

G-Frame Representation and Invertibility of G-Bessel Multipliers

A. ABDOLLAHI¹, E. RAHIMI^{2,*}

1. *Department of Mathematics, College of Sciences, Shiraz University, Shiraz 71454, Iran;*

2. *Department of Mathematics, Shiraz Branch, Islamic Azad University, Shiraz, Iran*

Abstract In this paper we show that every g-frame for an infinite dimensional Hilbert space \mathcal{H} can be written as a sum of three g-orthonormal bases for \mathcal{H} . Also, we prove that every g-frame can be represented as a linear combination of two g-orthonormal bases if and only if it is a g-Riesz basis. Further, we show each g-Bessel multiplier is a Bessel multiplier and investigate the inversion of g-frame multipliers. Finally, we introduce the concept of controlled g-frames and weighted g-frames and show that the sequence induced by each controlled g-frame (resp., weighted g-frame) is a controlled frame (resp., weighted frame).

Keywords g-frames; g-orthonormal basis; controlled g-frames; weighted g-frames; g-frame multipliers.

MR(2010) Subject Classification 42C15; 41A58

1. Introduction

Frames for a separable Hilbert space were first introduced in 1952 by Duffin and Schaeffer [11]. In [14], a generalization of the frame concept was introduced. Sun introduced g-frames and g-Riesz bases in a complex Hilbert space and discussed some properties of them. G-frames and g-Riesz bases in complex Hilbert spaces have some properties similar to those of frames, Riesz bases, but not all the properties are similar [14]. In this paper we generalize some results in [5–7], from frame theory to g-frames.

Throughout this paper, \mathcal{H} and \mathcal{K} are separable Hilbert spaces and $\{\mathcal{H}_i\}_{i \in J} \subseteq \mathcal{K}$ is a sequence of separable Hilbert spaces, where J is a subset of \mathbb{Z} , $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from \mathcal{H} to \mathcal{H}_i . For each sequence $\{\mathcal{H}_i\}_{i \in J}$, we define the space $(\bigoplus_{i \in J} \mathcal{H}_i)_{l_2}$ by

$$\left(\bigoplus_{i \in J} \mathcal{H}_i\right)_{l_2} = \left\{ \{f_i\}_{i \in J} : f_i \in \mathcal{H}_i, i \in J \text{ and } \sum_{i \in J} \|f_i\|^2 < \infty \right\}.$$

With the inner product defined by

$$\langle \{f_i\}, \{g_i\} \rangle = \sum_{i \in J} \langle f_i, g_i \rangle,$$

it is clear that $(\bigoplus_{i \in J} \mathcal{H}_i)_{l_2}$ is a Hilbert space.

Received June 25, 2012; Accepted November 22, 2012

* Corresponding author

E-mail address: abdollahi@shirazu.ac.ir (A. ABDOLLAHI); rahimie@shirazu.ac.ir (E. RAHIMI)

A frame for a complex Hilbert space \mathcal{H} is a family of vectors $\{f_i\}_{i \in J}$ so that there are two positive constants A and B satisfying

$$A\|f\|^2 \leq \sum_{i \in J} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}.$$

The constants A and B are called lower and upper frame bounds. The frame possesses many nice properties which makes it very useful in wavelet analysis and many other fields. We refer to [1, 2, 8–10, 15].

The notion of frames has been generalized to *g*-frames by Sun [14] in the following way: A sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is called a generalized frame, or simply a *g*-frame, for \mathcal{H} with respect to $\{\mathcal{H}_i\}_{i \in J}$ if there exist two positive constants A and B such that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i \in J} \|\Lambda_i f\|^2 \leq B\|f\|^2.$$

The constants A and B are called the lower and upper *g*-frame bounds, respectively. The supremum of all such A and the infimum of all such B are called the optimal bounds. If $A = B$ we call this *g*-frame a tight *g*-frame and if $A = B = 1$ it is called a normalized tight *g*-frame. We say simply a *g*-frame for \mathcal{H} , and denote it by $\{\Lambda_i\}_{i \in J}$, whenever the space sequence \mathcal{H}_i and the index set J are clear. If we only have the upper bound, we call $\{\Lambda_i\}_{i \in J}$ a *g*-Bessel sequence with bound B . We say that $\{\Lambda_i\}_{i \in J}$ is *g*-complete, if $\bigcap_{i \in J} \{f : \Lambda_i f = 0\} = \{0\}$ and is called *g*-orthonormal basis for \mathcal{H} , if

$$\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in J, \quad g_i \in \mathcal{H}_i, \quad g_j \in \mathcal{H}_j,$$

and

$$\sum_{i \in J} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in \mathcal{H}$$

We say that $\{\Lambda_i\}_{i \in J}$ is a *g*-Riesz basis for \mathcal{H} , if it is *g*-complete and there exist constants $0 < A \leq B < \infty$, such that for any finite subset $I \subseteq J$ and $g_i \in \mathcal{H}_i, i \in I$,

$$A \sum_{i \in I} \|g_i\|^2 \leq \left\| \sum_{i \in I} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i \in I} \|g_i\|^2.$$

In [14], for the *g*-frame $\{\Lambda_i\}_{i \in J}$, the *g*-frame operator S is defined by

$$S : \mathcal{H} \rightarrow \mathcal{H}, \quad Sf = \sum_{i \in J} \Lambda_i^* \Lambda_i f,$$

which is a bounded, self-adjoint, positive and invertible operator and

$$A\|f\| \leq \|Sf\| \leq B\|f\|.$$

The canonical dual *g*-frame for $\{\Lambda_i\}_{i \in J}$ is defined by $\{\widetilde{\Lambda}_i\}_{i \in J}$, where $\widetilde{\Lambda}_i = \Lambda_i S^{-1}$, which is also a *g*-frame for \mathcal{H} with $\frac{1}{B}$ and $\frac{1}{A}$ as its lower and upper *g*-frame bounds, respectively. Also every $f \in \mathcal{H}$ has an expansion

$$f = \sum_{i \in J} S^{-1} \Lambda_i^* \Lambda_i f = \sum_{i \in J} \Lambda_i^* \Lambda_i S^{-1} f.$$

Let $\{\Lambda_i\}_{i \in J}$ be a sequence in $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$, $\{e_{i,k} : k \in K_i\}$ be an orthonormal basis for \mathcal{H}_i , $i \in J$ where K_i is a subset of \mathbb{Z} and let $\psi_{i,k} = \Lambda_i^* e_{i,k}$. We have $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$. We call $\{\psi_{i,k} : i \in J, k \in K_i\}$ the sequence induced by $\{\Lambda_i\}_{i \in J}$ with respect to $\{e_{i,k} : k \in K_i\}$.

