Journal of Mathematical Research with Applications
Jul., 2013, Vol. 33, No. 4, pp. 412–418
DOI:10.3770/j.issn:2095-2651.2013.04.004
Http://jmre.dlut.edu.cn

On $\mathfrak{F}_s$-Quasinormality of 2-Maximal Subgroups

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Abstract Let $\mathfrak{F}$ be a class of finite groups. A subgroup H of a finite group G is said to be $\mathfrak{F}_s$-quasinormal in G if there exists a normal subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. In this paper, we use $\mathfrak{F}_s$-quasinormal subgroups to study the structure of finite groups. Some new results are obtained.

Keywords $\mathfrak{F}_s$-quasinormal subgroup; Sylow subgroup; maximal subgroup; 2-maximal subgroup.

MR(2010) Subject Classification 20D10; 20D20; 20D25

1. Introduction

All groups considered in the paper are finite and G denotes a finite group, the notations and terminology in this paper are standard, as in [2] and [8].

Recall that a subgroup H of G is called an s-quasinormal subgroup (or s-permutable subgroup [6]) in G if H is permutable with every Sylow subgroup P of G (that is, $HP = PH$). Wang [10] defined c-normal subgroup: A subgroup H of a group G is said to be c-normal if there exists a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. Moreover, Feng and Guo [1] defined the concept of $\mathfrak{F}_h$-normal subgroup: A subgroup H of a group G is said to be $\mathfrak{F}_h$-normal in G if there exists a normal subgroup K of G such that HK is a normal Hall subgroup of G and $(H \cap K)H_G/H_G$ is contained in the $\mathfrak{F}$-hypercenter $Z_\infty^{\mathfrak{F}}(G/H_G)$ of G/H_G. By using these concepts mentioned above, many interesting results have been obtained (see, for example, [1, 3, 7, 10]). Recently, Huang [6] introduced the following concept:

Definition 1.1 *Let $\mathfrak{F}$ be a non-empty class of groups and H a subgroup of a group G. H is said to be $\mathfrak{F}_s$-quasinormal in G if there exists a normal subgroup T of G such that HT is s-permutable in G and $(H \cap T)H_G/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$.*

Received April 8, 2012; Accepted November 25, 2012
Supported by the National Natural Science Foundation of China (Grant No. 11071147) and Doctoral Program Foundation of Institutions of Higher Education of China (Grant No. 20113402110036).
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Recall that, for a class $\mathfrak{F}$ of groups, a chief factor H/K of a group G is called $\mathfrak{F}$-central (see [9] or [2, Definition 2.4.3]) if $[H/K](G/C_G(H/K)) \in \mathfrak{F}$. The symbol $Z_\infty^{\mathfrak{F}}(G)$ denotes the $\mathfrak{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathfrak{F}$-central. A subgroup H of G is said to be $\mathfrak{F}$-hypercenter in G if $H \leq Z_\infty^{\mathfrak{F}}(G)$. We use $\mathfrak{N}$, $\mathfrak{U}$, and $\mathfrak{S}$ to denote the formations of all nilpotent groups, supersoluble groups and soluble groups, respectively.

Obviously, all subgroups, whether they are c-normal, s-quasinormal or $\mathfrak{F}_h$-normal, are all $\mathfrak{F}_s$-quasinormal in G, for any nonempty saturated formation $\mathfrak{F}$. For example, if a subgroup H is c-normal in G, then there exists a normal subgroup K such that $G = HK$ and $(H \cap K)H_G/H_G = 1 \leq Z_\infty^{\mathfrak{F}}(G/H_G)$. However, the converse is not true (see Example 1.2 in [6]).

By using this new concept, Huang [6] has given some conditions under which a finite group belongs to some formations. In this article, we study further the influence of $\mathfrak{F}_s$-quasinormal subgroups on the structure of finite groups. Some new results are obtained and a series of known results are generalized.

2. Preliminaries

The following known results are useful in our proof.

