The Characterization of Primitive Symmetric Signed Digraphs with the Second Maximum Base

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Abstract Recently, the primitive symmetric signed digraphs on n vertices with the maximum base $2n$ and the primitive symmetric loop-free signed digraphs on n vertices with the maximum base $2n - 1$ are characterized, respectively. In this paper, the primitive symmetric signed digraphs with loops on n vertices with the base $2n - 1$ are characterized, and then the primitive symmetric signed digraphs on n vertices with the second maximum base $2n - 1$ are characterized.

Keywords matrix; symmetric; primitive; non-powerful; base; signed digraph.

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1. Introduction

A sign pattern matrix is a matrix each of whose entries is a sign 1, -1 or 0. For a square sign pattern matrix M, notice that in the computations of the entries of the power M^k, an "ambiguous sign" may arise when we add a positive sign 1 to a negative sign -1. So a new symbol "#" was introduced in [1] to denote the ambiguous sign. The set $\Gamma = \{0, 1, -1, \#\}$ is defined as the generalized sign set and the addition and multiplication involving the symbol # are defined as follows:

$$(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \# \ \text{(for all } a \in \Gamma)$$

$$0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \# \ \text{(for all } b \in \Gamma \setminus \{0\})$$

In $[1, 2]$, the matrices with entries in the set Γ are called generalized sign pattern matrices. The addition and multiplication of generalized sign pattern matrices are defined in the usual way, so that the sum and product of the generalized sign pattern matrices are still generalized sign pattern matrices. In this paper, we assume that all the matrix operations considered are operations of the matrices over Γ.

Definition 1 ([1]) *A square generalized sign pattern matrix M is called powerful if each power*

of M contains no $\#$ entry.

Definition 2 ([3]) *Let M be a square generalized sign pattern matrix of order n and $M, M^2, M^3, \ldots$ be the sequence of powers of M. Suppose M^b is the first power that is repeated in the sequence. Namely, suppose b is the least positive integer such that there is a positive integer p such that*

$$M^b = M^{b+p}. \tag{1}$$

Then b is called the generalized base (or simply base) of M, and is denoted by $b(M)$. The least positive integer p such that (1) holds for $b = b(M)$ is called the generalized period (or simply period) of M, and is denoted by $p(M)$.

We now introduce some concepts of graph theory.

Let $D = (V, A)$ denote a digraph on n vertices. Loops are permitted, but no multiple arcs. A $u \to v$ walk in D is a sequence of vertices $u, u_1, \ldots, u_k = v$ and a sequence of arcs $e_1 = (u, u_1), e_2 = (u_1, u_2), \ldots, e_k = (u_{k-1}, v)$, where the vertices and the arcs are not necessarily distinct. A closed walk is a $u \to v$ walk where $u = v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W, denoted by $l(W)$. A k-cycle is a cycle of length k, denoted by C_k.

A signed digraph S is a digraph where each arc of S is assigned a sign 1 or -1. A generalized signed digraph S is a digraph where each arc of S is assigned a sign 1, -1 or $\#$.

The sign of the walk W in a (generalized) signed digraph, denoted by sgn W, is defined to be $\prod_{i=1}^{k} \text{sgn}(e_i)$, where $e_1, e_2, \ldots, e_k$ is the sequence of arcs of W.

For a cycle C in a (generalized) signed digraph S, if sgn $C = 1$ (or -1), then we call C a positive (or negative) cycle.

Let $M = (m_{ij})$ be a square (generalized) sign pattern matrix of order n. The associated digraph $D(M) = (V, A)$ of M (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $A = \{(i, j) | m_{ij} \neq 0\}$. The *associated (generalized) signed digraph* $S(M)$ of M is obtained from $D(M)$ by assigning the sign of m_{ij} to each arc (i, j) in $D(M)$, and we say $D(M)$ is the underlying digraph of $S(M)$.

Let S be a (generalized) signed digraph on n vertices. Then there is a (generalized) sign pattern matrix M of order n whose associated (generalized) signed digraph $S(M)$ is S. We say that S is powerful if M is powerful. Also the base $b(S)$ and period $p(S)$ are defined to be those of M. Namely, we define $b(S) = b(M)$ and $p(S) = p(M)$.

A digraph D is said to be strongly connected if there exists a path from u to v for all $u, v \in V$, and D is called primitive if there is a positive integer k such that for each vertex x and each vertex y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the primitive exponent (or exponent) of D, denoted by $\exp(D)$. It is also well-known that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor (g.c.d.) of the lengths of all the cycles of D is 1. A (generalized) signed digraph S is called primitive if the underlying digraph D is primitive, and in this case we define $\exp(S) = \exp(D)$.

A digraph D is symmetric if for every arc (u, v) in D, the arc (v, u) is also in D. A (generalized) signed digraph S is called combinatorially symmetric (or symmetric) if the underlying digraph D is symmetric. A digraph D is loop-free if D has no loops. If a digraph D is symmetric and loop-free, we regard D as a simple graph.

Let $\mathcal{S}_n = \{S | S$ is a primitive symmetric signed digraph on n vertices$\}$, $\mathcal{S}_n^\star = \{S | S$ is a primitive symmetric loop-free signed digraph on n vertices$\}$. Clearly, $\mathcal{S}_n^\star \subset \mathcal{S}_n$.