In order to present the main results of this paper, we need the following theorem that describes the relationship between frames (resp., Bessel sequences, tight frames, Riesz bases, orthonormal basis) and g-frame (resp., g-Bessel sequence, tight g-frame, g-Riesz bases, g-orthonormal bases), which can be found in [14].

Theorem 1.1 *Let $\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$, and $\psi_{i,k}$ be defined as above. Then we have the followings:*

(i) $\{\Lambda_i\}_{i \in J}$ is a g-frame (resp., g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis) for \mathcal{H} if and only if $\{\psi_{i,k} : i \in J, k \in K_i\}$ is a frame (resp., Bessel sequence, tight frame, Riesz basis, orthonormal basis) for \mathcal{H} .

(ii) The g-frame operator for $\{\Lambda_i\}_{i \in J}$ coincides with the frame operator for $\{\psi_{i,k} : i \in J, k \in K_i\}$.

(iii) Moreover, $\{\Lambda_i\}_{i \in J}$ and $\{\tilde{\Lambda}_i\}_{i \in J}$ are a pair of (canonical) dual g-frames if and only if the induced sequences are a pair of (canonical) dual frames.

We call $\{\psi_{i,k} : i \in J, k \in K_i\}$ the sequence induced by $\{\Lambda_i\}_{i \in J}$ with respect to $\{e_{i,k} : k \in K_i\}$. If $\{e'_{i,k} : i \in J, k \in K_i\}$ is an orthonormal basis for \mathcal{H} and $\Theta_i f = \sum_{k \in K_i} \langle f, e'_{i,k} \rangle e_{i,k}$, then $\{\Theta_i\}_{i \in J}$ is a g-orthonormal basis for \mathcal{H} .

Given two Bessel sequences, $\Psi = (\psi_i)$ and $\Phi = (\phi_i)$, and the weight sequence $m = (m_i)$, the Bessel multiplier for these sequences is an operator defined by

$$\mathbb{M}_{m, \Psi, \Phi} f = \sum_{i \in J} m_i \langle f, \psi_i \rangle \phi_i.$$

We shorten the notation by setting $\mathbb{M}_{m, \Psi} = \mathbb{M}_{m, \Psi, \Psi}$ (see [3]). Bessel multipliers and, in particular, frame multipliers have useful applications. For example, in [4], frame multipliers are used to solve approximation problems.

The concept of Bessel multipliers can be generalized to g-Bessel as follows: For given g-Bessel sequences $\Lambda = \{\Lambda_i\}_{i \in J}$ and $\Theta = \{\Theta_i\}_{i \in J}$, and the weight sequence $m = (m_i)$, the g-Bessel multiplier is defined by

$$\mathbb{M}_{m, \Lambda, \Theta} f = \sum_{i \in J} m_i \Lambda_i^* \Theta_i f.$$

We shorten the notation by setting $\mathbb{M}_{m, \Lambda} = \mathbb{M}_{m, \Lambda, \Lambda}$. (see [13])

A sequence $(m_i : i \in J)$ is called semi-normalized if there are bounds $b \geq a > 0$, such that $a \leq |m_i| \leq b$ for all $i \in J$.

We define $GL(\mathcal{H})$ as the set of all bounded linear operators with a bounded inverse. A frame controlled by an operator $C \in GL(\mathcal{H})$ is a family of vectors $\Psi = (\psi_i \in \mathcal{H} : i \in J)$, such that there exist two constants $m_{CL} > 0$ and $M_{CL} < \infty$ satisfying

$$m_{CL} \|f\|^2 \leq \sum_{i \in J} \langle f, \psi_i \rangle \langle C \psi_i, f \rangle \leq M_{CL} \|f\|^2, \quad f \in \mathcal{H}.$$

We call

$$S_C f = \sum_{i \in J} \langle f, \psi_i \rangle C \psi_i,$$

the controlled frame operator [6].

We also generalize this concept to *g*-frames. A *g*-frame controlled by the operator $C \in GL(\mathcal{H})$ or *C*-controlled *g*-frame is a family of operators $\Lambda = \{\Lambda_i\}_{i \in J}$, such that there exist two constants $m_{CL} > 0$ and $M_{CL} < \infty$ satisfying

$$m_{CL} \|f\|^2 \leq \sum_{i \in J} \langle \Lambda_i C^* f, \Lambda_i f \rangle \leq M_{CL} \|f\|^2.$$

The paper is organized as follows. In Section 2 we show that every *g*-frame for an infinite dimensional Hilbert space \mathcal{H} can be written as a sum of three *g*-orthonormal bases for \mathcal{H} . We next show that a *g*-frame can be represented as a linear combination of two *g*-orthonormal bases if and only if it is a *g*-Riesz basis. We further show that every *g*-frame can be written as a sum of two tight *g*-frames with *g*-frame bounds one or a sum of a *g*-orthonormal basis and a *g*-Riesz basis for \mathcal{H} . In Section 3 we show each *g*-Bessel multiplier is a Bessel multiplier and investigate the inversion of *g*-frame multipliers. Also sufficient conditions for invertibility of multipliers are determined. In Section 4 we introduce controlled *g*-frames and show that the sequence induced by each controlled *g*-frame is a controlled frame and controlled *g*-frames are equivalent to standard *g*-frames. Finally, in the last section, we investigate the concept of weighted *g*-frames, and show that the sequence induced by each weighted *g*-frame is a weighted frame.