Lemma 2.1 ([5, Lemma 2.2]) *Let G be a group and $H \leq K \leq G$.*

(1) If H is s-permutable in G, then H is s-permutable in K;

(2) Suppose that H is normal in G. Then K/H is s-permutable in G/H if and only if K is s-permutable in G;

(3) If H is s-permutable in G, then H is subnormal in G;

(4) If H and F are s-permutable in G, the $H \cap F$ is s-permutable in G;

(5) If H is s-permutable in G and $M \leq G$, then $H \cap M$ is s-permutable in M.

Lemma 2.2 ([4, Lemma 2.2]) *If H is a p-subgroup of G for some prime p and H is s-permutable in G, then the following properties hold:*

(1) $H \leq O_p(G)$;

(2) $O^p(G) \leq N_G(H)$.

Lemma 2.3 ([12]) *Let G be a group and $A \leq G$. If A is subnormal in G and A is soluble, then A is contained in some normal soluble subgroup of G.*

Lemma 2.4 ([11, Theorem 4.1]) *Let p be a prime number divisor of $|G|$ such that $(|G|, p-1) = 1$. Assume that the order of G is not divisible by p^3 and G is A_4-free. Then G is p-nilpotent.*

Lemma 2.5 ([4, Lemma 2.5]) *Let G be a group and p a prime number such that $p^{n+1} \nmid |G|$ for some integer $n \geq 1$. If $(|G|, (p-1)(p^2-1)\cdots(p^n-1)) = 1$, then G is p-nilpotent.*

Lemma 2.6 ([6, Lemma 2.3]) *Let G be a group and $H \leq K \leq G$.*

(1) H is $\mathfrak{F}_s$-quasinormal in G if and only if there exists a normal subgroup T of G such

that HT is s-permutable in G, $H_G \leq T$ and $H/H_G \cap T/H_G \leq Z_\infty^{\mathfrak{F}}(G/H_G)$;

(2) Suppose that H is normal in G. Then K/H is $\mathfrak{F}_s$-quasinormal in G/H if and only if K is $\mathfrak{F}_s$-quasinormal in G;

(3) Suppose that H is normal in G. Then, for every $\mathfrak{F}_s$-quasinormal subgroup E of G satisfying $(|H|,|E|)=1$, HE/H is $\mathfrak{F}_s$-quasinormal in G/H;

(4) If H is $\mathfrak{F}_s$-quasinormal in G and $\mathfrak{F}$ is S-closed, then H is $\mathfrak{F}_s$-quasinormal in K;

(5) If H is $\mathfrak{F}_s$-quasinormal in G, K is normal in G and $\mathfrak{F}$ is S_n-closed, then H is $\mathfrak{F}_s$-quasinormal in K;

(6) If $G \in \mathfrak{F}$, then every subgroup of G is $\mathfrak{F}_s$-quasinormal in G.

3. Main results and applications

Recall that a subgroup H is said to be a 2-maximal subgroup of G if H is a maximal subgroup of some maximal subgroup of G.

Theorem 3.1 *Let G be a group and P a Sylow p-subgroup of G, where p is the prime divisor of $|G|$ with $(|G|, p^2-1) = 1$. If every 2-maximal subgroup of P (if exists) is $\mathfrak{N}_s$-quasinormal in G, then G is soluble.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. If $p > 2$, then G is soluble by $(|G|, p^2-1) = 1$ and the well-known Feit-Thompson Theorem of odd groups. Hence we only need to consider the case that $p = 2$.

(1) $O_2(G) = 1$.

If $O_2(G) = P$ or $O_2(G)$ is a maximal subgroup or a 2-maximal subgroup of P, then $G/O_2(G)$ is 2-nilpotent by Lemma 2.5. It follows that G is soluble, a contradiction. Hence, there exists some 2-maximal subgroup P_2 such that $O_2(G) < P_2$. By Lemma 2.6 (2), we see that $G/O_2(G)$ satisfies the hypothesis. The minimal choice of G implies that $G/O_2(G)$ is soluble and thereby G is soluble, also a contradiction. Hence (1) holds.

