Rings in which Every Element Is A Left Zero-Divisor

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Abstract We introduce the concepts of left (right) zero-divisor rings, a class of rings without identity. We call a ring \( R \) left (right) zero-divisor if \( r_{R}(a) \neq 0 \) (\( l_{R}(a) \neq 0 \)) for every \( a \in R \), and call \( R \) strong left (right) zero-divisor if \( r_{R}(R) \neq 0 \) (\( l_{R}(R) \neq 0 \)). Camillo and Nielson called a ring right finite annihilated (RFA) if every finite subset has non-zero right annihilator. We present in this paper some basic examples of left zero-divisor rings, and investigate the extensions of strong left zero-divisor rings and RFA rings, giving their equivalent characterizations.

Keywords zero-divisor; left zero-divisor ring; strong left zero-divisor ring; RFA ring; extensions of rings.

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1. Some examples of left zero-divisor rings

Throughout this paper rings are general associative rings (with or without identity), \( \mathbb{Z} \) denotes the ring of integers and \( \mathbb{N} \) denotes the set of positive integers. Given a ring \( R \), the right (left) annihilator of a subset \( X \) of \( R \) is defined by \( r_{R}(X) = \{ a \in R \mid Xa = 0 \} \) (\( l_{R}(X) = \{ a \in R \mid aX = 0 \} \)), the polynomial ring over \( R \) in one indeterminate \( x \) is denoted by \( R[x] \).

Definition 1.1 A ring \( R \) is called left (right) zero-divisor if \( r_{R}(a) \neq 0 \) (\( l_{R}(a) \neq 0 \)) for every \( a \in R \), and a ring \( R \) is called zero-divisor if it is both left and right zero-divisor.

Obviously, any non-zero nil ring is zero-divisor; and rings with identity are never left (right) zero-divisor. If \( R \) is reversible (a ring \( R \) is called reversible if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \)), then \( R \) is left zero-divisor if and only if \( R \) is right zero-divisor. In general, a left (right) zero-divisor ring need not be a nil ring and the zero-divisor property for a ring is not left-right symmetric.

Proposition 1.2 If one of \( \{ R_{i} \}_{i \in W} \) is left zero-divisor, so is \( R = \bigoplus_{i \in W} R_{i} \) (\( R = \prod_{i \in W} R_{i} \)) is left zero-divisor does not imply that every \( R_{i} \) (\( i \in W \)) is left zero-divisor.

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For any ring \( R \), we define \( QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a + b = c + d, a, b, c, d \in R \right\} \), then \( QM_2(R) \) is a subring of \( M_2(R) \). Moreover, given an \((R,R)\)-bimodule \( M \), the trivial extension of \( R \) by \( M \) (see [4]) is the ring \( T(R,M) = R \bigoplus M \) with the usual addition and the following multiplication:

\[
(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + m_1r_2).
\]

This is isomorphic to the ring of all matrices \( \begin{pmatrix} r & m \\ 0 & r \end{pmatrix} \), where \( r \in R \) and \( m \in M \) and the usual matrix operations are used.

**Theorem 1.3** The following statements are equivalent for a ring \( R \):

1. \( R \) is left zero-divisor.
2. For any \( n \in \mathbb{N} \), the ring \( T_n(R) \) of \( n \times n \) upper triangular matrices over \( R \) is left zero-divisor.
3. \( QM_2(R) \) is left zero-divisor.
4. For any \( n \in \mathbb{N} \), \( S_n(R) = \left\{ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_0 \end{pmatrix} | a_i \in R, i = 0,1,\ldots,n-1 \right\} \) is left zero-divisor.
5. For any \( n \in \mathbb{N} \), \( R[x]/(x^n) \) is left zero-divisor, where \((x^n)\) is the ideal generated by \( x^n \).
6. \( T(R,R) \) is left zero-divisor.