2. Some *G*-frame representations

In [7], the author has shown, using operator theory, that every frame in a Hilbert space \mathcal{H} can be written as the sum of three orthonormal bases. More precisely, if $(x_i)_{i \in J}$ is a frame for \mathcal{H} , then there exist orthonormal bases (f_i) , (g_i) and (h_i) such that $x_i = a(f_i + g_i + h_i)$, $i \in J$, for some constant a . Furthermore, the author provided an example of a tight frame $(x_i)_{i \in J}$ that cannot be written in the form $x_i = a f_i + b g_i$, $i \in J$, for any orthonormal sequences (f_i) , (g_i) and any choice of constants a and b . The author also proved related results, in particular the following one: a frame in \mathcal{H} can be written as a linear combination of two orthonormal bases if and only if it is a Riesz basis. In this section we generalize some of these results from the frame case to the *g*-frame case.

Proposition 2.1 *If $\{\Lambda_i\}_{i \in J}$ is a *g*-frame for a Hilbert space \mathcal{H} , there are *g*-orthonormal bases $\{\Upsilon_i\}$, $\{\Gamma_i\}$, $\{\Psi_i\}$ for \mathcal{H} and a constant a so that $\Lambda_i = a(\Upsilon_i + \Gamma_i + \Psi_i)$ for all $i \in J$.*

Proof Let $\{\psi_{i,k}\}$ be the sequence induced by $\{\Lambda_i\}_{i \in J}$ with respect to an orthonormal basis $\{e_{i,k} : k \in K_i\}$. Hence $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$, and so, by Corollary 2.2 of [7], there are constant a and orthonormal bases $\{f_{i,k}\}$, $\{g_{i,k}\}$, $\{h_{i,k}\}$ such that $\psi_{i,k} = a(f_{i,k} + g_{i,k} + h_{i,k})$. Since $\{f_{i,k}\}$, $\{g_{i,k}\}$, $\{h_{i,k}\}$ are orthonormal bases for \mathcal{H} , by Theorem 1.1, Υ_i , Γ_i , Ψ_i are *g*-orthonormal bases, where $\Upsilon_i f = \sum_{k \in K_i} \langle f, f_{i,k} \rangle e_{i,k}$, $\Gamma_i f = \sum_{k \in K_i} \langle f, g_{i,k} \rangle e_{i,k}$, and $\Psi_i f = \sum_{k \in K_i} \langle f, h_{i,k} \rangle e_{i,k}$.

The proof is completed by noting that $\Lambda_i = a(\Upsilon_i + \Gamma_i + \Psi_i)$ for all $i \in J$. \square

Proposition 2.2 *A g-frame $\{\Lambda_i\}_{i \in J}$ can be written as a linear combination of two g-orthonormal bases for \mathcal{H} if and only if $\{\Lambda_i\}_{i \in J}$ is a g-Riesz basis for \mathcal{H} .*

Proof Let $\{\psi_{i,k}\}$ be the sequence induced by $\{\Lambda_i\}_{i \in J}$ with respect to an orthonormal basis $\{e_{i,k} : k \in K_i\}$. Then $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$. Suppose that there are g-orthonormal bases $\{\Upsilon_i\}$, $\{\Gamma_i\}$ for \mathcal{H} and constants a, b such that $\Lambda_i = a\Upsilon_i + b\Gamma_i$ for all $i \in J$. So, by Theorem 1.1, there are orthonormal bases $\{f_{i,k}\}$, $\{g_{i,k}\}$ for \mathcal{H} such that $\Upsilon_i f = \sum_{k \in K_i} \langle f, f_{i,k} \rangle e_{i,k}$, $\Gamma_i f = \sum_{k \in K_i} \langle f, g_{i,k} \rangle e_{i,k}$. Therefore, $\psi_{i,k} = af_{i,k} + bg_{i,k}$, and Proposition 2.5 of [7] implies that $\{\psi_{i,k}\}$ is a Riesz basis. So $\{\Lambda_i\}_{i \in J}$ is a g-Riesz basis for \mathcal{H} by Theorem 1.1. Conversely, if $\{\Lambda_i\}_{i \in J}$ is a g-Riesz basis, we have $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$, where $\{\psi_{i,k}\}$ is a Riesz basis. So by Proposition 2.5 of [7], for some constants a, b , and orthonormal bases $\{f_{i,k}\}$ and $\{g_{i,k}\}$, $\psi_{i,k} = af_{i,k} + bg_{i,k}$. Hence $\Lambda_i = a\Upsilon_i + b\Gamma_i$, where Υ_i and Γ_i are g-orthonormal bases and $\Upsilon_i f = \sum_{k \in K_i} \langle f, f_{i,k} \rangle e_{i,k}$, $\Gamma_i f = \sum_{k \in K_i} \langle f, g_{i,k} \rangle e_{i,k}$. \square

Proposition 2.3 *If K is a co-isometry on \mathcal{H} , and if $\{\Theta_i\}_{i \in J}$ is a g-orthonormal basis for \mathcal{H} , then $\{\Theta_i K^* : i \in J\}$ is a normalized tight g-frame for \mathcal{H} .*

Proof Since K is a co-isometry, K^* is an isometry. Hence, for all $f \in \mathcal{H}$,

$$\sum_{i \in J} \|\Theta_i K^* f\|^2 = \|K^* f\|^2 = \|f\|^2. \quad \square$$

By the same argument as above, we obtain the following results.

