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Solving a Class of Generalized Nash Equilibrium Problems

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Abstract Generalized Nash equilibrium problem (GNEP) is an important model that has many applications in practice. However, a GNEP usually has multiple or even infinitely many Nash equilibrium points and it is not easy to choose a favorable solution from those equilibria. This paper considers a class of GNEP with some kind of separability. We first extend the so-called normalized equilibrium concept to the stationarity sense and then, we propose an approach to solve the normalized stationary points by reformulating the GNEP as a single optimization problem. We further demonstrate the proposed approach on a GNEP model in similar product markets.

Keywords generalized Nash equilibrium problem; normalized equilibrium; normalized stationarity; separability.

MR(2010) Subject Classification 65K10; 90C33

1. Introduction

Generalized Nash equilibrium problem (GNEP) is a Nash equilibrium problem, in which each player's strategy set may depend on the rivals' strategies. The early study of such games dates back at least to Debreu [2] and Arrow [1], where the GNEP was called social equilibrium problem or abstract economy. In the sequel, Harker [10] studied GNEP via (quasi-)variational inequality reformulation and Pang et al. [16] proposed some reformulations for multi-leader-follower games and proposed a penalty method for solving GNEP. More recently, Fukushima [8] proposed another penalty method for finding the restricted generalized Nash equilibrium points and Kubota et al. [13] studied GNEP by using the regularized gap function for quasi-variational inequality problem. See also [3, 5–7] for more details about numerical methods for GNEP.

In this paper, we consider the following non-cooperative game with N players and shared constraints: For each $\nu = 1, \ldots, N$, player ν solves the following optimization problem with the other players' strategies $x^{-\nu} = (x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^N) \in \mathbb{R}^{n-n_\nu}$ being regarded as

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exogenous:

$$\begin{aligned} \min_{x^\nu} \quad & f^\nu(x^\nu, x^{-\nu}) \\ \text{s.t} \quad & g^\nu(x^\nu) \leq 0, \\ & G(x^\nu, x^{-\nu}) \leq 0, \end{aligned} \tag{1}$$

here $f^\nu : \mathbb{R}^n \to \mathbb{R}$, $g^\nu : \mathbb{R}^{n_\nu} \to \mathbb{R}^{p_\nu}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable functions with $n = \sum_{\nu=1}^{N} n_\nu$. Note that there is not any difference in our analysis if some equality constraints are added.

Since the GNEP may have many equilibrium points, in order to make optimal choices, we may expect to find as many equilibrium points as possible. However, it is usually impossible to find all equilibrium points in practice. In some cases, we may find some equilibrium points with special properties. The normalized equilibrium given in [17], at which the Lagrange multipliers associated with the shared constraints are proportionable among all players, is such an equilibrium point in game theory. In economic terms, it means that the relative values of shadow prices associated with the common resources are identical for all players at any normalized equilibrium point.

The aim of this paper is to find the normalized equilibrium points of the GNEP (1). Since the GNEP (1) may be nonconvex, we extend the normalized equilibrium concept given in [17] to the stationarity sense. Then, we reformulate the GNEP (1) as a single optimization problem by imposing some additional separability conditions.

Throughout the paper, for a given differentiable function $h : \mathbb{R}^s \to \mathbb{R}^t$ and a given point $z \in \mathbb{R}^s$, we denote by $\nabla h(z) \in \mathbb{R}^{s \times t}$ the transposed Jacobian matrix of h at z. Moreover, for two vectors $a, b \in \mathbb{R}^s$, $a \perp b$ means that the vector a is perpendicular to the vector b.

2. Normalized stationarity for GNEP

We first introduce a separability property that will be used later on.

Definition 2.1 *We say that the GNEP (1) is separable with positive weights $\beta \in \mathbb{R}^N$ if each objective function consists of a separable term and a parametrized common term across all players, that is,*

$$f^\nu(x^\nu, x^{-\nu}) = \bar{f}^\nu(x^\nu) + \beta_\nu \tilde{f}(x^\nu, x^{-\nu})$$

for each $\nu = 1, \ldots, N$.