Let $\mathcal{E}_n = \{\exp(S) | S \in \mathcal{S}_n\}$, $\mathcal{E}_n^\star = \{\exp(S) | S \in \mathcal{S}_n^\star\}$, and $\mathcal{B}_n = \{b(S) | S \in \mathcal{S}_n\}$, $\mathcal{B}_n^\star = \{b(S) | S \in \mathcal{S}_n^\star\}$. The primitive exponent, exponent sets $\mathcal{E}_n$ and $\mathcal{E}_n^\star$ were discussed in [4–7]. The base, the base sets $\mathcal{B}_n$ and $\mathcal{B}_n^\star$ were discussed in [8–11].

Theorem 1 ([5]) *Let D be a primitive symmetric digraph on n vertices. Then*

(1) $\exp(D) \leq 2n - 2$ and the equality holds if and only if D is isomorphic to G_1, where $G_1 = (V, A)$, $V = \{1, 2, \ldots, n\}$, $A = \{(i, i + 1), (i + 1, i) | 1 \leq i \leq n - 1\} \bigcup \{(1, 1)\}$.

(2) $\mathcal{E}_n = \{1, 2, \ldots, 2n - 2\} \backslash \mathcal{D}$ where $\mathcal{D}$ is the set of odd numbers in $\{n, n + 1, \ldots, 2n - 2\}$.

Theorem 2 ([7]) *Let D be a primitive symmetric loop-free digraph on n vertices. Then*

(1) $\exp(D) \leq 2n - 4$.

(2) $\mathcal{E}_n^\star = \{2, 3, \ldots, 2n - 4\} \backslash \mathcal{D}$ where $\mathcal{D}$ is the set of odd numbers in $\{n - 2, n - 1, \ldots, 2n - 5\}$.

The primitive symmetric signed digraphs on n vertices with the maximum base $2n$ and the base set $\mathcal{B}_n$ were characterized in [8, 9].

Theorem 3 ([8, 9]) *Let S be a primitive symmetric signed digraph on n vertices. Then*

(1) $b(S) \leq 2n$ and the equality holds if and only if S has at least one negative 2-cycle and D is isomorphic to G_1 where D is the underlying digraph of S.

(2) $\mathcal{B}_n = \{1, 2, \ldots, 2n\}$.

The primitive symmetric loop-free signed digraphs on n vertices with the maximum base $2n - 1$ and the base set $\mathcal{B}_n^\star$ were characterized in [10–12].

Let $n \geq 4$, l $(3 \leq l \leq n)$ be odd, $D_l = (V, A)$ be a digraph on n vertices with vertex set $V = \{1, 2, \ldots, n\}$ and arc set $A = \{(i, i + 1), (i + 1, i) | 1 \leq i \leq n - 1\} \cup \{(1, l), (l, 1)\}$. Clearly, D_l is a primitive symmetric loop-free digraph.

Let $n \geq 4$, l $(3 \leq l \leq n)$ be odd, SD_l be a signed digraph on n vertices with D_l as its underlying digraph, where every 2-cycle in SD_l is negative. Then SD_l is a primitive symmetric loop-free non-powerful signed digraph on n vertices and $b(SD_l) = 2n - 1$.

Theorem 4 ([10–12]) *Let S be a primitive symmetric loop-free signed digraph on n vertices. Then*

(1) $b(S) \leq 2n - 1$ and the equality holds if and only if $S \in SD_L = \{SD_l | 3 \leq l \leq n$ and l is odd $\}$.

(2) $\mathcal{B}_n^\star = \{2, \ldots, 2n - 1\}$.

A natural question is what primitive symmetric signed digraphs on n vertices are of the second maximum base $2n - 1$? We answer this in Section 3.

2. Some preliminaries

In this section, we introduce some needed definitions, theorems and lemmas. Other definitions and results not in this article can be found in [13–15].

Definition 3 ([3]) *Two walks W_1 and W_2 in a signed digraph are called a pair of SSSD walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.*

It is easy to see from the above relation between matrices and signed digraphs that a (generalized) sign pattern matrix M is powerful if and only if the associated (generalized) signed digraph $S(M)$ has no pairs of $SSSD$ walks. Thus for a (generalized) signed digraph S, S is powerful if and only if S has no pairs of $SSSD$ walks.

In [3], You, Shao and Shan obtained an important characterization of primitive non-powerful signed digraphs from the characterization of powerful irreducible sign pattern matrices [1].

Theorem 5 ([3]) *If S is a primitive signed digraph, then S is non-powerful if and only if S has a pair of cycles C' and C'' (say, with lengths p_1 and p_2, respectively) satisfying one of the following conditions:*

(A_1) *p_1 is odd, p_2 is even and $\operatorname{sgn} C'' = -1$;*

(A_2) *Both p_1 and p_2 are odd and $\operatorname{sgn} C' = -\operatorname{sgn} C''$.*

A pair of cycles C' and C'' satisfying (A_1) or (A_2) is a "distinguished cycle pair". It is easy to check that if C' and C'' is a distinguished cycle pair with lengths p_1 and p_2, respectively, then the closed walks $W_1 = p_2 C'$ (walk around C' by p_2 times) and $W_2 = p_1 C''$ have the same length $p_1 p_2$ and different signs: $(\operatorname{sgn} C')^{p_2} = -(\operatorname{sgn} C'')^{p_1}$.