Proposition 2.4 *Every g-frame is the sum of two normalized tight g-frames for \mathcal{H} .*

Proposition 2.5 *Every g-frame for a Hilbert space \mathcal{H} is the sum of a g-orthonormal basis for \mathcal{H} and a g-Riesz basis for \mathcal{H} .*

3. Invertibility of multipliers

In this section we show each g-Bessel multiplier is a Bessel multiplier and investigate the inversion of g-frame multipliers. Also sufficient conditions for invertibility of multipliers are determined. Equivalent results as proved in [3] for Bessel multiplier can be shown for g-Bessel multiplier. We prove some of them, the proof of the others follows in the same manner.

The following proposition gives the connection between the g-Bessel sequences and Bessel sequences.

Proposition 3.1 *Each g-Bessel multiplier is a Bessel multiplier.*

Furthermore, if $m \in \ell^\infty$ and $\Lambda = \{\Lambda_i\}_{i \in J}$, $\Theta = \{\Theta_i\}_{i \in J}$ are g-Bessel sequences for \mathcal{H} with bounds B_Λ, B_Θ , respectively, then the multiplier $M_{m, \Lambda, \Theta}$ is well defined on \mathcal{H} and $\|M_{m, \Lambda, \Theta}\| \leq \sqrt{B_\Lambda B_\Theta} \|m\|_\infty$.

Proof Let $\Lambda = \{\Lambda_i\}_{i \in J}$ and $\Theta = \{\Theta_i\}_{i \in J}$ be g-Bessel sequences with induced sequences $\{\psi_{i,k}\}$

and $\{\phi_{i,k}\}$, respectively. Then

$$\mathbb{M}_{m,\Lambda,\Theta} f = \sum_{i \in J} m_i \Lambda_i^* \Theta_i f = \sum_{i \in J} \sum_{k \in K_i} m_i \langle f, \phi_{i,k} \rangle \psi_{i,k} = \mathbb{M}_{m',\Phi,\Psi},$$

where $\Psi = \{\psi_{i,k} : i \in J, k \in K_i\}$, $\Phi = \{\phi_{i,k} : i \in J, k \in K_i\}$ and $m' = \{m'_{i,k} = m_i : i \in J, k \in K_i\}$. For the proof of the second part, since the bounds of $\{\psi_{i,k}\}$ and $\{\phi_{i,k}\}$ are B_Λ and B_Θ , respectively, the assertion follows by the first part and Theorem 6.1 of [3]. \square

Theorem 3.2 *Let $M_{m,\Lambda,\Theta}$ be well defined and invertible on \mathcal{H} .*

(i) *If $\Theta = \{\Theta_i\}_{i \in J}$ (resp., $\Lambda = \{\Lambda_i\}_{i \in J}$) is a *g*-Bessel sequence for \mathcal{H} with bound B_Θ , then $m\Lambda = \{m_i \Lambda_i\}_{i \in J}$ (resp., $m\Theta$) satisfies the lower *g*-frame condition for \mathcal{H} with bound $\frac{1}{B_\Theta \|M_{m,\Lambda,\Theta}^{-1}\|^2}$.*

(ii) *If Θ (resp., Λ) and $m\Lambda$ (resp., $m\Theta$) are *g*-Bessel sequences for \mathcal{H} , then they are *g*-frames for \mathcal{H} .*

(iii) *If Θ (resp., Λ) is a *g*-Bessel sequence for \mathcal{H} and $m \in \ell^\infty$, then Λ (resp., Θ) satisfies the lower *g*-frame condition for \mathcal{H} .*

(iv) *If Θ and Λ are *g*-Bessel sequences for \mathcal{H} and $m \in \ell^\infty$, then Θ and Λ are *g*-frames for \mathcal{H} ; $m\Lambda$ and $m\Theta$ are also *g*-frames for \mathcal{H} .*

Proof (i) Since $\Theta = \{\Theta_i\}_{i \in J}$ is a *g*-Bessel sequence, by Theorem 1.1, $\Theta_i f = \sum_{k \in K_i} \langle f, \phi_{i,k} \rangle e_{i,k}$, where $\{\phi_{i,k}\}$ is a Bessel sequence for \mathcal{H} with bound B_Θ , and $\mathbb{M}_{m,\Lambda,\Theta} = \mathbb{M}_{m',\Phi,\Psi}$, where $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$, $\Psi = \{\psi_{i,k} : i \in J, k \in K_i\}$, $\Phi = \{\phi_{i,k} : i \in J, k \in K_i\}$ and $m' = \{m'_{i,k} = m_i : i \in J, k \in K_i\}$. Also we have

$$\sum_{i \in J} \|m_i \Lambda_i f\|^2 = \sum_{i \in J} \sum_{k \in K_i} |\langle f, m_i \psi_{i,k} \rangle|^2.$$

By Proposition 4.3 of [5], $m'\Psi$ satisfies the lower frame condition for \mathcal{H} with bound

$$\frac{1}{B_\Theta \|M_{m',\Phi,\Psi}^{-1}\|^2} = \frac{1}{B_\Theta \|M_{m,\Lambda,\Theta}^{-1}\|^2}.$$

(ii) and (iii) follow from (i).

(iv) Let Θ and Λ be *g*-Bessel sequences for \mathcal{H} and $m \in \ell^\infty$. Then $m\Lambda$ and $m\Theta$ are also *g*-Bessel for \mathcal{H} . \square

In the following proposition we give a sufficient condition for invertibility of *g*-multipliers.

