Journal of Mathematical Research with Applications
Jul., 2013, Vol. 33, No. 4, pp. 419–428
DOI:10.3770/j.issn:2095-2651.2013.04.005
Http://jmre.dlut.edu.cn

On the Property of Solutions for a Class of Higher Order Periodic Differential Equations

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Abstract In this paper, the property of linear dependence of solutions for higher order linear differential equation

$$f^{(k)}(z)+A_{k-2}(z)f^{(k-2)}(z)+\cdots+A_0(z)f(z)=0, \tag{$*$}$$

where $A_j(z)$ $(j=0,2,\ldots,k-2)$ are constants and A_1 is a non-constant entire function of period $2\pi i$ and rational in e^z, is investigated. Under certain condition, the representation of solution of Eq. $(*)$ is given, too.

Keywords differential equation; linearly dependent; periodic coefficients.

MR(2010) Subject Classification 30D35; 34M10

1. Introduction and main results

In this paper, we use the standard notations from the Nevanlinna's values distribution theory of meromorphic functions [10, 13]. In addition, we use the notation $\sigma(f)$ and $\lambda(f)$, respectively, to denote the order of growth and the exponent of convergence of the zeros of a meromorphic function f. $\sigma_2(f)$, the hyper-order of $f(z)$, is defined to be

$$\sigma_2(f)=\overline{\lim_{r\to+\infty}}\frac{\log\log T(r,f)}{\log r}.$$

We define as in [7]

$$\sigma_e(f)=\overline{\lim_{r\to+\infty}}\frac{\log T(r,f)}{r}$$

to be the e-type order of a meromorphic function $f(z)$. Obviously, if $f(z)$ is entire, then

$$\sigma_e(f)=\overline{\lim_{r\to+\infty}}\frac{\log\log M(r,f)}{r}.$$

We also define as in [7]

$$\lambda_e(f)=\overline{\lim_{r\to+\infty}}\frac{\log N(r,f)}{r}$$

to be the e-type exponent of convergence of the zeros of $f(z)$.

Received February 1, 2012; Accepted November 22, 2012
Supported by the National Natural Science Foundation of China (Grant Nos. 11126144; 11171119), the Youth Science Foundation of Education Bureau of Jiangxi Province (Grant No. GJJ12207) and the Natural Science Foundation of Jiangxi Province (No. 20132BAB211009).
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For a set $E \subset (1, +\infty)$, we denote

$$m(E) = \int_E \mathrm{d}r, \quad m_l(E) = \int_1^\infty \chi_E(t)\mathrm{d}t/t,$$

where $\chi_E(t)$ denotes the characteristic function of the set E. In accordance with the usual notation, we will use the abbreviation "n.e." (nearly everywhere) to mean "everywhere in $(0, +\infty)$ except in a set of finite linear measure".

The study of the properties of solutions of a linear differential equation with periodic coefficients is one of the difficult aspects in the complex oscillation theory of differential equations. However, it is also one of the important aspects since it relates to many special functions. Many important researches were done by various authors, see, for instance, [1, 2, 4–9, 11].

For the second-order periodic differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

In [1], Bank and Laine proved the following theorem.

Theorem A *Let $A(z)$ be a nonconstant periodic entire function of period ω, which is of finite order of growth and transcendental in $e^{\alpha z}$, where $\alpha = 2\pi i\omega^{-1}$. If $f(z) \not\equiv 0$ is a solution of the equation (1.1) with the property $\lambda(f) < \infty$, then $f(z)$ and $f(z + \omega)$ are linearly dependent.*

In [7], Chiang and Gao proved the following theorem.

Theorem B *Let $A(z) = B(e^z)$, where $B(\zeta) = g_1(1/\zeta) + g_2(\zeta), g_1$ and g_2 are entire functions with g_2 transcendental and $\sigma(g_2)$ not equal to a positive integer or infinity, and g_1 arbitrary.*

(i) Suppose $\sigma(g_2) > 1$. If f is a non-trivial solution of (1.1) with $\lambda_e(f) < \sigma(g_2)$, then $f(z)$ and $f(z + 2\pi i)$ are linearly dependent.

(ii) Suppose $\sigma(g_2) < 1$. If f is a non-trivial solutions of (1.1) with $\lambda_e(f) < 1$, then $f(z)$ and $f(z + 2\pi i)$ are linearly dependent.