**Proof** (1) \( \Rightarrow \) (2). Assume that \( R \) is left zero-divisor and \( A = (a_{ij}) \in T_n(R) \), where \( a_{ij} = 0 \) if \( i > j \). Then there exists \( 0 \neq t_{ii} \in R \) such that \( a_{ii}t_{ii} = 0 \) for any \( i, 1 \leq i \leq n \). Taking \( D = (d_{ij}) \), where \( d_{11} = t_{11} \neq 0, d_{ij} = 0, 1 < i, j \leq n \), we get \( 0 \neq D \in T_n(R) \) such that \( AD = 0 \). Hence \( T_n(R) \) is left zero-divisor.

(2) \( \Rightarrow \) (3). We construct a map \( f : QM_2(R) \rightarrow T_2(R), \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mapsto \left( \begin{array}{cc} a + b & b \\ 0 & d - b \end{array} \right) \), for any \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in QM_2(R) \). It is easy to verify that \( f \) is an injective and a ring homomorphism.

For any \( \left( \begin{array}{cc} x & z \\ 0 & y \end{array} \right) \in T_2(R) \), since

\[
f \left( \begin{pmatrix} x-z \\ x-y-z \\ y+z \end{pmatrix} \right) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix},
\]

\( f \) is a ring isomorphism. This completes the proof by (2).

(3) \( \Rightarrow \) (1). Let \( r \in R \). Then \( A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in QM_2(R) \). Since \( QM_2(R) \) is left zero-
Proposition 1.5 Let \( m \) be a divisor, there exists \( 0 \neq T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{QM}_2(R) \) such that \( AT = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} = 0 \), it follows that \( ra = rb = rc = rd = 0 \). Notice that \( T \neq 0 \), there must be \( 0 \neq s \in R \) such that \( rs = 0 \), as desired.

(1) \( \Rightarrow \) (4). Let \( A = (a_{ij}) \in S_n(R) \), where \( a_{ii} = a_0, 1 \leq i \leq n \). Since \( R \) is left zero-divisor, there exists \( 0 \neq t_0 \in R \) such that \( a_0t_0 = 0 \). Taking \( 0 \neq T = (t_{ij}) \in S_n(R) \), where \( t_{1n} = t_0 \) and \( t_{ij} = 0, 1 < i \leq n, 1 \leq j < n \), we get \( AT = 0 \). Thus, \( S_n(R) \) is left zero-divisor.

(4) \( \Rightarrow \) (5). Note that \( R[x]/(x^n) \cong S_n(R) \), we obtain the result by (4).

(5) \( \Rightarrow \) (6). This is obvious since \( T(R, R) \cong R[x]/(x^2) \).

(6) \( \Rightarrow \) (1). Let \( a \in R \). Then \( A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in T(R, R) \). Since \( T(R, R) \) is left zero-divisor, there exists \( 0 \neq T = \begin{pmatrix} t & m \\ 0 & t \end{pmatrix} \in T(R, R) \) such that \( AT = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} t & m \\ 0 & t \end{pmatrix} = \begin{pmatrix} at & am \\ 0 & at \end{pmatrix} = 0 \), it follows that \( at = 0 \) and \( am = 0 \). Notice that \( T \neq 0 \), we have \( t \neq 0 \) or \( m \neq 0 \). Consequently in any case there is \( 0 \neq s \in R \) such that \( as = 0 \), as asserted. \( \square \)

Let \( R[x; x^{-1}] \) be the ring of Laurent polynomials in one variable \( x \) with coefficients in a ring \( R \), i.e., \( R[x; x^{-1}] \) consists of all formal sums \( \sum_{i=k}^{n} m_i x^i \) with obvious addition and multiplication, where \( m_i \in R \) and \( k, n \) are (possible negative) integers.

Proposition 1.4 Let \( R \) be a ring. Then \( R[x] \) is left zero-divisor if and only if so is \( R[x; x^{-1}] \).

Proof Suppose that \( R[x] \) is left zero-divisor. Let \( f(x) \in R[x; x^{-1}] \). Then there exists an \( n \in \mathbb{N} \) such that \( f_1(x) = f(x)x^n \in R[x] \). Hence there exists \( 0 \neq g(x) \in R[x] \) such that \( f_1(x)g(x) = f(x)g(x)x^n = 0 \), it follows that \( f(x)g(x) = 0 \) and \( R[x; x^{-1}] \) is left zero-divisor.