(2) $2^3 \mid |G|$ (This follows directly from Lemma 2.5).

(3) Final contradiction.

Let P_2 be a 2-maximal subgroup of P. Then $P_2 \neq 1$ and $(P_2)_G = 1$. By the hypothesis, there exists a normal subroup K of G such that P_2K is s-permutable in G and $P_2 \cap K \leq Z_\infty^{\mathfrak{N}}(G)$. Hence $P_2 \cap K \leq Z_\infty^{\mathfrak{N}}(G)_p \leq O_2(G) = 1$. By Lemma 2.5, K is soluble. Since $P_2K/K \cong P_2/P_2 \cap K \cong P_2$, we have P_2K/K is soluble. Hence P_2K is soluble. Since P_2K is s-permutable in G, P_2K is subnormal in G by Lemma 2.1(3). It follows from Lemma 2.3 that P_2K is contained in some soluble normal subgroup L of G. Obviously, $p^3 \nmid |G/L|$. Hence G/L is soluble by Lemma 2.5. This implies that G is soluble, a contradiction.

Lemma 3.2 *Let p be the smallest prime dividing $|G|$ and P some Sylow p-subgroup of G. Then G is soluble if and only if every maximal subgroup of P is $\mathfrak{S}_s$-quasinormal in G.*

Proof The necessity part is obvious by Lemma 2.6(6). We only need to prove the sufficiency part. Suppose that the assertion is false and let G be a counterexample of minimal order. Then $p = 2$ by the well known Feit-Thompson theorem of odd group. We proceed with the proof by the following steps.

(1) $O_2(G) = 1$.

Assume that $N = O_2(G) \neq 1$. Then P/N is a Sylow 2-subgroup of G/N. Let M/N be a maximal subgroup of P/N. Then M is a maximal subgroup of P. By the hypothesis and Lemma 2.6(2), M/N is $\mathfrak{S}_s$-quasinormal in G/N. The minimal choice of G implies that G/N is soluble. It follows that G is soluble, a contradiction. Hence (1) holds.

(2) $O_{2'}(G) = 1$.

Assume that $D = O_{2'}(G) \neq 1$. Then PD/D is a Sylow 2-subgroup of G/D. Suppose that M/D is a maximal subgroup of PD/D. Then there exists a maximal subgroup P_1 of P such that $M = P_1D$. By the hypothesis and Lemma 2.6(3), $M/D = P_1D/D$ is $\mathfrak{S}_s$-quasinormal in G/D. Hence G/D is soluble by the choice of G. It follows that G is soluble, a contradiction.

(3) Final contradiction.

Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a normal subroup K of G such that P_1K is s-permutable in G and $(P_1 \cap K)(P_1)_G/(P_1)_G \leq Z_\infty^{\mathfrak{S}}(G/(P_1)_G)$. Note that $Z_\infty^{\mathfrak{S}}(G)$ is a soluble normal subgroup of G. By (1) and (2), we have $(P_1)_G = 1$ and $Z_\infty^{\mathfrak{S}}(G) = 1$. This induces that $P_1 \cap K = 1$. If $K = 1$, then P_1 is s-permutable in G and so $P_1 = 1$ by (1) (2) and Lemma 2.2(1). This means that $|P| = 2$. Then by [8, (10.1.9)], G is 2-nilpotent and so G is soluble, a contradiction. Now assume that $K \neq 1$. If $2 \mid |K|$, then $|K_2| = 2$, where K_2 is some Sylow 2-subgroup of K. By [8, (10.1.9)] again, we see that K is 2-nilpotent, and so K has a normal 2-complement $K_{2'}$. Since $K_{2'}$ char $K \trianglelefteq G$, $K_{2'} \trianglelefteq G$. Hence by (2), $K_{2'} = 1$ and so $|K| = 2$, which contradicts (1). If $2 \nmid |K|$, then K is a $2'$-group. Hence by (2), $K \leq O_{2'}(G) = 1$, also a contradiction. The theorem is proved. □