The following result can be used to determine the base.

Theorem 6 ([3]) *Let S be a primitive non-powerful signed digraph. Then*

(1) There is an integer k such that there exists a pair of SSSD walks of length k from each vertex x to each vertex y in S.

(2) If there exists a pair of SSSD walks of length k from each vertex x to each vertex y, then there also exists a pair of SSSD walks of length $k + 1$ from each vertex x to each vertex y in S.

(3) The minimal such k (as in (1)) is just $b(S)$-the base of S.

The following result will be useful.

Theorem 7 ([11, 12]) *Let D be a symmetric digraph on n vertices. Suppose that there exist a cycle C and an odd cycle C' with lengths of $k \geq 1$ and $k' \geq 1$ in D such that $C \cap C' = \emptyset$. Let P be the shortest path from C to C', $d(x, y)$ be the distance from x to y. Then for any two vertices $x, y \in D$, there exist $x' \in C, y' \in C'$ or $x' \in C', y' \in C$ such that*

$$d(x, x') + l(P) + d(y, y') \leq 2(n - k - k' + 1) + \max\{[\frac{k}{2}], \frac{k' - 1}{2}\}. \qquad (2)$$

3. Characterization with the second maximum base

It was shown in [1] that if a primitive signed digraph S is powerful, then $b(S) = \exp(D)$, where D is the underlying digraph of S. So for a primitive powerful symmetric (loop-free) signed digraph, Theorems 1 and 2 give the base. Therefore, if S is a primitive symmetric (loop-free) signed digraph on n vertices with base $2n - 1$, then S must be non-powerful. Furthermore, if $S \in \mathcal{S}_n^\star$ and $b(S) = 2n - 1$, then $S \in SD_L = \{SD_l | 3 \le l \le n$ and l is odd $\}$ by Theorem 4. So we only need to study the case for $S \in \mathcal{S}_n$ but $S \notin \mathcal{S}_n^\star$.

Let $D_{n,I} = (V, A)$ be a digraph on n vertices, where vertex set $V = \{1, 2, \ldots, n\}$, arc set $A = \{(i, i+1), (i+1, i) | 1 \le i \le n - 1\} \cup \{(j, j) | j \in I\}$, and $\{1\} \subset I \subseteq \{1, 2, \ldots, n\}$. Clearly, $D_{n,I}$ is a primitive symmetric digraph and $D_{n,I}$ has at least two loops.

Let $D = (V, A)$ be a digraph. For any vertex $v \in V$, if $(v, v) \in A$, we denote the loop on vertex v by $C_1^{(v)}$.

Lemma 8 *Let $SD_{n,I}$ be a signed digraph with $D_{n,I}$ as its underlying digraph, where all 2-cycles in $SD_{n,I}$ are positive, and for any $i, j \in I \setminus \{1\}$, $\operatorname{sgn} C_1^{(i)} = \operatorname{sgn} C_1^{(j)}$, $\operatorname{sgn} C_1^{(i)} = -\operatorname{sgn} C_1^{(1)}$. Then*

(1) $SD_{n,I} \in \mathcal{S}_n$ and $SD_{n,I}$ is non-powerful.

(2) $b(SD_{n,I}) = 2n - 1$.

Proof (1) is easy to verify by Theorem 5 and the definitions.

(2) It is obvious that $b(SD_{n,I}) \le 2n - 1$ by Theorem 3. On the other hand, there are no $SSSD$ walks of length $2n - 2$ from n to n, so $b(SD_{n,I}) \ge 2n - 1$. Combining the two inequalities, we obtain $b(SD_{n,I}) = 2n - 1$. $\square$

In the rest of the paper, for an undirected walk W of graph G and two vertices x, y on W, let $Q_W(x \to y)$ be the shortest path from x to y on W. Let $Q(x \to y)$ be the shortest path from x to y on G. For a cycle C, if x and y are two (not necessarily distinct) vertices on C and P is a path from x to y along C, then $C \backslash P$ denotes the path or cycle from x to y along C obtained by deleting the edges of P.

Lemma 9 *Let $n \ge 3$, $S = (V, A)$ be a signed digraph with $D_{n,I}$ as its underlying digraph. If there exists a negative 2-cycle in S, then $b(S) \le 2n - 2$.*

Proof Let $(i, i) \in A$ $(2 \le i \le n)$, say $C_1^{(i)}$, $(j, j+1)$ and $(j+1, j)$ $(1 \le j \le n - 1)$ be two arcs of a negative 2-cycle, say C_2. Then $2C_1^{(1)}$ $(2C_1^{(i)})$ and C_2 have the same length 2 and different signs in S.

Let x and y be any two (not necessarily distinct) vertices in V. We will show there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y. Since vertex 1 (or i) is on the loop $C_1^{(1)}$ (or $C_1^{(i)}$), we only need to show there exists a pair of $SSSD$ walks, say W_1, W_2, of length $l(W_1) = l(W_2) \le 2n - 2$ from x to y where both W_1 and W_2 meet vertex 1 (or i).

Case 1 $i \le j$.

Subcase 1.1 $1 \le x \le i$ and $1 \le y \le n$.