Conversely, assume that \( R[x; x^{-1}] \) is left zero-divisor, and let \( f(x) \in R[x] \). Then there exists \( 0 \neq g(x) \in R[x; x^{-1}] \) such that \( f(x)g(x) = 0 \) since \( R[x] \subseteq R[x; x^{-1}] \). As \( g(x) = x^{-m}g_1(x) \) for some \( m \in \mathbb{N} \) and \( 0 \neq g_1(x) \in R[x] \), \( f(x)g(x) = x^{-m}f(x)g_1(x) = 0 \), we obtain that \( f(x)g_1(x) = 0 \). \( \square \)

Proposition 1.5 Let \( R \) and \( S \) be rings and \( V =_R V_S \) be an \((R, S)\)-bimodule. If \( R \) is left zero-divisor, so is \( A = \begin{pmatrix} R & V \\ 0 & S \end{pmatrix} \).

Proof Take any \( \begin{pmatrix} r & v \\ 0 & s \end{pmatrix} \in A \). For \( r \in R \), there exists \( 0 \neq t \in R \) such that \( rt = 0 \) since \( R \) is left zero-divisor. Thus, we get \( 0 \neq \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} \in A \) such that \( \begin{pmatrix} r & v \\ 0 & s \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = 0 \), which implies that \( A \) is left zero-divisor. \( \square \)
Proposition 1.6 If a ring $R$ is left zero-divisor, so is the ring 

$$V(R) = \left\{ \begin{pmatrix} a & d & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & e & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & b & f \\ 0 & 0 & 0 & 0 & 0 & c \end{pmatrix} \mid a, b, c, d, e, f \in R \right\}.$$ 

Proof Fix $A \in V(R)$. Since $R$ is left zero-divisor, there exists $0 \neq a' \in R$ such that $aa' = 0$. Taking $0 \neq T = (t_{ij}) \in V(R)$, where $t_{12} = a'$ and 0 elsewhere, we obtain that $AT = 0$. □

Let $R$ be a commutative ring, $M$ an $R$-module and $\sigma$ an endomorphism of $R$. Recall that the Nagata extension of $R$ by $M$ and $\sigma$ (see [4]), denoted by $N(R, M, \sigma)$, is the ring $R \bigoplus M$ with the usual addition and the multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, \sigma(r_1)m_2 + r_2m_1)$, where $r_1 \in R$ and $m_i \in M, i = 1, 2$.

Proposition 1.7 Let $R$ be a commutative left zero-divisor ring. Then the Nagata extension $N(R, R, \sigma)$ of $R$ by $R$ and $\sigma$ is left zero-divisor.

Proof For any $(r, m) \in N(R, R, \sigma)$, we have $0 \neq t \in R$ such that $\sigma(r)t = 0$ since $R$ is left zero-divisor and $\sigma(R) \subseteq R$. Putting $0 \neq (0, t) \in N(R, R, \sigma)$, we get that $(r, m)(0, t) = (r0, \sigma(r)t + 0m) = (0, 0)$. Therefore $N(R, R, \sigma)$ is left zero-divisor. □

It is interesting to know if the polynomial ring of a ring share the same property with the ring. If $R[x]$ is left zero-divisor, then $R$ is again left zero-divisor. We raise the following question: if $R$ is left zero-divisor, is the polynomial ring $R[x]$ necessarily left zero-divisor?