Proposition 3.3 *Let $\Lambda = \{\Lambda_i\}_{i \in J}$ be a *g*-frame for \mathcal{H} , $G : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded bijective operator and $\Theta_i = \Lambda_i G, \forall i \in J$. Let m be positive (resp., negative) semi-normalized. Then Θ is a *g*-frame for \mathcal{H} and the *g*-frame multiplier $M_{m,\Lambda,\Theta}$ is invertible on \mathcal{H} with*

$$M_{m,\Lambda,\Theta}^{-1} = \begin{cases} G^{-1} S_{(\sqrt{m_i} \Lambda_i)}^{-1}, & \text{when } m_i > 0, \forall i, \\ -G^{-1} S_{(\sqrt{|m_i|} \Lambda_i)}^{-1}, & \text{when } m_i < 0, \forall i. \end{cases} \tag{1}$$

Proof We have $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$, where $\{\psi_{i,k}\}$ is a frame for \mathcal{H} . Since $\Theta_i = \Lambda_i G$,

$$\Theta_i f = \sum_{k \in K_i} \langle Gf, \psi_{i,k} \rangle e_{i,k} = \sum_{k \in K_i} \langle f, G^* \psi_{i,k} \rangle e_{i,k}.$$

We know G^* is a bounded bijective operator, and hence $\phi_{i,k} = G^*\psi_{i,k}$ is a frame for \mathcal{H} (see [8]). So Θ is a g-frame for \mathcal{H} . By Proposition 3.1, $\mathbb{M}_{m,\Lambda,\Theta} = \mathbb{M}_{m',\Phi,\Psi}$ where $\Psi = \{\psi_{i,k} : i \in J, k \in K_i\}$, $\Phi = \{\phi_{i,k} : i \in J, k \in K_i\}$ and $m' = \{m'_{i,k} = m_i : i \in J, k \in K_i\}$. If m is positive then, by Theorem 1.1, the g-frame operator for $\{\sqrt{m_i}\Lambda_i\}_{i \in J}$ coincides with the frame operator for $\{\sqrt{m'_{i,k}}\psi_{i,k} : i \in J, k \in K_i\}$. Therefore, by Lemma 4.4 of [6], $\mathbb{M}_{m,\Lambda,\Theta} = \mathbb{M}_{m',\Phi,\Psi}$ is invertible and equation (1) holds. \square

By the same method that we use to prove Proposition 3.3, one can prove the followings.

Proposition 3.4 *Let $\Lambda = \{\Lambda_i\}_{i \in J}$ be a g-frame for \mathcal{H} and let $\Lambda^d = \{\Lambda_i^d\}_{i \in J}$ be a dual g-frame of Λ . Let $0 \leq \lambda < \frac{1}{\sqrt{B_\Lambda B_{\Lambda^d}}} (\leq 1)$ and let (m_i) be such that $1 - \lambda \leq m_i \leq 1 + \lambda$, for all $i \in J$. Then M_{m,Λ,Λ^d} and $M_{m,\Lambda^d,\Lambda}$ are invertible on \mathcal{H} ,*

$$\frac{1}{1 + \lambda\sqrt{B_\Lambda B_{\Lambda^d}}} \|h\| \leq \|M_{m,\Lambda,\Lambda^d}^{-1}h\| \leq \frac{1}{1 - \lambda\sqrt{B_\Lambda B_{\Lambda^d}}} \|h\|, \quad \forall h \in \mathcal{H}, \tag{2}$$

and the same inequalities hold for $\|M_{m,\Lambda^d,\Lambda}^{-1}h\|$. Moreover,

$$M_{m,\Lambda,\Lambda^d}^{-1} = \sum_{k=0}^{\infty} (M_{m',\Lambda,\Lambda^d})^k \text{ and } M_{m,\Lambda^d,\Lambda}^{-1} = \sum_{k=0}^{\infty} (M_{m',\Lambda^d,\Lambda})^k, \tag{3}$$

where $m' = (1 - m_i)$.

Corollary 3.5 *Let Λ be a g-frame for \mathcal{H} and $\tilde{\Lambda}$ be the canonical dual of Λ . Let $0 \leq \lambda < \sqrt{\frac{A_\Lambda}{B_\Lambda}} (\leq 1)$ and $(m_i)_{i \in J}$ be such that $1 - \lambda \leq m_i \leq 1 + \lambda$, for all $i \in J$. Then $M_{m,\Lambda,\tilde{\Lambda}}$ and $M_{m,\tilde{\Lambda},\Lambda}$ are invertible on \mathcal{H} ,*

$$\frac{1}{1 + \lambda\sqrt{B_\Lambda/A_\Lambda}} \|h\| \leq \|M_{m,\Lambda,\tilde{\Lambda}}^{-1}h\| \leq \frac{1}{1 - \lambda\sqrt{B_\Lambda/A_\Lambda}} \|h\|, \quad \forall h \in \mathcal{H},$$

and the same inequalities hold for $\|M_{m,\tilde{\Lambda},\Lambda}^{-1}h\|$. Moreover,

$$M_{m,\Lambda,\tilde{\Lambda}}^{-1} = \sum_{k=0}^{\infty} (M_{m',\Lambda,\tilde{\Lambda}})^k \text{ and } M_{m,\tilde{\Lambda},\Lambda}^{-1} = \sum_{k=0}^{\infty} (M_{m',\tilde{\Lambda},\Lambda})^k,$$

where $m' = (1 - m_i)$.