Set $W = Q(x \to i) + Q(i \to j) + Q(j \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then for $k = 1, 2$,

$$l(W_k) = (i - x) + (j - i) + |j - y| + 2 = \begin{cases} 2j - x - y + 2 \leq 2n - 2, & \text{if } 1 \leq y \leq j; \\ y - x + 2 \leq 2n - 2, & \text{if } j + 1 \leq y \leq n. \end{cases}$$

Subcase 1.2 $i < x \leq j$ and $1 \leq y \leq i$.

Set $W = Q(x \to j) + Q(j \to i) + Q(i \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then $l(W_1) = l(W_2) = (j - x) + (j - i) + (i - y) + 2 = 2j - x - y + 2 \leq 2(n - 1) - 3 - 1 + 2 = 2n - 4$.

Subcase 1.3 $i < x \leq j$ and $i < y \leq j$.

Set

$$W = \begin{cases} Q(x \to i) + Q(i \to j) + Q(j \to y), & \text{if } x \leq y; \\ Q(x \to j) + Q(j \to i) + Q(i \to y), & \text{otherwise}, \end{cases}$$

$W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then

$$l(W_1) = l(W_2) = \begin{cases} (x - i) + (j - i) + (j - y) + 2 \leq 2n - 4, & \text{if } x \leq y; \\ (j - x) + (j - i) + (y - i) + 2 < 2n - 4, & \text{otherwise}. \end{cases}$$

Subcase 1.4 $i < x \leq j$ and $j + 1 \leq y \leq n$.

Set $W = Q(x \to i) + Q(i \to j) + Q(j \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then $l(W_1) = l(W_2) = (x - i) + (j - i) + (y - j) + 2 = x + y - 2i + 2 \leq n - 1 + n - 4 + 2 = 2n - 3$.

Subcase 1.5 $j + 1 \leq x \leq n$ and $1 \leq y \leq n$.

Set $W = Q(x \to j) + Q(j \to i) + Q(i \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then for $k = 1, 2$,

$$l(W_k) = (x - j) + (j - i) + |i - y| + 2 = \begin{cases} x - y + 2 \leq 2n - 2, & \text{if } 1 \leq y \leq i; \\ x + y - 2i + 2 \leq 2n - 2, & \text{if } i + 1 \leq y \leq n. \end{cases}$$

Case 2 $i \geq j + 1$.

Subcase 2.1 $j = 1$ and $2 \leq i \leq n$.

Subcase 2.1.1 $2 \leq x \leq n$, $y = n$ and $x = n, 2 \leq y \leq n$.

Set $W = Q(x \to 2) + Q(2 \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then $l(W_1) = l(W_2) = (x - 2) + (y - 2) + 2 = x + y - 2 \leq 2n - 2$.

Subcase 2.1.2 Otherwise.

Set $W = Q(x \to 1) + Q(1 \to y)$, $W_1 = W + 2C_1^{(1)}$, and $W_2 = W + C_2$. Then $l(W_1) = l(W_2) = (x - 1) + (y - 1) + 2 = x + y \leq 2n - 2$.

Subcase 2.2 $2 \leq j \leq n - 1$ and $3 \leq j + 1 \leq i \leq n$.

Subcase 2.2.1 $1 \leq x \leq j$, $1 \leq y \leq j$.

Set

$$W = \begin{cases} Q(x \to 1) + Q(1 \to j) + Q(j \to y), & \text{if } x \leq y; \\ Q(x \to j) + Q(j \to 1) + Q(1 \to y), & \text{otherwise}, \end{cases}$$

$W_1 = W + 2C_1^{(1)}$, and $W_2 = W + C_2$. Then for $k = 1, 2$,

$$l(W_k) = \begin{cases} (x-1) + (j-1) + (j-y) + 2 = 2j + x - y \leq 2j \leq 2n - 2, & \text{if } x \leq y; \\ (j-x) + (j-1) + (y-1) + 2 = 2j + y - x < 2j \leq 2n - 2, & \text{otherwise.} \end{cases}$$

Subcase 2.2.2 $1 \leq x \leq j, j + 1 \leq y \leq n$.

Set $W = Q(x \to j) + Q(j \to i) + Q(i \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then for $k = 1, 2$,

$$l(W_k) = (j-x) + (i-j) + |i-y| + 2 = \begin{cases} 2i - x - y + 2 \leq 2n - 2, & \text{if } j + 1 \leq y \leq i; \\ y - x + 2 \leq 2n - 2, & \text{if } i + 1 \leq y \leq n. \end{cases}$$

Subcase 2.2.3 $j + 1 \leq x \leq i, 1 \leq y \leq j$.

Set $W = Q(x \to i) + Q(i \to j) + Q(j \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then $l(W_1) = l(W_2) = (i-x) + (i-j) + (j-y) + 2 = 2i - x - y + 2 \leq 2n - 2$.

Subcase 2.2.4 $j + 1 \leq x \leq i, j + 1 \leq y \leq i$.

Set

$$W = \begin{cases} Q(x \to j) + Q(j \to i) + Q(i \to y), & \text{if } x \leq y; \\ Q(x \to i) + Q(i \to j) + Q(j \to y), & \text{otherwise,} \end{cases}$$

$W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then for $k = 1, 2$,

$$l(W_k) = \begin{cases} (x-j) + (i-j) + (i-y) + 2 = 2i - 2j + x - y + 2 \leq 2n - 2, & \text{if } x \leq y; \\ (i-x) + (i-j) + (y-j) + 2 = 2i - 2j + y - x + 2 < 2n - 2, & \text{otherwise.} \end{cases}$$

Subcase 2.2.5 $j + 1 \leq x \leq i, i + 1 \leq y \leq n$.