We do not know whether $R$ is left zero-divisor when both $R/I$ and $I$ are left zero-divisor for an ideal $I$ of $R$. In view of this question, the following proposition may be of some interest. According to Lambek [5], a ring $R$ is called symmetric if $abc = 0 \Leftrightarrow acb = 0$ for all $a, b, c \in R$, i.e., if $bc \in r_R(a) \Leftrightarrow cb \in r_R(a)$. We call a ring $R$ left symmetric if $rst = 0$ implies $srt = 0$ for all $r, s, t \in R$. For example, let $R = 2\mathbb{Z}$. Then $T(R, R) \cong \left\{ \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \mid r, s \in R \right\}$ is left symmetric. Note that this definition is equivalent to that of symmetric rings for rings with identity, but in general they are different. For instance, $R = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ is symmetric but not left symmetric.

Proposition 1.8 Let $R$ be left symmetric and $I$ a non-trivial ideal of $R$ which is a right annihilator in $R$. If $R/I$ is left zero-divisor, then $R$ is left zero-divisor.

Proof Since $I$ is non-trivial, we assume that $I = r_R(S)$ where $0 \neq S \subseteq R$. For any $a \in R$, there exists $0 \neq I \in R/I$ such that $\overline{aI} = 0$, i.e., $at \in I = r_R(S)$ since $R/I$ is left zero-divisor. It follows that $Sat = 0$. Consequently $aSt = 0$ since $R$ is left symmetric. Note that $t \notin I$, we have $St \neq 0$. This implies that there exists $s_0 \in S$ such that $s_0t \neq 0$ and $a(s_0t) = 0$. Thus $r_R(a) \neq 0$, as required. □
It is natural to conjecture that the homomorphic image \(R/I\) of \(R\) and \(eR, eRe\) may also be left (right) zero-divisor for a left (right) zero-divisor ring \(R, I \triangleleft R\) and \(e = e^2 \in R\). We have, however, a negative answer to these situations by the following example.

**Example 1.9** The ring \(R = \left\{ \left( \begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \mid a, b \in \mathbb{Z} \right\}\) is left zero-divisor. We have \(I = \left( \begin{array}{cc} 0 & 0 \\ \mathbb{Z} & 0 \end{array} \right) \triangleleft R, e = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) = e^2 \in R\) and \(ReR = R, \) but \(R/I \cong \left( \begin{array}{cc} \mathbb{Z} & 0 \\ 0 & 0 \end{array} \right)\) is not left zero-divisor.

From the above example it also follows that the left zero-divisor property of rings is not a radical property in the sense of Amitsur and Kurosh.

2. **Strong left zero-divisor rings and RFA rings**

Observe that for some rings, they not only satisfy \(r_R(a) \neq 0\) for any \(a \in R\) but also have \(r_R(R) \neq 0\). In this section, we will focus on these rings.

**Definition 2.1** A ring \(R\) is called strong left (right) zero-divisor if \(r_R(R) \neq 0 (l_R(R) \neq 0)\).

Any strong left (right) zero-divisor ring is left (right) zero-divisor, but the converse does not hold.

**Example 2.2** Let \(R = \sum_{i=2}^{\infty} \mathbb{Z}x_i\) be a countably infinite dimensional vector space over the field \(\mathbb{Z}_2 = \{0, 1\}\), with basis \(T = \{x_2, x_3, \ldots, x_n, \ldots\}\). Multiplication of the base vectors is defined as

\[
x_i x_j = \begin{cases} 0, & \text{if } (i, j) \neq 1, \\
x_{ij}, & \text{if } (i, j) = 1,
\end{cases}
\]

where \((i, j)\) is the maximal prime divisor of \(i\) and \(j\). Thought of as a ring, \(R\) is the set of all finite sums \(\sum a_i x_i\), where \(a_i\) are elements in the field \(\mathbb{Z}_2\). Addition is defined articulately as \(a_i x_i + a_j x_j\) just written together, if \(i \neq j\); and if \(i = j\), then \(a_i x_i + a_i' x_i = (a_i + a_i') x_i\). Multiplication is distributive and defined as above. The ring \(R\) is then commutative. Moreover, for any \(a = a_1 x_{i_1} + a_2 x_{i_2} + \cdots + a_n x_{i_n} \in R\), we have \(a^2 = 0\), and hence \(R\) is zero-divisor.

