i_j}) = \emptyset.$$

Therefore,

$$\varphi_{X_0}(\Delta_k) \subset \bigcup_{j=0}^{k} (Y \backslash M_{i_j}) = \bigcup_{j=0}^{k} G(x_{i_j}).$$

Then $G : X_0 \to 2^Y$ is a generalized L-KKM mapping. According to Theorem 3.2 (or Lemma 2.2),

$$\bigcap_{i=0}^{n} G(x_i) \neq \emptyset,$$

which means that $Y \neq \bigcup_{i=0}^{n} M_i$. This is a contradiction and the proof is completed. □

Remark 4.1 Theorem 4.1 extends Theorem 3.1 of Ding [8] and Theorem 3 of Piao [18] from single-valued continuous mapping to set-valued lower semicontinuous mapping.

Theorem 4.2 *Let* D *be a nonempty subset of a topological space* X, $S : D \to 2^X$, $T : X \to 2^X$ *be two set-valued mappings such that for any* $x \in X$, $S^{-1}(x) \subset T^{-1}(x)$. *For each* $A = \{x_0, \dots, x_n\} \in \langle D \rangle$, *there exists at least a lower semicontinuous mapping* $\varphi_A : \Delta_n \to 2^X$. *Let* $G : D \to 2^X$ *be a set-valued mapping with finitely closed (or finitely open) values. Suppose that there exists* $X_0 = \{x_0, x_1, \dots, x_n\} \in \langle D \rangle$, *such that* $X = \bigcup_{i=0}^{n} G(x_i)$ *and* $G(x_i) \subset S(x_i)$. *for any* $y \in X$ *and any* $\{x_{i_0}, \dots, x_{i_k}\} \subset \{x_0, \dots, x_n\}$,

$$\varphi_{X_0}(\Delta_k) \subset T^{-1}(y) = \{x \in D : y \in T(x)\},$$

where $\Delta_k = co(\{e_{i_0}, \dots, e_{i_k}\})$. *Then there exists* $\hat{x} \in D$ *such that* $\hat{x} \in T(\hat{x})$.

Proof It is easy to see that all conditions of Theorem 4.1 are satisfied. By Theorem 4.1, for any lower semicontinuous mapping $\varphi_{X_0} : \Delta_n \to 2^X$, there exists $\{e_{i_0}, \dots, e_{i_k}\} \subset \{e_0, \dots, e_n\}$ such that

$$\varphi_{X_0}(\Delta_k) \bigcap (\bigcap_{j=0}^{k} G(x_{i_j})) \neq \emptyset,$$

where $\Delta_k = co(\{e_{i_0}, \ldots, e_{i_k}\})$.

By the assumption, we obtain

$$\varphi_{X_0}(\Delta_k) \bigcap \big(\bigcap_{j=0}^{k} G(x_{i_j}) \big) \subset \varphi_{X_0}(\Delta_k) \bigcap \big(\bigcap_{j=0}^{k} S(x_{i_j}) \big) \neq \emptyset.$$

Taking any $\hat{x} \in \varphi_{X_0}(\Delta_k) \bigcap \big(\bigcap_{j=0}^{k} S(x_{i_j}) \big)$, then

$$\hat{x} \in \varphi_{X_0}(\Delta_k) \subset X \text{ and } x_{i_j} \in S^{-1}(\hat{x}) \subset T^{-1}(\hat{x}).$$

Therefore, we have $\hat{x} \in \varphi_{X_0}(\Delta_k) \subset T^{-1}(\hat{x})$ is a fixed point of T.

Remark 4.2 Theorem 4.2 generalizes Theorem 6 of Piao [18] from single-valued continuous mapping to set-valued lower semicontinuous mapping. When S is an identity map and $G(x) = T(x)$, Theorem 4.2 goes back to Theorem 3.2 of Ding [8].

Theorem 4.3 *Let X be a nonempty set, Y be a topological space and $f, g : X \times Y \to R \bigcup \{-\infty, +\infty\}$ and $s : X \to Y$ be three functions such that*

(i) For each $x \in X$, $\alpha \leq f(x, s(x)) \leq \beta$;

(ii) For each $A = \{x_0, \ldots, x_n\} \in \langle D \rangle$ such that for each $A_1 = \{x_{i_0}, \ldots, x_{i_k}\} \in A$ and each $y \in \varphi_A(\Delta_k)$, there exists $j \in \{0, \ldots, k\}$, $\alpha \leq g(x_{i_j}, y) \leq \beta$;

(iii) For each $(x, y) \in X \times Y$, $\alpha \leq g(x, y) \leq \beta \Rightarrow \alpha \leq f(x, y) \leq \beta$;

(iv) The set $\{y \in Y : \alpha \leq f(x, y) \leq \beta\}$ is nonempty finitely closed valued in Y;

(v) There exists $\{x_0, \ldots, x_n\} \in \langle X \rangle$ such that $\bigcap_{j=0}^{k} \{y \in Y : \alpha \leq f(x_j, y) \leq \beta\}$ is compact subset of Y.

Then there exists $\hat{y} \in Y$ such that $\alpha \leq f(x, \hat{y}) \leq \beta, \forall x \in X$.

Proof Define $F, G : X \to 2^Y$ as follows:

$$F(x) = \{y \in Y : \alpha \leq g(x, y) \leq \beta\}, \quad \forall x \in X$$

and

$$G(x) = \{y \in Y : \alpha \leq f(x, y) \leq \beta\}, \quad \forall x \in X.$$

By (i), $s(x) \in G(x)$ and $G(x) \neq \emptyset$ for each $x \in X$. The condition (iv) implies that G has nonempty finitely closed values. By (iii) of this theorem, we obtain that $F(x) \subseteq G(x)$.

Next, we prove that F is a generalized L-KKM mapping. Suppose, on the contrary to the conclusion, that there exists $N = \{x_0, \ldots, x_n\} \in \langle X \rangle$ and any lower semicontinuous mapping $\varphi_N : \Delta_n \to 2^Y$ such that $\{e_{i_0}, \ldots, e_{i_k}\} \subset \{e_0, \ldots, e_n\}$,

$$\bar{y} \in \varphi_N(\Delta_k) \text{ and } \bar{y} \notin \bigcup_{j=0}^{k} F(x_{i_j}).$$

This in turn means that

$$\bar{y} \in \varphi_N(\Delta_k) \text{ but } g(x_{i_j}, \bar{y}) > \alpha \text{ or } g(x_{i_j}, \bar{y}) < \beta,$$

which contradicts (ii). Then F is a generalized L-KKM mapping and so G is also a generalized L-KKM mapping. From the condition (v), it follows that $\bigcap_{j=0}^{k} G(x_j)$ is compact. Then all the assumptions of Lemma 2.2 have been checked, and we obtain that

$$\bigcap_{x\in X} G(x) \neq \emptyset.$$

Taking $\hat{y} \in \bigcap_{x\in X} G(x)$, we get $\hat{y} \in Y$ such that $\alpha \leq f(x, \hat{y}) \leq \beta$ for each $x \in X$.

Remark 4.3 Theorem 4.3 extends Theorem 5.1 of Zhang [20], and Theorem 3.1 of Li [16] in several aspects.

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