Set $W = Q(x \to j) + Q(j \to i) + Q(i \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then $l(W_1) = l(W_2) = (x-j) + (i-j) + (y-i) + 2 = x + y - 2j + 2 \leq 2n - 3$.

Subcase 2.2.6 $i + 1 \leq x \leq n, 1 \leq y \leq n$.

Set $W = Q(x \to i) + Q(i \to j) + Q(j \to y)$, $W_1 = W + 2C_1^{(i)}$, and $W_2 = W + C_2$. Then for $k = 1, 2$,

$$l(W_k) = (x-i) + (i-j) + |j-y| + 2 = \begin{cases} x - y + 2 \leq 2n - 2, & \text{if } 1 \leq y \leq j; \\ x + y - 2j + 2 \leq 2n - 2, & \text{if } j + 1 \leq y \leq n. \end{cases}$$

From the above arguments, there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y, so we have $b(S) \leq 2n - 2$ by Theorem 6. $\square$

Lemma 10 *Let $S = (V, A) \in \mathcal{S}_n$. If S has at least one loop and $b(S) = 2n - 1$, then S has no negative 2-cycles.*

Proof Suppose S has at least a negative 2-cycle, denoted by C_2. Since S has at least one loop, let $(v, v) \in A$, denoted by $C_1^{(v)}$. Let x and y be any two (not necessarily distinct) vertices in V.

Case 1 Every 2-cycle is negative.

Let P_1 be the shortest path from x to v, P_2 the shortest path from y to v.

Subcase 1.1 There exists x (or y) satisfying $l(P_1) = n - 1$ (or $l(P_2) = n - 1$). Without loss of generality, let $l(P_1) = n - 1$.

Since $b(S) = 2n - 1$, D is not isomorphic to G_1 where D is the underlying digraph of S. But $S \in \mathcal{S}_n$, $l(P_1) = n - 1$, and the terminal vertex (or the initial vertex) is the loop vertex v, so there exists a set I where $\{1\} \subset I \subseteq \{1, 2, \ldots, n\}$ such that D is isomorphic to $D_{n,I}$. Thus $b(S) \leq 2n - 2$ by Lemma 9, a contradiction.

Subcase 1.2 For all vertices x, y, $l(P_1) \leq n - 2$ and $l(P_2) \leq n - 2$.

Let $W_1 = P_1 + 2C_1^{(v)} + P_2$, and $W_2 = P_1 + C_2 + P_2$. Then W_1, W_2 are a pair of $SSSD$ walks from x to y with the length $l(W_1) = l(W_2) \leq 2n - 2$. Then $b(S) \leq 2n - 2$ by Theorem 6, a contradiction.

Case 2 There exists at least a positive 2-cycle.

Assume that u is contained in a positive 2-cycle C_2' and a negative 2-cycle C_2. Let P be the shortest path from v to u. Suppose there are k vertices on P where $k \geq 1$. Let $P_1(P_2)$ be the shortest path from $x(y)$ to P and $P_1(P_2)$ intersect P at $x'(y')$ where $0 \leq l(P_i) \leq n - k$ $(i = 1, 2)$.

Subcase 2.1 There exists x (or y) satisfying $l(P_1) = n - k$ (or $l(P_2) = n - k$).

Without loss of generality, we suppose $l(P_1) = n - k$. In this case, we have $x' = u$ because there are k vertices on P and u is contained at least two 2-cycles, and thus we obtain a contradiction by the same proof of Subcase 1.1.

Subcase 2.2 For any vertices x, y, $l(P_1) < n - k$ and $l(P_2) < n - k$.

Set $a = l(Q_P(x' \to v))$, and $b = l(Q_P(y' \to v))$,

$$W = \begin{cases} P_1 + Q_P(x' \to v) + P + Q_P(u \to y') + P_2, & \text{if } a \leq b; \\ P_1 + Q_P(x' \to u) + P + Q_P(v \to y') + P_2, & \text{otherwise.} \end{cases}$$

Let $W_1 = W + C_2'$, $W_2 = W + C_2$. Then W_1 and W_2 are a pair of $SSSD$ walks from x to y with length $l(W_1) = l(W_2) \leq 2(n - k - 1) + 2(k - 1) + 2 = 2n - 2$. Therefore there exists $SSSD$ walks of length $2n - 2$ from x to y, and thus $b(S) \leq 2n - 2$ by Theorem 6, a contradiction. $\square$

Lemma 11 Let $S = (V, A) \in \mathcal{S}_n$ and S has at least one loop. If $b(S) = 2n - 1$, then S has no negative even cycles.

Proof Suppose S has at least a negative even cycle, denoted by C with length k. Clearly, all 2-cycles are positive in S by Lemma 10 and $k \geq 4$. Since S has at least one loop, let $(v, v) \in A$, denoted by $C_1^{(v)}$. Then $kC_1^{(v)}$ and C have the same length k and different signs. Let x and y be any two (not necessarily distinct) vertices in V.

Case 1 $v \in C$.

Let $P_1(P_2)$ be the shortest path from $x(y)$ to C and let $P_1(P_2)$ intersect C at $x'(y')$ where $0 \leq l(P_i) \leq n - k$ $(i = 1, 2)$.