For any \(a = x_{i_1} + x_{i_2} + \cdots + x_{i_n} \in R\) and any positive integer \(n \geq 2\), since \((n, n + 1) = 1\), we get that

\[
x_{i_1} a = x_{i_1} (i_1 + 1) + \cdots, \quad x_{[i_1(i_1 + 1) + 1]} a = x_{j_1} + \cdots,
\]

where \(j_1 = i_1(i_1 + 1)[i_1 (i_1 + 1) + 1], \ldots\). Thus, if \(a \in r_R(T)\), then necessarily \(a = 0\), whence \(r_R(R) \subseteq r_R(T) = 0\). So \(R\) is not strong left zero-divisor.

For a ring \(R\) with a ring endomorphism \(\alpha : R \rightarrow R\), a skew polynomial ring \(R[x; \alpha]\) of \(R\) is the ring obtained by giving the polynomial ring over \(R\) with the new multiplication \(xr = \alpha(r)x\) for all \(r \in R\).

**Theorem 2.3** Let \(R\) be a ring and \(\alpha : R \rightarrow R\) an epimorphism. Then \(R\) is strong left zero-divisor if and only if so is \(R[x; \alpha]\).
Proof If $Rb = 0$, then $R\alpha(b) = \alpha(R)\alpha(b) \subseteq \alpha(Rb) = 0$, hence $\alpha(r_R(R)) \subseteq r_R(R)$. Now assume that $R$ is strong left zero-divisor, then $T = r_R(R) \neq 0$. For every $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x; \alpha]$, taking any $0 \neq t(x) = \sum_{j=0}^{m} b_j x^j \in T[x; \alpha] \subseteq R[x; \alpha]$, we obtain

$$f(x)t(x) = \sum_{k=0}^{m+n} \sum_{i+j=k} a_i \alpha^j(b_j) x^k = 0.$$  

Hence $r_{R[x; \alpha]}(R[x; \alpha]) \neq 0$.

Conversely, assume that $R[x; \alpha]$ is strong left zero-divisor. For any $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in r_{R[x; \alpha]}(R[x; \alpha])$, there exists at least one $a_{ik} \neq 0, 0 \leq i_k \leq n, a_{ik} \in R$. Note that $R \subseteq R[x; \alpha]$ and $Rf(x) = 0$. It follows that $Ra_{ik} = 0$ and $r_R(R) \neq 0$. □

Theorem 2.3 answers partially the question raised in the above section.

Recall that for an infinite set of commuting indeterminates $\{x_\lambda\}$ over $R$, Gilmer-Grams [3] defined rings

$$R[[x_\lambda]] = \bigcup \{R[F] \mid F \text{ is a finite subset of } \{x_\lambda\}\} \quad \text{and}$$

$$R[[\{x_\lambda\}]] = \bigcup \{R[[F]] \mid F \text{ is a finite subset of } \{x_\lambda\}\}.$$

**Theorem 2.4** Let $R$ be a ring. Then the following statements are equivalent:

1. $R$ is strong left zero-divisor.
2. $T_n(R)$ is strong left zero-divisor for any $n \in \mathbb{N}$.
3. $QM_2(R)$ is strong left zero-divisor.
4. $S_n(R)$ is strong left zero-divisor for any $n \in \mathbb{N}$.
5. $R[x]/(x^n)$ is strong left zero-divisor for any $n \in \mathbb{N}$.
6. $T(R, R)$ is strong left zero-divisor.
7. $R[x; x^{-1}]$ is strong left zero-divisor.
8. $R[[x_\lambda]]$ is strong left zero-divisor.
9. $R[[\{x_\lambda\}]]$ is strong left zero-divisor.

**Proof** Note that if $R$ is strong left zero-divisor, then $S = r_R(R) \neq 0$ and there exists $0 \neq t(\{x_\lambda\}) \in S[[x_\lambda]] \subseteq S[[\{x_\lambda\}]]$ such that $R[\{x_\lambda\}]t(\{x_\lambda\}) = 0$ ($R[[\{x_\lambda\}]][t(\{x_\lambda\})] = 0$), hence $R[[\{x_\lambda\}]] (R[[\{x_\lambda\}]]$ is strong left zero-divisor. It follows that (1) ⇔ (8) and (1) ⇔ (9).