Proposition 3.6 *Let Λ be a g-frame for \mathcal{H} . Assume that $\Theta - \Lambda$ is a g-Bessel sequence for \mathcal{H} with bound $B_{\Theta - \Lambda} < \frac{A_\Lambda^2}{B_\Lambda}$. For every positive (or negative) semi-normalized sequence m , satisfying*

$$0 < a \leq |m_i| \leq b, \forall i, \text{ and } \frac{b}{a} < \frac{A_\Lambda}{\sqrt{B_{\Theta - \Lambda} B_\Lambda}},$$

it follows that Θ is a g-frame for \mathcal{H} , the multipliers $M_{m,\Lambda,\Theta}$ and $M_{m,\Theta,\Lambda}$ are invertible on \mathcal{H} ,

$$\frac{1}{bB_\Lambda + b\sqrt{B_\Lambda B_{\Theta - \Lambda}}} \|h\| \leq \|M_{m,\Lambda,\Theta}^{-1}h\| \leq \frac{1}{aA_\Lambda - b\sqrt{B_\Lambda B_{\Theta - \Lambda}}} \|h\|,$$

and the same inequalities hold for $\|M_{m,\Theta,\Lambda}^{-1}h\|$. Moreover,

$$M_{m,\Lambda,\Theta}^{-1} = \begin{cases} \sum_{k=0}^{\infty} [S_{(\sqrt{m_i}\Lambda_i)}^{-1} (S_{(\sqrt{m_i}\Lambda_i)} - M_{m,\Lambda,\Theta})]^k S_{(\sqrt{m_i}\Lambda_i)}^{-1}, & \text{if } m_i > 0, \forall i, \\ -\sum_{k=0}^{\infty} [S_{(\sqrt{|m_i|\Lambda_i})}^{-1} (S_{(\sqrt{|m_i|\Lambda_i})} + M_{m,\Lambda,\Theta})]^k S_{(\sqrt{|m_i|\Lambda_i})}^{-1}, & \text{if } m_i < 0, \forall i, \end{cases}$$

$$M_{m,\Theta,\Lambda}^{-1} = \begin{cases} \sum_{k=0}^{\infty} [S_{(\sqrt{|m_i|\Lambda_i})}^{-1} (S_{(\sqrt{|m_i|\Lambda_i})} - M_{m,\Theta,\Lambda})]^k S_{(\sqrt{|m_i|\Lambda_i})}^{-1}, & \text{if } m_i > 0, \forall i, \\ -\sum_{k=0}^{\infty} [S_{(\sqrt{|m_i|\Lambda_i})}^{-1} (S_{(\sqrt{|m_i|\Lambda_i})} + M_{m,\Theta,\Lambda})]^k S_{(\sqrt{|m_i|\Lambda_i})}^{-1}, & \text{if } m_i < 0, \forall i. \end{cases}$$

Proposition 3.7 Let Λ be a g-frame for \mathcal{H} . Assume that $\exists \mu \in [0, \frac{A_\Lambda^2}{B_\Lambda})$ such that $\sum \|(m_i \Theta_i - \Lambda_i) f\|^2 \leq \mu \|f\|^2, \forall f \in \mathcal{H}$. Then $m\Theta$ is a g-frame for \mathcal{H} , the multipliers $M_{m,\Lambda,\Theta}$ and $M_{m,\Theta,\Lambda}$ are invertible on \mathcal{H} ,

$$\frac{1}{B_\Lambda + \sqrt{\mu B_\Lambda}} \|h\| \leq \|M_{m,\Lambda,\Theta}^{-1} h\| \leq \frac{1}{A_\Lambda - \sqrt{\mu B_\Lambda}} \|h\|, \forall h \in \mathcal{H},$$

and the same inequalities hold for $\|M_{m,\Theta,\Lambda}^{-1} h\|$. Moreover,

$$M_{m,\Lambda,\Theta}^{-1} = \sum_{k=0}^{\infty} [S_\Lambda^{-1} (S_\Lambda - M_{m,\Lambda,\Theta})]^k S_\Lambda^{-1},$$

and

$$M_{m,\Theta,\Lambda}^{-1} = \sum_{k=0}^{\infty} [S_\Lambda^{-1} (S_\Lambda - M_{m,\Theta,\Lambda})]^k S_\Lambda^{-1}.$$

As a consequence, if m is semi-normalized, then Θ is also a g-frame for \mathcal{H} .

Proposition 3.8 Let Λ be a g-frame for \mathcal{H} . Assume that there exists $\mu \in [0, \frac{1}{B_\Lambda})$ such that $\sum \|(m_i \Theta_i - \Lambda_i^d) f\|^2 \leq \mu \|f\|^2$, for all $f \in \mathcal{H}$, for some dual g-frame $\Lambda^d = (\Lambda_i^d)$ of Λ . Then $m\Theta$ is a g-frame for \mathcal{H} , the bounded multipliers $M_{m,\Lambda,\Theta}$ and $M_{m,\Theta,\Lambda}$ are invertible on \mathcal{H} ,

$$\frac{1}{1 + \sqrt{\mu B_\Lambda}} \|h\| \leq \|M_{m,\Lambda,\Theta}^{-1} h\| \leq \frac{1}{1 - \sqrt{\mu B_\Lambda}} \|h\|, \forall h \in \mathcal{H},$$

and the same inequalities hold for $\|M_{m,\Theta,\Lambda}^{-1} h\|$. Moreover,

$$M_{m,\Lambda,\Theta}^{-1} = \sum_{k=0}^{\infty} (I_{\mathcal{H}} - M_{m,\Lambda,\Theta})^k \quad \text{and} \quad M_{m,\Theta,\Lambda}^{-1} = \sum_{k=0}^{\infty} (I_{\mathcal{H}} - M_{m,\Theta,\Lambda})^k.$$

As a consequence, if m is semi-normalized, then Θ is also a g-frame for \mathcal{H} .

4. Controlled G-frames

In this section we introduce the concept of controlled g-frames and we show that the sequence induced by each controlled g-frame is a controlled frame. We also show that controlled g-frames are equivalent to standard g-frames.