For second-order differential equation (1.1), if f_1 and f_2 are two linearly independent solutions, then

$$-4A = \frac{c^2}{E^2} - \frac{E'}{E}^2 + 2\frac{E''}{E},$$

where $E = f_1 f_2$. This formula plays an important role in the proofs of Theorems A and B. But for a higher-order differential equation, there does not exist such formula. So it is more difficult to investigate the properties of solutions for higher-order periodic differential equations.

For a higher order periodic differential equation only with two terms, Gao Shi-An proved the following result in [9].

Theorem C *Let $A(z) = B(e^z)$, where $B(\zeta) = g_1(\frac{1}{\zeta}) + g_2(\zeta), g_1(t)$ and $g_2(t)$ are entire functions, $g_1(t)$ (or $g_2(t)$) is transcendental and $\sigma(g_1)$ (or $\sigma(g_2)$) $< \frac{1}{2}$. If f is a non-trivial solution of the differential equation*

$$f^{(k)} + A(z)f = 0, \tag{1.2}$$

with

$$\log^{+} N(r, \frac{1}{f}) = O(r), \tag{1.3}$$

then $f(z)$ and $f(z + 2\pi i)$ are linearly dependent.

For a general higher-order periodic differential equation, Bank and Langley proved the following theorem in [2].

Theorem D *Suppose that $k \geq 2$ and that $A_0, \ldots, A_{k-2}$ are entire functions of period $2\pi i$, and that f is a non-trivial solution of the differential equation*

$$f^{(k)} + A_{k-2} f^{(k-2)} + \cdots + A_0 f = 0. \tag{1.4}$$

Suppose further that f satisfies $\log^{+} N(r, 1/f) = o(r)$, that A_0 is non-constant and rational in e^z, and that if $k \geq 3$, then $A_1, \ldots, A_{k-2}$ are constants. Then there exists an integer q with $1 \leq q \leq k$ such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent. The same conclusion holds if A_0 is transcendental in e^z, and f satisfies

$$\log^{+} N(r, 1/f) = O(r),$$

and if $k \geq 3$, then as $r \to +\infty$ through a set L_1 of infinite linear measure, we have

$$T(r, A_j) = o(T(r, A_0))$$

for $j = 1, \ldots, k-2$.

Later, Chen proved the following theorem in [5].

Theorem E *Let A_j $(j = 0, \ldots, k-2)$ be entire functions of period $2\pi i, A_j(z) = C_j(\frac{1}{\zeta}) + B_j(\zeta), \zeta = e^z$, and $C_j(t), B_j(t)$ be entire functions with finite order of growth. Let $B_0(t)$ be transcendental with $\sigma(B_0) < \frac{1}{2}, \sigma(B_j) < \sigma(B_0)$ $(j = 1, \ldots, k-2)$ and $\sigma(C_s) < \sigma(B_0)$ $(s = 0, 1, \ldots, k-2)$ if $\sigma(B_0) > 0$; or B_j $(j = 1, \ldots, k-2)$ and C_s $(s = 0, 1, \ldots, k-2)$ be polynomials if $\sigma(B_0) = 0$. If $f(z)$ is a non-trivial solution of (1.4) and satisfies (1.3), then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.*

We can see that the results of Theorem C to Theorem E are under the hypothesis that A_0 of (1.4) is the dominant coefficient. A natural question is what can be said when $A_s(s \in \{1, \ldots, k-2\})$ of (1.4) is the dominant coefficient. When A_1 is the dominant coefficient and transcendental in e^z, the author and Chen have obtained following result recently in [12].

Theorem F *Let $k \geq 3, A_0, \ldots, A_{k-2}$ $(A_0 \not\equiv 0)$ be entire function of period $2\pi i$ satisfying*

$$\max\{\sigma_e(A_j)(j \neq 1)\} < \sigma_e(A_1) < +\infty.$$

If $f(z)$ is a non-trivial solution of Eq. (1.4) satisfying $\lambda_e(f) < \sigma_e(A_1)$, then there exists an integer q with $1 \leq q \leq k$ such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent.

Remark 1 A_1 of Theorem F must be transcendental in e^z.

In this paper, we continue to study the properties of solutions of (1.4) when A_1 is the dominant coefficient and rational in e^z. One of our results is similar to Theorem C. Another result is the representation of solution of Eq. (1.4). We will prove the following Theorems.