Making a little modification in the proof of Theorem 1.3, we can prove that (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6).

By Theorem 2.3, we know that $R$ is strong left zero-divisor if and only if so is $R[x]$. By analogy with the proof of Proposition 1.4, it is easy to prove that (1) ⇔ (7). □

**Theorem 2.5** A ring $R$ is strong left zero-divisor if and only if so is $M_n(R)$, the ring of $n \times n$ matrices over $R$, for any positive integer $n$.

**Proof** Assume that $R$ is strong left zero-divisor and $A = (a_{ij}) \in M_n(R)$. Then $r_{R}(R) \neq 0$. For any $0 \neq r \in r_{R}(R)$, we have $a_{ij}r = 0, \forall i \leq i, j \leq n$. Putting $T = (t_{ij}) \in M_n(R)$, where $t_{ii} = r$ and $t_{ij} = 0$ if $i \neq j, 1 \leq i, j \leq n$, we get that $T \neq 0$ and $AT = 0$. Hence $r_{M_n(R)}(M_n(R)) \neq 0$ because $A$ is arbitrary.
Conversely, assume that $M_n(R)$ is strong left zero-divisor and $r \in R$. Take any $0 \neq A = (a_{ij}) \in r_{M_n(R)}(M_n(R)) \neq 0$, and suppose that some $a_{kl} \neq 0, 1 \leq k, l \leq n$. If we put $T = (t_{ij})$ as above, then from $TA = 0$ one can get that $r_{a_{kl}} = 0$. This implies that $a_{kl} \in r_{R}(R) \neq 0$. □

Given a monoid $G$ and a ring $R$, we use $R[G]$ to denote the monoid ring of $G$ over $R$.

**Theorem 2.6** A ring $R$ is strong left zero-divisor if and only if so is $R[G]$ for any monoid $G$.

**Proof** Assume that $r_{R}(R) \neq 0$ and $\sum r_{i}g_{i} \in R[G]$. For any $0 \neq a \in r_{R}(R)$, we have $(\sum r_{i}g_{i})(ae) = \sum (r_{i}a)g_{i} = 0$, where $e$ is the identity of $G$. Thus $0 \neq ae \in r_{R[G]}(R[G])$.

Conversely, assume that $R[G]$ is strong left zero-divisor and $a \in R$. If $0 \neq \sum r_{i}g_{i} \in r_{R[G]}(R[G])$, then from $0 = (ae)(\sum r_{i}g_{i}) = \sum (a_{ir_{i}})g_{i}$ we get that $ar_{i} = 0$ for any $i$. This shows that $r_{i} \in r_{R}(R)$ for any $i$, and $r_{R}(R) \neq 0$. □

Let $G$ denote a group with identity $e$, and $R = \bigoplus_{g \in G} R_{g}$ be a $G$-graded ring. Beattie [1] defined the generalized smash product $R \# G^{*}$ of $R$ and $G$ to be the free left $R$-module $\bigoplus_{g \in G} R P_{g}$ with multiplication defined for elements $aP_{g}$ and $bP_{h}$ by $(aP_{g})(bP_{h}) = ab_{gh^{-1}}P_{h}$, and extended to general elements of $R \# G^{*}$ by linearity.

**Theorem 2.7** Let $R = \bigoplus_{g \in G} R_{g}$ be a $G$-graded ring. Then $R$ is strong left zero-divisor if and only if so is $R \# G^{*}$.

**Proof** Assume that $r_{R}(R) \neq 0$ and $\sum a_{i}P_{g_{i}} \in R \# G^{*}$. Take any $0 \neq r \in r_{R}(R)$. Since $r_{R}(R)$ is a graded ideal of $R$, $(\sum a_{i}P_{g_{i}})rP_{e} = \sum a_{ir_{i}}P_{e} = 0$. This implies that $0 \neq rP_{e} \in r_{R \# G^{*}}(R \# G^{*})$.