Definition 4.1 A g-frame controlled by an operator $C \in GL(\mathcal{H})$ or C -controlled g-frame is a family of vectors $\Lambda = \{\Lambda_i\}_{i \in J}$, such that there exist two constants $m_{CL} > 0$ and $M_{CL} < \infty$ satisfying

$$m_{CL} \|f\|^2 \leq \sum_{i \in J} \langle \Lambda_i C^* f, \Lambda_i f \rangle \leq M_{CL} \|f\|^2,$$

for all $f \in \mathcal{H}$. The controlled g-frame operator is defined by

$$S_C f = \sum_{i \in J} \Lambda_i^* \Lambda_i C^* f, \quad f \in \mathcal{H}.$$

Proposition 4.2 *The sequence induced by each controlled g-frame is a controlled frame.*

Proof Let $\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ and $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$, where $\{\psi_{i,k} : i \in J, k \in K_i\}$ is the sequence induced by $\{\Lambda_i : i \in J\}$ with respect to $\{e_{i,k} : k \in K_i\}$. Hence $\Lambda_i C^* f = \sum_{k \in K_i} \langle f, C\psi_{i,k} \rangle e_{i,k}$. Then

$$\sum_{i \in J} \langle \Lambda_i C^* f, \Lambda_i f \rangle = \sum_{i \in J} \sum_{k \in K_i} \langle f, C\psi_{i,k} \rangle \langle \psi_{i,k}, f \rangle,$$

and

$$S_C f = \sum_{i \in J} \Lambda_i^* \Lambda_i C^* f = \sum_{i \in J} \sum_{k \in K_i} \langle f, C\psi_{i,k} \rangle \psi_{i,k}. \quad \square$$

Proposition 4.3 *Let $C \in GL(\mathcal{H})$ and $\Lambda = \{\Lambda_i\}_{i \in J}$ be a C -controlled g-frame in \mathcal{H} . Then Λ is a classical g-frame. Furthermore if S is a g-frame operator, we have $SC^* = CS$ and so*

$$\sum_{i \in J} \Lambda_i^* \Lambda_i C^* f = C \sum_{i \in J} \Lambda_i^* \Lambda_i f.$$

Proof Since $\Lambda = \{\Lambda_i\}_{i \in J}$ is a C -controlled g-frame, by Proposition 4.2, we have $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$, where $\{\psi_{i,k}\}$ is a C -controlled frame. By Proposition 3.2 of [6], $\{\psi_{i,k}\}$ is a classical frame and so $\Lambda = \{\Lambda_i\}_{i \in J}$ is a classical g-frame. By Theorem 1.1, the g-frame operator for $\{\Lambda_i\}_{i \in J}$ coincides with the frame operator for $\{\psi_{i,k} : i \in J, k \in K_i\}$ and the proof is completed by using Proposition 3.2 of [6] and Proposition 4.2. \square

Proposition 4.4 *Let $C \in GL(\mathcal{H})$ be self-adjoint. The family Λ is a g-frame for \mathcal{H} controlled by C if and only if it is a (classical) g-frame for H , C is positive and commutes with the g-frame operator S .*

Proof The assertion follows from Proposition 3.3 of [6] and Propositions 4.2, 4.3. \square

Corollary 4.5 *Let C be a self-adjoint operator and Λ be a C -controlled g-frame. Denote by (m_{CS}, M_{CS}) , (m, M) and (m_C, M_C) any bounds for the controlled g-frame operator S_C , the g-frame operator S , and the operator C , respectively. Then,*

- (i) $m' = \frac{m_{CL}}{M_C}$, $M' = \frac{M_{CL}}{m_C}$ are bounds for S ;
- (ii) $m'_C = \frac{m_{CL}}{M}$, $M'_C = \frac{M_{CL}}{m}$ are bounds for C ;
- (iii) $m'_{CL} = mm_C$, $M'_{CL} = MM_C$ are bounds for S_C .

5. Weighted G-frames

In this section we investigate the concept of weighted g-frames, and show that the sequence induced by each weighted g-frame is a weighted frame.

Definition 5.1 *Let $\{\Lambda_i\}_{i \in J}$ and $(w_i : i \in J)$ be a sequence of positive weights. This pair is called a weighted g-frame or a w -g-frame of \mathcal{H} if there exist constants $m > 0$, $M < \infty$ such that*

$$m\|f\|^2 \leq \sum_{i \in J} w_i^2 \|\Lambda_i f\|^2 \leq M\|f\|^2, \quad f \in \mathcal{H}.$$

Assume now that the restriction on the weights is lifted, i.e., $(w_i) \subseteq \mathbb{C}$. Then we call $(w_i \Lambda_i)$ a weighted *g*-frame if this sequence forms a *g*-frame, i.e.,

$$m\|f\|^2 \leq \sum_{i \in J} |w_i|^2 \|\Lambda_i f\|^2 \leq M\|f\|^2, f \in \mathcal{H}.$$

Proposition 5.2 *The sequence induced by each weighted *g*-frame is a weighted frame.*

Proof If $\{w_i \Lambda_i\}_{i \in J}$ is a weighted *g*-frame, then we have

$$w_i \Lambda_i f = \sum_{k \in K_i} \langle f, w_i \psi_{i,k} \rangle e_{i,k}$$

and

$$\sum_{i \in J} |w_i|^2 \|\Lambda_i f\|^2 = \sum_{i \in J} \sum_{k \in K_i} |w_i|^2 |\langle f, \psi_{i,k} \rangle|^2.$$

So $\{w'_{i,k} \psi_{i,k} : i \in J, k \in K_i\}$ is a weighted frame with $w'_{i,k} = w_i$ for $i \in J, k \in K_i$. \square

Proposition 5.3 *Let $C \in GL(\mathcal{H})$ be self-adjoint and $\{\Lambda_i\}_{i \in J}$ be a controlled *g*-frame and assume $C \Lambda_i^* = w_i \Lambda_i^*$. Then the sequence (w_i) is semi-normalized and positive. Furthermore $C = M_{w, \Lambda, \tilde{\Lambda}}$.*

Proof We have $\Lambda_i f = \sum_{k \in K_i} \langle f, \psi_{i,k} \rangle e_{i,k}$ and so $\Lambda_i^* f_i = \sum_{k \in K_i} \langle f_i, e_{i,k} \rangle \psi_{i,k}$ for all $f_i \in \mathcal{H}_i$. Since $C \Lambda_i^* = w_i \Lambda_i^*$, it is easy to show that $C \psi_{i,k} = w_i \psi_{i,k}$. The conclusions follow from Proposition 4.2 of [6] and Propositions 3.1, 5.2. \square

The following Lemma can be proved by the same manner.