In the rest of this section, we assume that the GNEP (1) is separable with positive weights $\beta \in \mathbb{R}^N$. Then (1) becomes

$$(NLP(x^{-\nu})) \qquad \begin{aligned} \min_{x^\nu} \quad & \bar{f}^\nu(x^\nu) + \beta_\nu \tilde{f}(x^\nu, x^{-\nu}) \\ \text{s.t} \quad & g^\nu(x^\nu) \leq 0, \\ & G(x^\nu, x^{-\nu}) \leq 0. \end{aligned}$$

Suppose that $\hat{x}$ is a Nash equilibrium point of the above GNEP and, for each ν, the Guignard constraint qualification [9] holds at $\hat{x}^\nu$ for $(NLP(\hat{x}^{-\nu}))$. Then, for $\nu = 1, \dots, N$, we have

$$\begin{aligned} &\nabla \bar{f}^\nu(\hat{x}^\nu) + \beta_\nu \nabla_{x^\nu} \tilde{f}(\hat{x}^\nu, \hat{x}^{-\nu}) + \nabla g^\nu(\hat{x}^\nu)\lambda_g^\nu + \nabla_{x^\nu} G(\hat{x}^\nu, \hat{x}^{-\nu})\lambda_G^\nu = 0, \\ &0 \le \lambda_g^\nu \perp g^\nu(\hat{x}^\nu) \le 0, \\ &0 \le \lambda_G^\nu \perp G(\hat{x}^\nu, \hat{x}^{-\nu}) \le 0. \end{aligned} \tag{2}$$

Definition 2.2 *(a) A vector $\hat{x}$ is said to be a normalized equilibrium point with positive weights $\beta \in \mathbb{R}^N$ of the above GNEP if it is a Nash equilibrium point and the Lagrange multipliers associated with the shared constraints are proportionable among all players, i.e., there exist multipliers $\{\lambda_g, \lambda_G\}$ such that (2) holds and*

$$\lambda_G^\nu = \beta_\nu \lambda^0, \quad \nu = 1, \dots, N.$$

(b) A vector $\hat{x}$ is said to be a normalized stationary point with positive weights $\beta \in \mathbb{R}^N$ of the above GNEP if there exist multipliers $\{\lambda_g, \lambda_G\}$ such that (2) holds and

$$\lambda_G^\nu = \beta_\nu \lambda^0, \quad \nu = 1, \dots, N.$$

We have the following result.

Theorem 2.3 *The vector $\hat{x}$ is a normalized stationary point with positive weights $\beta \in \mathbb{R}^N$ of the above GNEP if and only if it is a stationary point of the optimization problem*

$$\begin{aligned} \min_x \quad & \sum_{\nu=1}^N \frac{1}{\beta_\nu} \bar{f}^\nu(x^\nu) + \tilde{f}(x^\nu, x^{-\nu}) \\ \text{s.t} \quad & g^\nu(x^\nu) \le 0 \ (\nu = 1, \dots, N), \\ & G(x^\nu, x^{-\nu}) \le 0. \end{aligned} \tag{3}$$

Proof "only if" part: If $\hat{x}$ is a normalized stationary point with positive weights $\beta \in \mathbb{R}^N$ of the GNEP, then there exist multipliers $\{\lambda_g, \lambda_G\}$ such that, for each $\nu = 1, \dots, N$,

$$\begin{aligned} &\nabla \bar{f}^\nu(\hat{x}^\nu) + \beta_\nu \nabla_{x^\nu} \tilde{f}(\hat{x}^\nu, \hat{x}^{-\nu}) + \nabla g^\nu(\hat{x}^\nu)\lambda_g^\nu + \nabla_{x^\nu} G(\hat{x}^\nu, \hat{x}^{-\nu})\lambda_G^\nu = 0, \\ &0 \le \lambda_g^\nu \perp g^\nu(\hat{x}^\nu) \le 0, \\ &0 \le \lambda_G^\nu \perp G(\hat{x}^\nu, \hat{x}^{-\nu}) \le 0. \end{aligned} \tag{4}$$

Since $\beta_\nu > 0$ and $\lambda_G^\nu = \beta_\nu \lambda^0$ for each $\nu = 1, \dots, N$, we can reformulate (4) as

$$\begin{aligned} &\frac{1}{\beta_\nu} \nabla \bar{f}^\nu(\hat{x}^\nu) + \nabla_{x^\nu} \tilde{f}(\hat{x}^\nu, \hat{x}^{-\nu}) + \nabla g^\nu(\hat{x}^\nu) \frac{\lambda_g^\nu}{\beta_\nu} + \nabla_{x^\nu} G(\hat{x}^\nu, \hat{x}^{-\nu})\lambda^0 = 0, \\ &0 \le \frac{\lambda_g^\nu}{\beta_\nu} \perp g^\nu(\hat{x}^\nu) \le 0, \\ &0 \le \lambda^0 \perp G(\hat{x}^\nu, \hat{x}^{-\nu}) \le 0, \\ &\nu = 1, \dots, N. \end{aligned} \tag{5}$$

Setting $\mu^v = \frac{\lambda_g^\nu}{\beta_\nu}$ for all $\nu = 1, \dots, N$, we see that (5) is equivalent to the KKT conditions of problem (3) at $\hat{x}$ with multipliers $\{\mu, \lambda^0\}$.