Theorem 1 *Let $k \geq 3$. Suppose $A_0(\not\equiv 0), A_2, \ldots, A_{k-2}$ are constants, and A_1 is a non-constant entire function of period $2\pi i$ and rational in e^z. If $f(z)$ is a solution of (1.4) satisfying $\lambda(f) < 1$, then there exists an integer q with $1 \leq q \leq k$ such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent.*

Theorem 2 *Suppose that $k \geq 3$, and A_1 is a non-constant periodic entire function, rational in e^z. Suppose further that $A_0(\not\equiv 0), A_2, \ldots, A_{k-2}$ are constants. If $f(z)$ is a solution of (1.4) satisfying $\lambda(f) < 1$, then there exists an integer q with $1 \leq q \leq k$, a constant d, and rational functions $R(\xi), S(\xi)$, analytic on $0 < |\xi| < +\infty$, such that*

$$f(z) = R(e^{z/q}) \exp(\mathrm{d}z + S(e^{z/q})).$$

2. Lemmas for the proof of Theorems

Lemma 1 ([3]) *Let $g(z)$ be an entire function of infinite order, with the hyper-order $\sigma_2(g) = \sigma$, and ν denote the central index of g. Then*

$$\overline{\lim_{r\to+\infty}} \frac{\log\log \nu(r)}{\log r} = \sigma.$$

Lemma 2 ([10]) *Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f(z)^n P(f) = Q(f),$$

where $P(f), Q(f)$ are differential polynomials in f with meromorphic coefficients b_j, and the degree of $Q(f)$ is at most n. Then

$$m(r, P(f)) = O\{\sum_j m(r, b_j) + S(r, f)\},$$

where $S(r, f) = O\{\log T(r, f) + \log r\}$, n.e..

Lemma 3 ([2]) *Let $A(z)$ be a non-constant entire function with period $2\pi i$. Then*

$$c = \lim_{r\to+\infty} \frac{T(r, A)}{r} > 0.$$

If c is finite, then $A(z)$ is rational in e^z.

Remark 2 When $A(z)$ is a non-constant, entire function and rational in e^z, then $T(r, A) \sim cr$; when $A(z)$ is an entire function and transcendental in e^z, then $r = o\{T(r, A)\}$.

Lemma 4 *Suppose that $k \geq 3, A_0(\not\equiv 0), A_2, \ldots, A_{k-2}$ are constants. A_1 is a non-constant entire function of period $2\pi i$ and rational in e^z. Suppose further that $f, g, f_1, \ldots, f_k$ are all non-trivial solutions of (1.4) satisfying*

$$\max\{\lambda(f), \lambda(g), \lambda(f_i)\ (i = 1, \ldots, k)\} < 1.$$

Then there exist a constant d $(0 < d < 1)$ and a set $L \subset (0, +\infty)$ of infinite linear measure such *that the following hold:*

(1) $\sigma_2(f)=1$;

(2) $\log T(r,f)\neq o(r^d)$ *as* $r\to+\infty$ *in* L;

(3) If f,g *are linearly independent, then we have* $\log T(r,f/g)\neq o(r^d)$ *as* $r\to+\infty$;

(4) $\log T(r,f'/f)=o(r^d)$ *as* $r\to+\infty$ *n.e.;*

(5) If $f_1,\dots,f_k$ *are linearly independent, then the product* $E=f_1\cdots f_k$ *satisfies* $\log T(r,E)=o(r^d)$ *as* $r\to+\infty$ *n.e..*

Proof (1) It is easy to see that every solution f ($\not\equiv 0$) of (1.4) is entire and transcendental. From Wiman-Valiron theory, there exists a set $E_1\subset(1,+\infty)$ with $m_l(E_1)<+\infty$, such that for $j=1,2,\dots,k$ and for z satisfying $|z|=r\notin[0,1]\cup E_1$ and $|f(z)|=M(r,f)$, we have

$$\frac{f^{(j)}(z)}{f(z)}=(\frac{\nu_f(r)}{z})^j(1+o(1)),\tag{2.1}$$

where $\nu_f(r)$ denotes the central index of $f(z)$. For any given $\varepsilon>0$, we have for sufficiently large r,

$$|A_j(z)|\le\exp\{r^{1+\varepsilon}\},\tag{2.2}$$

for $j=0,1,\dots,k-2$. Now, we take a z satisfying $|z|=r\notin[0,1]\cup E_1$ and $|f(z)|=M(r,f)$. Substituting (2.1),(2.2) into (1.4), we obtain

$$(\frac{\nu_f(r)}{|z|})^k|1+o(1)|\le k(\frac{\nu_f(r)}{|z|})^{k-2}|1+o(1)|\exp\{r^{1+\varepsilon}\}.$$