Subcase 1.1 $v \in Q_C(x' \to y')$.

Set $W = P_1 + Q_C(x' \to y') + P_2$, $W_1 = W + kC_1^{(v)}$, and $W_2 = W + C$. Then W_1 and W_2 are a pair of $SSSD$ walks from x to y with length $l(W_1) = l(W_2) \leq 2(n-k) + \frac{k}{2} + k = 2n - \frac{k}{2} \leq 2n - 2$. Therefore there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y.

Subcase 1.2 $v \in C \setminus Q_C(x' \to y')$.

Set $a = l(Q_C(x' \to v))$, $b = l(Q_C(y' \to v))$,

$$W_1 = \begin{cases} P_1 + Q_C(x' \to v) + Q_C(v \to x') + Q_C(x' \to y') + P_2, & \text{if } a \leq b; \\ P_1 + Q_C(x' \to y') + Q_C(y' \to v) + Q_C(v \to y') + P_2, & \text{otherwise} \end{cases}$$

and $W_2 = P_1 + C \setminus Q_C(x' \to y') + P_2$.

So $l(W_1)$ and $l(W_2)$ have the same parity since $l(Q_C(x' \to y'))$ and $l(Q_C \setminus Q_C(x' \to y'))$ have the same parity, and $\text{sgn} \, Q_C(x' \to y') = -\text{sgn} \, C \setminus Q_C(x' \to y')$. Thus $\text{sgn} \, W_1 = -\text{sgn} \, W_2$ by $\text{sgn} \, C = -1$ and all 2-cycles in S are positive. Therefore there exists a pair of $SSSD$ walks from x to y with length $\leq \max\{l(W_1), l(W_2)\} \leq 2(n-k) + k = 2n - k < 2n - 2$, and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y since both W_1 and W_2 meet the loop vertex v.

Case 2 $v \notin C$.

Let P be the shortest path from v to C and P intersect C at v'. By Theorem 7, there exist $x' \in C_1^{(v)}, y' \in C$ or $x' \in C, y' \in C_1^{(v)}$ such that (2) holds. Without loss of generality, suppose there exist $x' \in C_1^{(v)}, y' \in C$ such that (2) holds. For convenience, let P_1 be the shortest path from x to x' and P_2 be the shortest path from y to y'.

Set $W_1 = P_1 + P + Q_C(v' \to y') + P_2$, $W_2 = P_1 + P + C \setminus Q_C(v' \to y') + P_2$. Then $\text{sgn} \, W_1 = -\text{sgn} \, W_2$ and $l(W_1)$ and $l(W_2)$ have the same parity with $l(W_1) \leq l(W_2) \leq 2(n - k - 1 + 1) + \frac{k}{2} + k = 2n - \frac{k}{2} \leq 2n - 2$ by (2). So $W_2, W_3 = W_1 + (l(W_2) - l(W_1))C_1^{(v)}$ are a pair of $SSSD$ walks from x to y and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y.

From the above arguments, we have $b(S) \leq 2n-2$ by Theorem 6, contradicting $b(S) = 2n-1$. $\square$

Lemma 12 *Let $S = (V, A) \in \mathcal{S}_n$. Suppose that all 2-cycles are positive, S has a loop $C_1^{(v)}$ and an odd cycle C with length $k (\geq 3)$ such that $\text{sgn} \, C_1^{(v)} \times \text{sgn} \, C = -1$. Then $b(S) \leq 2n - 2$.*

Proof Let x and y be any two (not necessarily distinct) vertices in V. We consider the following two cases.

Case 1 $\text{sgn} \, C_1^{(v)} = 1$ and $\text{sgn} \, C = -1$.

Subcase 1.1 $v \notin C$.

Let P, P_1, P_2, v', x', y' be defined as in Case 2 of Lemma 11.

Set $W_1 = P_1 + [l(C \setminus Q_C(v' \to y')) - l(Q_C(v' \to y'))]C_1^{(v)} + P + Q_C(v' \to y') + P_2$, $W_2 = P_1 + P + C \setminus Q_C(v' \to y') + P_2$. Then $\text{sgn} \, C \setminus Q_C(v' \to y') = -\text{sgn} \, Q_C(v' \to y')$ by all 2-cycles are positive in S and $\text{sgn} \, C = -1$, and thus $\text{sgn} \, W_1 = -\text{sgn} \, W_2$ by $\text{sgn} \, C_1^{(v)} = 1$.

So W_1, W_2 are a pair of $SSSD$ walks from x to y with length $l(W_1) = l(W_2) \le 2(n - k - 1 + 1) + \frac{k-1}{2} + k = 2n - \frac{k+1}{2} \le 2n - 2$ by (2). Therefore there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y.

Subcase 1.2 $v \in C$.

Let $P_1(P_2)$ be the shortest path from $x(y)$ to C and $P_1(P_2)$ intersect C at $x'(y')$ where $0 \le l(P_i) \le n - k$ $(i = 1, 2)$.

Subcase 1.2.1 $v \in Q_C(x' \to y')$.