Conversely, assume that $R \# G^{*}$ is strong left zero-divisor. Taking

$$0 \neq \sum a^{(r)}_{i}P_{g_{i}} \in r_{R \# G^{*}}(R \# G^{*}),$$

we get that $0 = r_{g}P_{h}(\sum a^{(r)}_{i}P_{g_{i}}) = \sum r_{g}a^{(r)}_{h}P_{g_{i}}$ for any $g, h \in G$ and $r_{g} \in R_{g}$. Thus for every $g_{i} \in G, r_{g}a^{(r)}_{h}P_{g_{i}} = 0$. If $a^{(r)}_{h} \neq 0$, then there exists an $h_{0} \in G$ such that $a^{(r)}_{h_{0}} \neq 0$, and hence $a^{(r)}_{h_{0}} \neq 0$. Since $g \in G$ and $r_{g} \in R_{g}$ are arbitrary, we have $a^{(r)}_{h_{0}} \neq 0$. □

Camillo-Nielson [2] introduced the concept of right finite annihilated rings (in short, RFA rings) to describe exactly when a direct product or direct sum of rings is right McCoy. A ring $R$ is called RFA if every finite subset of $R$ has a nonzero right annihilator.

Clearly, strong left-zero divisor rings are RFA rings, but the converse does not hold.

**Example 2.8** Let $R = \mathbb{Z}[x_{1}, x_{2}, x_{3}, \ldots]/(x_{1}^{2}, x_{2}^{2}, x_{3}^{2}, \ldots)$, and $A = \langle x_{1}, x_{2}, x_{3}, \ldots \rangle$ be the ideal of $R$ generated by $x_{1}, x_{2}, x_{3}, \ldots$. Then $A$ is nil, left zero-divisor and RFA. But $A$ is neither nilpotent nor strong left zero-divisor.

For RFA rings, we have the following

**Proposition 2.9** Let $R$ be a ring and $S = \{(a_{n})_{n=1}^{\infty} \in \prod R \mid a_{n}$ is a eventually constant}, a subring of the countable direct product $\prod_{n=1}^{\infty} R$. Then ring $R$ is RFA if and only if so is $S$.

**Proof** It is a trivial verification.
Theorem 2.10 Let $R$ be a ring. Then the following statements are equivalent:

(1) $R$ is RFA.
(2) $T_n(R)$ is RFA for any $n \in \mathbb{N}$.
(3) $QM_2(R)$ is RFA.
(4) $S_n(R)$ is RFA for any $n \in \mathbb{N}$.
(5) $R[x]/(x^n)$ is RFA for any $n \in \mathbb{N}$.
(6) $T(R, R)$ is RFA.
(7) $R[x; x^{-1}]$ is RFA.
(8) $R[[x_\lambda]]$ is RFA.

Proof (1) $\Rightarrow$ (2). Assume that $F = \{A_k = (a^k_{ij}) \in T_n(R), k = 1, 2, \ldots, m\}$ is a finite subset of $T_n(R)$. Then $E = \{a^k_{ij}1 \leq i, j \leq n, k = 1, 2, \ldots, m\}$ is a finite subset of $R$, there exists $0 \neq t \in R$ such that $a^k_{ij}t = 0$ for every $a^k_{ij}$ ($1 \leq i, j \leq n, 1 \leq k \leq m$) since $R$ is RFA. Putting $D = (d_{ij}) \in T_n(R)$ with $d_{11} = t$ and zeros elsewhere, we have that $A_kD = 0$ for $1 \leq k \leq m$.

(2) $\Rightarrow$ (3). Holds since $QM_2(R) \cong T_2(R)$.