Corollary 3.3 *Let M be a maximal subgroup of G with $|G : M| = r$, where r is a prime. Let p be the smallest prime dividing $|M|$. If there exists a Sylow p-subgroup P of M such that every maximal subgroup of P is $\mathfrak{S}_s$-quasinormal in G, then G is soluble.*

Proof By the well known Feit-Thompson's theorem, we may assume that $2 \mid |G|$. If $r = 2$, then M is normal in G. By Lemma 2.6(4), every maximal subgroup of P is $\mathfrak{S}_s$-quasinormal in M. Hence by Lemma 3.2, M is soluble. It follows that G is soluble. If $r \neq 2$, then $p = 2$ and P is a Sylow 2-subgroup of G. By using Lemma 3.2, we obtain that G is soluble.

Corollary 3.4 ([1, Theorem 4.2]) *Let p be the smallest prime dividing $|G|$ and P some Sylow p-subgroup of G. Then G is soluble if and only if every maximal subgroup of P is $\mathfrak{S}_h$-normal in G.*

Theorem 3.5 *Let P be some Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Assume that G is A_4-free and every 2-maximal subgroup of P (if exists) is $\mathfrak{S}_s$-quasinormal in G. Then G is soluble.*

Proof Suppose that the assertion is false and let G be a counterexample of minimal order. Then $p = 2$ by the well-known Feit-Thompson theorem. We proceed with the proof via the following steps.

(1) $O_2(G) = 1$.

Assume that $N = O_2(G) \neq 1$. Then P/N is a Sylow 2-subgroup of G/N. Let M/N be a 2-maximal subgroup of P/N. Then M is a 2-maximal subgroup of P. By the hypothesis and Lemma 2.6(2), M/N is $\mathfrak{S}_s$-quasinormal in G/N. The minimal choice of G implies that G/N is soluble. It follows that G is soluble, a contradiction. Hence (1) holds.

(2) $O_{2'}(G) = 1$.

Assume that $D = O_{2'}(G) \neq 1$. Then PD/D is a Sylow 2-subgroup of G/D. Suppose that M/D is a 2-maximal subgroup of PD/D. Then there exists a 2-maximal subgroup P_2 of P such that $M = P_2D$. By the hypothesis and Lemma 2.6(3), $M/D = P_2D/D$ is $\mathfrak{S}_s$-quasinormal in G/D. Hence G/D is soluble by the choice of G. It follows that G is soluble, a contradiction.

(3) Final contradiction.

Let P_2 be a 2-maximal subgroup of P. By the hypothesis, there exists a normal subroup K of G such that P_2K is s-permutable in G and $(P_2 \cap K)(P_2)_G/(P_2)_G \leq Z_{\infty}^{\mathfrak{S}}(G/(P_2)_G)$. If $K = 1$, then as the same proof in Lemma 3.2, we may assume that $K \neq 1$. Note that $Z_{\infty}^{\mathfrak{S}}(G)$ is a soluble normal subgroup of G. By (1) and (2), we have $(P_2)_G = 1$ and $Z_{\infty}^{\mathfrak{S}}(G) = 1$. This induces that $P_2 \cap K = 1$. If $2 \nmid |K|$, then K is a $2'$-group, Hence by (2), $K \leq O_{2'}(G) = 1$, a contradiction. If $2 \mid |K|$ and $2^2 \nmid |K|$, then by [8, (10.1.9)], K has a normal Hall $2'$-subgroup $K_{2'}$. Since $K_{2'}$ char $K \trianglelefteq G$, $K_{2'} \trianglelefteq G$. Then by (2), $K_{2'} = 1$. It follows that $|K| = 2$, which is impossible by (1). Finally assume that $2^2 \mid |K|$, then $|K_2| = 2^2$ since $P_2 \cap K = 1$, where K_2 is some Sylow 2-subgroup of K. By Lemma 2.4, K is 2-nilpotent. Hence K has a normal 2-complement $K_{2'}$. Since $K_{2'}$ char $K \trianglelefteq G$, $K_{2'} \trianglelefteq G$. Hence $K_{2'} = 1$ by (2) and so $|K| = 2^2$, which contradicts (1). The theorem is proved. $\square$