Set $a = l(Q_C(x' \to v))$, $b = l(Q_C(y' \to v))$, $W_1 = P_1 + Q_C(x' \to y') + P_2$, and

$$W_2 = \begin{cases} P_1 + Q_C(x' \to v) + Q_C(v \to x') + C \setminus Q_C(x' \to y') + P_2, & \text{if } a \le b; \\ P_1 + C \setminus Q_C(x' \to y') + Q_C(y' \to v) + Q_C(v \to y') + P_2, & \text{otherwise.} \end{cases}$$

It is easy to see that $\operatorname{sgn} Q_C(x' \to y') = -\operatorname{sgn} C \setminus Q_C(x' \to y')$ and thus $\operatorname{sgn} W_1 = -\operatorname{sgn} W_2$. So W_2, $W_3 = W_1 + (l(W_2) - l(W_1))C_1^{(v)}$ are a pair of $SSSD$ walks from x to y with length $l(W_2) \le 2(n - k) + k = 2n - k < 2n - 2$, and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y.

Subcase 1.2.2 $v \in C \setminus Q_C(x' \to y')$.

It is similar to Subcase 1.2.1.

Case 2 $\operatorname{sgn} C_1^{(v)} = -1$ and $\operatorname{sgn} C = 1$.

Subcase 2.1 $v \notin C$.

Let P, P_1, P_2, v', x', y' be defined as in Case 2 of Lemma 11.

Set $W_1 = P_1 + P + C \setminus Q_C(v' \to y') + P_2$, $W_2 = P_1 + [l(C \setminus Q_C(v' \to y')) - l(Q_C(v' \to y'))]C_1^{(v)} + P + Q_C(v' \to y') + P_2$.

Because $\operatorname{sgn} C = 1$, we have $\operatorname{sgn} C \setminus Q_C(v' \to y')) = \operatorname{sgn} Q_C(v' \to y')$. Because $\operatorname{sgn} C_1^{(v)} = -1$ and $l(C \setminus Q_C(v' \to y')) - l(Q_C(v' \to y'))$ is odd, W_1, W_2 are a pair of $SSSD$ walks from x to y with length $l(W_1) = l(W_2) \le 2(n - k - 1 + 1) + \frac{k-1}{2} + k = 2n - \frac{k+1}{2} \le 2n - 2$ by (2). Thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y.

Subcase 2.2 $v \in C$.

Let $P_1(P_2)$ be the shortest path from $x(y)$ to C and $P_1(P_2)$ intersect C at $x'(y')$ where $0 \le l(P_i) \le n - k$ $(i = 1, 2)$.

Subcase 2.2.1 $v \in Q_C(x' \to y')$.

Set $a = l(Q_C(x' \to v))$, $b = l(Q_C(y' \to v))$, $W_1 = P_1 + Q_C(x' \to y') + P_2 + [l(C \setminus Q_C(v' \to y')) - l(Q_C(v' \to y'))]C_1^{(v)}$, and

$$W_2 = \begin{cases} P_1 + Q_C(x' \to v) + Q_C(v \to x') + C \setminus Q_C(x' \to y') + P_2, & \text{if } a \le b; \\ P_1 + C \setminus Q_C(x' \to y') + Q_C(y' \to v) + Q_C(v \to y') + P_2, & \text{otherwise.} \end{cases}$$

Clearly, $l(W_1)$, $l(W_2)$ have the same parity. Similar to the Subcase 2.1, we have $\operatorname{sgn} W_1 = -\operatorname{sgn} W_2$. So W_2, $W_3 = W_1 + (l(W_2) - l(W_1))C_1^{(v)}$ are a pair of $SSSD$ walks from x to y with

length $l(W_2) \leq 2(n-k) + k + \frac{k-1}{2} = 2n - \frac{k+1}{2} \leq 2n - 2$. Thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y.

Subcase 2.2.2 $v \in C \setminus Q_C(x' \to y')$.

It is similar to Subcase 2.2.1.

From the above arguments, we have $b(S) \leq 2n - 2$ by Theorem 6. $\square$

Let $T_{n,3,2}$ be a tree on n vertices, where T has three pendant vertices, and two loops are on the two pendant vertices, respectively. Clearly, $T_{n,3,2}$ is a primitive symmetric digraph.

Lemma 13 *Let $ST_{n,3,2}$ be a signed digraph with $T_{n,3,2}$ as its underlying digraph, where all 2-cycles in $ST_{n,3,2}$ are positive, and the only two loops satisfy that one is positive and the other is negative. Then*

(1) $ST_{n,3,2} \in \mathcal{S}_n$ *and $ST_{n,3,2}$ is non-powerful.*

(2) $b(ST_{n,3,2}) = 2n - 1$.

Proof (1) is easy to verify by Theorem 5 and the definitions.

(2) It is obvious that $b(ST_{n,3,2}) \leq 2n - 1$ by Theorem 3. On the other hand, there are no $SSSD$ walks of length $2n - 2$ from u to u where u is the pendant vertex which is not loop vertex, so $b(ST_{n,3,2}) \geq 2n - 1$. Combining the two inequalities, we obtain $b(ST_{n,3,2}) = 2n - 1$. $\square$

Theorem 14 *Suppose $S \in \mathcal{S}_n$, and S has at least one loop. Then $b(S) = 2n - 1$ if and only if S is one of the following signed digraphs:*

(1) $SD_{n,I}$, *where $\{1\} \subset I \subseteq \{1, 2, \ldots, n\}$;*

(2) $ST_{n,3,2}$.

Proof Sufficiency is easy by Lemmas 8 and 13.