In a similar way, we can show the "if" part.

Theorem 2.3 indicates that solving the GNEP $\{(NLP(x^{-\nu}))\}_{\nu=1}^N$ is equivalent to solving the single optimization problem (3) in the stationarity sense.

Example 2.4 This problem is taken from [14]. Suppose that there are two players and they solve the following problems, respectively:

$$\begin{array}{ll} \min\limits_{x_1} & x_1^2 + ax_1x_2 \\ \text{s.t} & x_1 + x_2 = c, \end{array} \qquad \begin{array}{ll} \min\limits_{x_2} & x_2^2 + bx_1x_2 \\ \text{s.t} & x_1 + x_2 = c, \end{array}$$

where $a > 0$, $b > 0$, and c are given parameters such that $ab - a - b \neq 0$. The above GNEP is obviously separable with positive weights $\beta = (a, b)$. By solving

$$\begin{aligned} 2x_1 + ax_2 - a\lambda^0 &= 0, \\ 2x_2 + bx_1 - b\lambda^0 &= 0, \\ x_1 + x_2 &= c, \end{aligned}$$

we know that the normalized stationary point with positive weights β of the above GNEP is $x^* = (\frac{ac(b-2)}{2(ab-a-b)}, \frac{bc(a-2)}{2(ab-a-b)})$ with $\lambda^0 = \frac{abc-4c}{2(ab-a-b)}$. It is easy to verify that x^* is a stationary point of the following single optimization problem:

$$\begin{array}{ll} \min\limits_{(x_1,x_2)} & \dfrac{1}{a}x_1^2 + \dfrac{1}{b}x_2^2 + x_1x_2 \\ \text{s.t} & x_1 + x_2 = c. \end{array}$$

Example 2.5 This problem is taken from [10]. Suppose that there are two players and they solve the following problems, respectively:

$$\begin{array}{ll} \min\limits_{x_1} & x_1^2 - 34x_1 + \frac{8}{3}x_1x_2 \\ \text{s.t} & 0 \leq x_1 \leq 10, \\ & x_1 + x_2 \leq 15, \end{array} \qquad \begin{array}{ll} \min\limits_{x_2} & x_2^2 - 24.25x_2 + \frac{5}{4}x_1x_2 \\ \text{s.t} & 0 \leq x_2 \leq 10, \\ & x_1 + x_2 \leq 15. \end{array}$$

This is a GNEP with one shared constraint and the solution set is given by

$$S^* = \{(5, 9)\} \cup \{(t, 15 - t) | 9 \leq t \leq 10\}.$$

It is not difficult to see that (5,9) is a normalized equilibrium point with wights $(\frac{8}{3}, \frac{5}{4})$, which is a stationary point of the single optimization problem

$$\begin{array}{ll} \min\limits_{(x_1,x_2)} & \dfrac{3}{8}x_1^2 - \dfrac{51}{4}x_1 + \dfrac{4}{5}x_2^2 - \dfrac{97}{5}x_2 + x_1x_2 \\ \text{s.t} & 0 \leq x_1 \leq 10, \\ & 0 \leq x_2 \leq 10, \\ & x_1 + x_2 \leq 15. \end{array}$$

3. Applications in similar products markets

Consider an oligopoly consisting of N manufacturers that produce similar products noncooperatively before the market demand is realized. The market demand is characterized by inverse demand functions $p_\nu(x), \nu = 1, \dots, N$, where $p_\nu(x)$ denotes the market price of the product made by the manufacturer ν and $x = (x^\nu)_{\nu=1}^N$ with x^ν being the supply quantity of the manufacturer ν,