Corollary 3.6 ([1, Theorem 4.3]) *Let P be some Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Assume that G is A_4-free and every 2-maximal subgroup of P (if exists) is $\mathfrak{S}_h$-normal in G. Then G is soluble.*

Theorem 3.7 *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every 2-maximal subgroup of P (if exists) is $\mathfrak{U}_s$-quasinormal in G and G is A_4-free, then G is p-nilpotent.*

Proof Suppose that the assertion is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G) = 1$, and if $|P| = p^{\alpha}$, then $\alpha \geq 3$.

If $O_{p'}(G) \neq 1$, then by Lemma 2.6(3), we see that every 2-maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$ is $\mathfrak{U}_s$-quasinormal in $G/O_{p'}(G)$. By the minimal choice of G, $G/O_{p'}(G)$ is p-nilpotent and so G is p-nilpotent, a contradiction. Hence, $O_{p'}(G) = 1$. By Lemma 2.4, we have $\alpha \geq 3$.

(2) G is soluble and $O_p(G) \neq 1$.

Obviously, a $\mathfrak{U}_s$-quasinormal subgroup of G is $\mathfrak{S}_s$-quasinormal in G. Hence by Theorem 3.5, we see that G is soluble. It follows from (1) that $O_p(G) \neq 1$.

(3) $O_p(G)$ is a minimal normal subgroup of G and $G/O_p(G)$ is p-nilpotent.

Let N be a minimal normal subgroup of G contained in $O_p(G)$. By Lemma 2.6(2), G/N satisfies the hypothesis. The minimal choice of G implies that G/N is p-nilpotent. If G has another minimal normal subgroup N_1 contained in $O_p(G)$, then G/N_1 is also p-nilpotent. It follows that $G \simeq G/(N \cap N_1)$ is p-nilpotent, a contradiction. Thus N is the uinque minimal normal subgroup of G contained in $O_p(G)$.

Let T/N be a normal p-complement of G/N. By the well known Schur-Zassenhaus theorem, N has a complement H in T and any two complements are conjugate in T. Then $G = PT = PNH = PH$ and $G = TN_G(H) = NN_G(H)$ by Frattini argument. Assume that $\Phi(O_p(G)) \neq 1$. Then $N \leq \Phi(O_p(G))$, and so $G = N_G(H)$ since $\Phi(O_p(G)) \leq \Phi(G)$. This means that H is a normal Hall p'-subgroup of G. It follows that G is p-nilpotent. The contradiction shows that $\Phi(O_p(G)) = 1$. Hence $O_p(G)$ is an elementary abelian group ([2, Theorem 1.8.17]). If $N < O_p(G)$, then $1 \neq O_p(G) \cap N_G(H)$ is normal in $NN_G(H) = G$. Obviously, $N \nsubseteq N_G(H)$, so $N \nsubseteq O_p(G) \cap N_G(H)$, which contradicts the fact that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Hence $O_p(G) = N$ is a minimal normal subgroup of G.

(4) Final contradiction.