Lemma 5.4 *Let $(w_i : i \in J)$ be a semi-normalized real sequence with bounds a, b . Then if $\{\Lambda_i\}_{i \in J}$ is a *g*-frame with bounds m and M , then $\{w_i \Lambda_i\}_{i \in J}$ is also a *g*-frame with bounds $a^2 m$ and $b^2 M$. The sequence $\{w_i^{-1} \tilde{\Lambda}_i\}_{i \in J}$ is a dual *g*-frame of $\{w_i \Lambda_i\}_{i \in J}$.*

Lemma 5.5 *Let $\Lambda = \{\Lambda_i\}_{i \in J}$ be a *g*-frame and $w = (w_i : i \in J)$ be a positive semi-normalized sequence. Then the multiplier $M_{w, \Lambda}$ is the *g*-frame operator of the *g*-frame $\{\sqrt{w_i} \Lambda_i\}_{i \in J}$ and therefore it is positive, self-adjoint and invertible.*

Proof By using Lemma 5.4, $\{\sqrt{w_i} \Lambda_i\}_{i \in J}$ is a *g*-frame and if S is the *g*-frame operator of it, then

$$Sf = \sum_{i \in J} (\sqrt{w_i} \Lambda_i)^* \sqrt{w_i} \Lambda_i f = \sum_{i \in J} w_i \Lambda_i^* \Lambda_i f$$

and

$$\mathbb{M}_{w, \Lambda} f = \sum_{i \in J} w_i \Lambda_i^* \Lambda_i f.$$

Therefore, $\mathbb{M}_{w, \Lambda} = S$ is positive, self-adjoint and invertible. \square

We end this section with the following theorem, which can be proved in exactly the same fashion as above.

Theorem 5.6 *Let $\Lambda = \{\Lambda_i\}_{i \in J}$ be a sequence and $w = (w_i : i \in J)$ be a positive, semi-normalized sequence. Then the following properties are equivalent:*

- (i) $\Lambda = \{\Lambda_i\}_{i \in J}$ is a g -frame;
- (ii) $\mathbb{M}_{w, \Lambda}$ is positive, self-adjoint and invertible operator;
- (iii) There are constants $m > 0$, $M < \infty$ such that

$$m\|f\|^2 \leq \sum_{i \in J} w_i \|\Lambda_i f\|^2 \leq M\|f\|^2,$$

for all $f \in \mathcal{H}$;

- (iv) $\{\sqrt{w_i} \Lambda_i\}_{i \in J}$ is a g -frame;
- (v) $\mathbb{M}_{w', \Lambda}$ is a positive and invertible operator, for any positive, semi-normalized sequence $w' = (w'_i : i \in J)$;
- (vi) $\{w_i \Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g -frame.

Acknowledgements The authors would like to thank the referee for useful and helpful comments and suggestions.

References

- [1] R. BALAN, P. G. CASAZZA, C. HEIL, et al. *Deficits and excesses of frames*. Adv. Comput. Math., 2003, **18**(2-4): 93–116.
- [2] R. BALAN, P. G. CASAZZA, C. HEIL, et al. *Density, overcompleteness, and localization of frames, I, theory*, J. Fourier Anal. Appl., 2006, **12**(2): 105–143.
- [3] P. BALAZS. *Basic definition and properties of Bessel multipliers*. J. Math. Anal. Appl., 2007, **325**(1): 571–585.
- [4] P. BALAZS. *Hilbert Schmidt operators and frames classification, approximation by multipliers and algorithms*. Int. J. Wavelets Multiresolut. Inf. Process., 2008, **6**(2): 315–330.
- [5] P. BALAZS, D. T. STOEVA. *Unconditional convergence and invertibility of Multipliers*. arXiv: 0911.2783, 2009, Preprint.
- [6] P. BALAZS, J. P. ANTOINE, A. GRYBOS. *Weighted and controlled frames: mutual relationship and first numerical Properties*. Int. J. Wavelets Multiresolut. Inf. Process., 2010, **8**(1): 109–132.
- [7] P. G. CASAZZA. *Every frame is a sum of three (but not two) orthonormal bases-and other frame representations*. J. Fourier Anal. Appl., 1998, **4**(6): 727–732.
- [8] O. CHRISTENSEN. *An Introduction to Frames and Riesz Bases*. Birkhäuser Boston, Inc., Boston, MA, 2003.
- [9] I. DAUBECHIES, A. GROSSMANN, Y. MEYER. *Painless nonorthogonal expansions*. J. Math. Phys., 1986, **27**(5): 1271–1283.
- [10] I. DAUBECHIES. *Ten Lectures on Wavelets*. SIAM, Philadelphia, PA, 1992.
- [11] R. J. DUFFIN, A. C. SCHAEFFER. *A class of non-harmonic Fourier series*, Trans. Amer. Math. Soc., 1952, **72**: 341–366.
- [12] K. GRÖCHENIG. *Foundations of Time-Frequency Analysis*. Birkhäuser Boston, Inc., Boston, MA, 2001.
- [13] A. RAHIMI. *Multipliers of generalized frames in Hilbert spaces*. Bull. Iranian Math. Soc., 2011, **37**(1): 63–80.
- [14] Wenchang SUN. *G-frames and g-Riesz bases*. J. Math. Anal. Appl., 2006, **322**(1): 437–452.
- [15] R. YOUNG. *An Introduction to Nonharmonic Fourier Series*. Academic Press, Inc., San Diego, CA, 2001.