(3) $\Rightarrow$ (1). For any finite subset $F$ of $R$, $E = \{A_r = \left( \begin{array}{cc} r & 0 \\ 0 & r \end{array} \right) | r \in F\}$ is a finite subset of $QM_2(R)$. Then there exists $0 \neq T = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in QM_2(R)$ such that $A_rT = 0$ for every $r \in F$, it follows that $ra = rb = rc = rd = 0$. Notice that $T \neq 0$, there is $0 \neq s \in R$ such that $Fs = 0$, as desired.

(1) $\Rightarrow$ (4). Let $F = \{A_k = (a^k_{ij}) \in S_n(R), k = 1, 2, \ldots, m\}$ be a finite subset of $S_n(R)$, then $E = \{a^k_{ij}1 \leq i, j \leq n, k = 1, 2, \ldots, m\}$ is a finite subset of $R$, there exists $0 \neq t \in R$ such that $a^k_{ij}t = 0$ for every $a^k_{ij}$ ($1 \leq i, j \leq n, 1 \leq k \leq m$) since $R$ is RFA. Taking $0 \neq T = (t_{ij}) \in S_n(R)$ with $t_{11} = t$ and zeros elsewhere, we obtain that $A_kT = 0$ for $1 \leq k \leq m$.

(4) $\Rightarrow$ (5). Holds by $R[x]/(x^n) \cong S_n(R)$.

(5) $\Rightarrow$ (6). Follows from $T(R, R) \cong R[x]/(x^2)$.

(6) $\Rightarrow$ (1). Let $F$ be a finite subset of $R$ and $E = \{A_r = \left( \begin{array}{cc} r & 0 \\ 0 & r \end{array} \right) | r \in F\}$. Then there exists $0 \neq T = \left( \begin{array}{cc} t & m \\ 0 & t \end{array} \right) \in T(R, R)$ such that $A_rT = 0$ for any $r \in F$, it follows that $rt = 0$ and $rm = 0$. Notice that $T \neq 0$, we have that $t \neq 0$ or $m \neq 0$. Consequently in any case there is $0 \neq s \in R$ such that $Fs = 0$, as desired.

(1) $\Rightarrow$ (8). Let $E = \{f_i(x_\lambda) | i = 1, 2, \ldots, m\}$ be a finite subset of $R[\{x_\lambda\}]$, then $E \subseteq R[F]$ for some finite subset $F$ of $\{x_\lambda\}$, and the set $H$ of coefficients of all $f_i(x_\lambda) \subseteq E$ is a finite subset of $R$. Hence there exists $0 \neq t \in R$ such that $Ht = 0$, it follows that $f_i(x_\lambda)t = 0$ for $1 \leq k \leq m$.

(8) $\Rightarrow$ (1). Let $E$ be a finite subset of $R$. Then $E \subseteq R[\{x_\lambda\}]$, and there exists $0 \neq f(x_\lambda) \in R[\{x_\lambda\}]$ such that $Ef(x_\lambda) = 0$. Thus $Ea = 0$ for any nonzero coefficient $a$ of $f(x_\lambda)$.

(1) $\Leftrightarrow$ (7). The proof is analogous to that of (1) $\Leftrightarrow$ (8). □
Example 2.11 Consider the ring \( R = \begin{pmatrix} 0 & Z & Z \\ 0 & 0 & Z \\ 0 & 0 & Z \end{pmatrix} \). For any \( A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & d \end{pmatrix} \in R \) and \( T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R \), we have \( AT = 0 \), which implies that \( R \) is strong left zero-divisor.

We conclude this paper with the following chart:

\[
\begin{array}{c}
\text{nilpotent} \\
\Rightarrow \\
\text{locally nilpotent} \\
\Rightarrow \\
\text{nil} \\
\Rightarrow \\
\text{strong left zero-divisor} \\
\Rightarrow \\
\text{RFA} \\
\Rightarrow \\
\text{left zero-divisor}
\end{array}
\]

No other implications hold (except by transitivity). Note that Example 2.11 shows that a strong left zero-divisor, left zero-divisor and RFA ring are not necessarily nilpotent, nil and locally nilpotent, respectively; and Example 20.2 in Szasz [6] also shows that a left zero-divisor ring is not necessarily an RFA ring.

References