Necessity. Since S has at least one loop and $b(S) = 2n-1$, all even cycles in S are positive by Lemma 11, and S must be non-powerful. Thus there exists a distinguished cycle pair satisfying (A_2) of Theorem 5.

Let C_l and C_k be a distinguished cycle pair satisfying (A_2) of Theorem 5. Then l, k are odd, and $\operatorname{sgn} C_l = -\operatorname{sgn} C_k$.

Case 1 $l, k \geq 3$.

Then C_1 and C_l or C_1 and C_k is a distinguished cycle pair where C_1 is any loop in S, and thus we have $b(S) \leq 2n - 2$ by Lemma 12, leading to a contradiction.

Case 2 $l = 1$, $k \geq 3$ or $l \geq 3$, $k = 1$.

Then we have $b(S) \leq 2n - 2$ by Lemma 12, leading to a contradiction.

Case 3 $l = k = 1$.

Then there exist two loops C_1 and C_1' such that $\operatorname{sgn} C_1 = -\operatorname{sgn} C_1'$ where $V(C_1) = v_1$ and $V(C_1') = v_k$. Let $P = v_1 v_2 \cdots v_k$ be the shortest path from C_1 to C_1'.

Since $b(S) = 2n - 1$, there exist two vertices x, y such that there are no $SSSD$ walks of

length $2n - 2$ from x to y. Let $P_1(P_2)$ be the shortest path from $x(y)$ to P and $P_1(P_2)$ intersect P at $x'(y')$ where $0 \leq l(P_i) \leq n - k$ $(i = 1, 2)$.

Now we prove $l(P_1) = n - k$.

If $k = n$, $l(P_1) = n - k$ holds clearly. If $k < n$, we suppose $l(P_1) \leq n - k - 1$, and set $a = l(Q_P(x' \to v_1))$, $b = l(Q_P(y' \to v_1))$,

$$W = \begin{cases} P_1 + Q_P(x' \to v_1) + P + Q_P(v_k \to y') + P_2, & \text{if } a \leq b; \\ P_1 + Q_P(x' \to v_k) + P + Q_P(v_1 \to y') + P_2, & \text{otherwise} \end{cases}$$

and $W_1 = W + C_1$, $W_2 = W + C_1'$.

Then $l(W_1) = l(W_2) \leq (n - k - 1) + 2(k - 1) + (n - k) + 1 = 2n - 2$, and $\operatorname{sgn} W_1 = -\operatorname{sgn} W_2$. So W_1, W_2 are a pair of $SSSD$ walks from x to y and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y. It is a contradiction.

Therefore $l(P_1) = n - k$. Similarly, $l(P_2) = n - k$ and $x = y$.

Now we show $x' = y'$. If $x' \neq y'$, suppose $a < b$ where a, b defined as above, then $l(Q_P(x' \to v_1)) + l(Q_P(v_k \to y')) < k - 1$. Let W, W_1, W_2 be defined as above. We have $l(W_1) = l(W_2) < 2n - 1$, and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y. It is a contradiction. So $x' = y'$.

Notice that S has no more arcs except loops because P is the shortest path from v_1 to v_k and P_1 is the shortest path from x to P. So we consider the following two cases.

Subcase 3.1 If $x' = y' = v_1$ or $x' = y' = v_k$.

Without loss of generality, we let $x' = y' = v_k$. In this case, the length of the longest path of S is $n - 1$ and S has at least two loops. One loop is on the vertex v_1 which is the initial vertex (or terminal vertex) of the longest path, and the other loop is on the v_k where $2 \leq k \leq n$.

If S has at least three loops, let $C_1^{(w)}$ be a loop on the vertex w. If $\operatorname{sgn} C_1' = -\operatorname{sgn} C_1^{(w)}$, then $C_1', C_1^{(w)}$ is a distinguished cycle pair, and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y, a contradiction. So $\operatorname{sgn} C_1' = \operatorname{sgn} C_1^{(w)}$.

Combining the above arguments, we see that there exists set I such that $\{1\} \subset I \subseteq \{1, 2, \ldots, n\}$, S is some signed digraph $SD_{n,I}$.

Subcase 3.2 If $x' = y' \neq v_1$ and $x' = y' \neq v_k$.

In this case, the underlying digraph of S is isomorphic to the tree $T_{n,3,2}$. If S has at least three loops, let $C_1^{(w)}$ be a loop on the vertex w. Then $\operatorname{sgn} C_1 = -\operatorname{sgn} C_1^{(w)}$ or $\operatorname{sgn} C_1' = -\operatorname{sgn} C_1^{(w)}$, and thus there exists a pair of $SSSD$ walks of length $2n - 2$ from x to y, a contradiction. So S has no more loops and thus S is some signed digraph $ST_{n,3,2}$.

Combining the two cases, we get the desired conclusion. $\square$

By Theorems 4 and 14, we can characterize the primitive symmetric signed digraphs with the second maximum base as follows.

Theorem 15 *Suppose $S \in \mathcal{S}_n$. Then $b(S) = 2n - 1$ if and only if S is one of the following signed digraphs:*

(1) $SD_{n,I}$, where $\{1\} \subset I \subseteq \{1, 2, \ldots, n\}$;

(2) $ST_{n,3,2}$.

(3) SD_l, where $3 \le l \le n$ and l is odd.

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