Let $T/O_p(G)$ be the normal p'-complement of $G/O_p(G)$. Then by Schur-Zassenhaus theorem, $N = O_p(G)$ has a complement H in T, which is a Hall p'-subgroup of G and any two complements of $O_p(G)$ are conjugate in T. This implies that $G = TN_G(H)$ by Frattini argument. Let P^* be a Sylow p-subgroup of $N_G(H)$ which is contained in P. Thus $P^* = P \cap N_G(H)$. By the choice of G, we have $N_G(H) < G$. Hence $P^* < P$. If $|P : P^*| = p$, then $|G : N_G(H)| = p$. Since p is the smallest prime divisor of $|G|$, $N_G(H)$ is a normal subgroup of G. It follows that H is a normal subgroup of G, a contradiction. Thus $|P : P^*| \geq p^2$. Let P_2 be a 2-maximal subgroup of P and P_1 a maximal subgroup of P with $P^* \leq P_2 < P_1$. If $O_p(G) \leq P_1$, then $O_p(G)P^* \leq P_1$. But since $G = TN_G(H) = O_p(G)N_G(H)$, $P = O_p(G)P^* \leq P_1$, a contradiction. Hence $O_p(G) \nleq P_1$ and so $(P_2)_G = (P_1)_G = 1$ by (3). Then by the hypothesis, there exists a normal subgroup K of G such that P_2K is s-permutable in G and $P_2 \cap K \leq Z_\infty^{\mathfrak{U}}(G)$. If $Z_\infty^{\mathfrak{U}}(G) \neq 1$, then there exists $L \leq Z_\infty^{\mathfrak{U}}(G)$ such that L is a minimal normal subgroup of G with $|L| = r$, for some prime $r \in \pi(G)$. Since $O_{p'}(G) = 1$ by (1), $r = p$, which is impossible. Hence $Z_\infty^{\mathfrak{U}}(G) = 1$. It follows that $P_2 \cap K = 1$. Assume that $P_2K < G$. If $p \nmid |K|$, then K is a p'-group. By (1), $K \leq O_{p'}(G) = 1$. Hence, P_2 is s-permutable in G. By Lemma 2.2(2), we have $O^p(G) \leq N_G(P_2)$. By (1), $p^3 \mid |G|$. Hence, $P_2 \neq 1$. Then $O_p(G) \leq P_2^G \leq P_2^{O^p(G)P} \leq P_1^P \leq P_1$, which contradicts the fact that $O_p(G) \nleq P_1$. If $p \mid |K|$ and $p^2 \nmid |K|$, then by [8, (10.1.9)], K has a normal Hall p'-subgroup of $K_{p'}$. Since $K_{p'}$ char $K \trianglelefteq G$, $K_{p'} \trianglelefteq G$. Then by (1), $K_{p'} = 1$. It follows that $|K| = p$ and so $K = O_p(G)$. Hence by Lemma 2.2(1), $P_2K \leq O_p(G) = K$. This induces that $P_2 \leq K$. Obviously, $P_2 < K$. Hence $|P_2| = 1$, which is impossible by (1). Now assume $p^2 \mid |K|$. Then since $K \trianglelefteq G$, $K_p \leq P$, where K_p is some Sylow p-subgroup of K. Hence $P_2K_p = P$. This induces that $P \subseteq P_2K$. Hence P_2K satisfies the hypothesis by Lemma 2.6(4).

The minimal choice of G implies that P_2K is p-nilpotent. Let H_1 be a p-complement of P_2K. Then by (1), $H_1 = 1$ and so $P_2K = P$. By Lemma 2.2(1), $O_p(G) \leq P_2K = P \leq O_p(G)$. It follows from (3) that $K = P = O_p(G)$. Consequently $P_2 = 1$ and so $|P| = p^2$, which contradicts (1). Therefore $P_2K = G$. In this case, the order of Sylow p-subgroup of K is p^2. By Lemma 2.4, K is p-nilpotent and so K has a normal p-complement H_1. By (1), $H_1 = 1$ and so $|K| = p^2$. It follows that $G = P_2K$ is a p-group. The final contradiction completes the proof. □

Corollary 3.8 ([1, Theorem 5.1]) *Let p be the smallest prime number dividing the order of G and P a Sylow p-subgroup of G. If every 2-maximal subgroup of P is $\mathfrak{U}_h$-normal in G and G is A_4-free, then G is p-nilpotent.*

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