Before market demand is realized, the manufacturer ν chooses his quantity x^ν and his profit can be formulated as

$$R_\nu(x^\nu, x^{-\nu}) = x^\nu p_\nu(x^\nu, x^{-\nu}) - c_\nu(x^\nu),$$

where $x^{-\nu}$ denotes the total bids by the other manufacturers, $x^\nu p_\nu(x^\nu, x^{-\nu})$ means the total revenue for the manufacturer ν, and $c_\nu(x^\nu)$ denotes the cost function of the manufacturer ν. The ν-th manufacturer's decision problem is to choose the supply quantity x^ν that maximizes his profit, that is,

$$\max_{x^\nu \in \mathcal{X}_\nu} R_\nu(x^\nu, x^{-\nu}) = x^\nu p_\nu(x^\nu, x^{-\nu}) - c_\nu(x^\nu),$$

suppose $\mathcal{X}_\nu := \{x^\nu \in [0, +\infty) \mid g^\nu(x^\nu) \le 0, G(x^\nu, x^{-\nu}) \le 0\}$ is a nonempty and bounded convex set for each $\nu = 1, \dots, N$.

Suppose that, for each $\nu = 1, 2, \dots, N$,

(A1) $p_\nu(\cdot)$ is twice continuously differentiable and decreasing;

(A2) $p_\nu{}'(q) + q p_\nu{}''(q) \le 0$ holds for any $q \ge 0$;

(A3) the cost function c_ν is twice continuously differentiable and its first and second derivatives are always nonnegative.

Under the assumptions (A1)–(A3), one can easily show that $R_\nu(x^\nu, x^{-\nu})$ is concave, which guarantees the existence of generalized Nash equilibrium of the model. Additionally, we suppose that the multipliers of the manufacturers corresponding to the shared constraints are proportionable, that is,

$$\lambda_G^\nu = \beta_\nu \lambda^0, \quad \nu = 1, \dots, N,$$

where $\beta_\nu > 0$ for each ν.

We next report our numerical experience for the above model. In our test, we set the data as follows:

The inverse demand functions are given by

$$p_1(x_1, x_2) := a_1 - \beta^1(x_1 + x_2), \quad p_2(x_1, x_2) := \alpha_2 - \beta^2(x_1 + x_2).$$

The cost functions are given by

$$c_1(x_1) := \gamma_1 x_1, \quad c_2(x_2) := \gamma_2 x_2.$$

The constraint functions are given by

$$g^1(x_1) := x_1 - u_1, \quad g^2(x_2) := x_2 - u_2, \quad G(x_1, x_2) := x_1 + x_2 - u.$$

Then the GNEP is written as

$$\begin{array}{llcll}
\min\limits_{x_1} & \beta^1 x_1^2 - (\alpha_1 - \gamma_1)x_1 + \beta^1 x_1 x_2 & \qquad & \min\limits_{x_2} & \beta^2 x_2^2 - (\alpha_2 - \gamma_2)x_2 + \beta^2 x_1 x_2 \\
\text{s.t} & x_1 - u_1 \le 0, & & \text{s.t} & x_2 - u_2 \le 0, \\
 & x_1 + x_2 - u \le 0, & & & x_1 + x_2 - u \le 0.
\end{array}$$

By Theorem 2.3, solving this GNEP is equivalent to solving the single optimization problem

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 - \frac{\alpha_1 - \gamma_1}{\beta^1} x_1 - \frac{\alpha_2 - \gamma_2}{\beta^2} x_2 + x_1 x_2 \\ \text{s.t} \quad & x_1 - u_1 \le 0, \\ & x_2 - u_2 \le 0, \\ & x_1 + x_2 - u \le 0. \end{aligned} \tag{6}$$

We first solved this GNEP using the method proposed by Li [11] and then solved the optimization problem (6) by the solver `fmin` in Matlab R2010a. We get the same results shown in the table, which reveal the proposed approach is applicable.

α_1	48	64	68	76	84	98
α_2	52	67	78	82	88	128
γ_1	23	28	26	32	36	48
γ_2	25	32	23	36	42	54
β^1	5	4	2	5	4	2
β^2	6	5	3	6	5	3
u_1	2.0	4.0	3.5	3.5	5.0	5.0
u_2	1.5	2.0	2.0	2.5	3.0	4.0
u	3.5	5.5	5.5	5.5	7.5	8.0
(x_1, x_2)	(1.833,1.333)	(3.667,1.667)	(3.333,1.667)	(3.311,2.177)	(4.933,2.133)	(4.166,3.833)

Table 1 Numerical results

4. Conclusions

We have discussed a GNEP with some kind of separability. We first extended the well-known normalized equilibrium concept to the stationarity sense and then, we proposed an approach to solve the normalized stationary points by reformulating the GNEP as a single optimization problem. We further demonstrated the proposed approach on a GNEP model in similar product markets.

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