This gives

$$\overline{\lim_{r\to+\infty}}\frac{\log\log\nu_f(r)}{\log r}\le 1+\varepsilon.\tag{2.3}$$

Since ε is arbitrary, by (2.3), we have $\sigma_2(f)\le 1$. We assert that $\sigma_2(f)=1$. Assume that $\sigma_2(f)<1$, since

$$T(r,\frac{f'}{f})=m(r,\frac{f'}{f})+N(r,\frac{f'}{f})=O\{\log T(r,f)+\log r+N(r,\frac{1}{f})\},\quad\text{n.e.},$$

this gives

$$\log T(r,\frac{f'}{f})\le\log\log T(r,f)+\log\log r+\log N(r,\frac{1}{f})+\log M,\quad\text{n.e.},$$

(We denote by M some fixed positive constant, M may be different at each occurrence). Using the fact that $\lambda(f)<1$, we obtain

$$\sigma(\frac{f'}{f})<1.\tag{2.4}$$

On the other hand, we obtain by rewriting (1.4),

$$-A_1=\frac{f}{f'}(\frac{f^{(k)}}{f}+A_{k-2}\frac{f^{(k-2)}}{f}+\cdots+A_2\frac{f''}{f}+\cdots+A_0).\tag{2.5}$$

Since

$$\frac{f^{(j)}}{f}=(\frac{f'}{f})^j+\frac{1}{2}j(j-1)(\frac{f'}{f})^{j-2}(\frac{f'}{f})'+H_{j-2}(\frac{f'}{f}),\tag{2.6}$$

for $j=2,\ldots,k$, where $H_{j-2}(\frac{f'}{f})$ is a differential polynomial in $\frac{f'}{f}$ and its derivatives with constant coefficients, and the degree of $H_{j-2}(\frac{f'}{f})$ is not greater than $j-2$, it follows from (2.6) that

$$\sigma(\frac{f^{(j)}}{f})\le\sigma(\frac{f'}{f}),\tag{2.7}$$

for $j=2,\ldots,k$. By (2.5), (2.7), we get

$$\sigma(\frac{f'}{f})\ge 1,$$

a contradiction to (2.4), so $\sigma_2(f)=1$ holds.

(2) We can choose a constant d satisfying

$$\max\{\lambda(f),\lambda(g),\lambda(f_i)\ (i=1,\ldots,k)\}<d<1.$$

From (1), there exists a sequence $\{r_n\}$ $(r_n\to\infty)$ such that

$$\lim_{r_n\to+\infty}\frac{\log\log T(r_n,f)}{\log r_n}=1.$$

We take $L=\bigcup_{n=1}^{+\infty}[r_n,r_n+1]$, then obviously $m(L)=+\infty$ and

$$\lim_{\substack{r\to+\infty\\ r\in L}}\frac{\log\log T(r,f)}{\log r}=1$$

holds, which gives

$$\log T(r,f)\ne o(r^d)\ \text{ as }\ r\to+\infty\ \text{ in }\ L.$$

(3) Assume that $U=f/g$ satisfies

$$\log T(r,U)=o(r^d)\tag{2.8}$$

as $r\to+\infty$. Substituting $f=gU$ into (1.4) yields,

$$kg^{(k-1)}+B_{k-2}g^{(k-2)}+\cdots+(B_0+A_1)g=0,\tag{2.9}$$

where each coefficient B_j is a polynomial in the logarithmic derivatives $\frac{U^{(m)}}{U'}$ for $m=1,\ldots,k$, and in $A_2,\ldots,A_{k-2}$, so by (2.8),

$$m(r,B_j)=o(r^d),\quad \text{n.e.}\tag{2.10}$$

holds. Since g is a solution of (1.4),

$$g^{(k)}+A_{k-2}g^{(k-2)}+\cdots+A_1g'+A_0g=0\tag{2.11}$$

holds. Eliminating A_1 from (2.9) and (2.11) yields,

$$\frac{g^{(k)}}{g}+A_{k-2}\frac{g^{(k-2)}}{g}+\cdots+A_2\frac{g''}{g}+(-k\frac{g^{(k-1)}}{g}-B_{k-2}\frac{g^{(k-2)}}{g}-\cdots-B_1\frac{g'}{g}-B_0)\frac{g'}{g}+A_0=0.\tag{2.12}$$

Setting $G=\frac{g'}{g}$ and combining (2.6) and (2.12) yields

$$G^k+C_{k-1}G^{k-1}+\cdots+C_0=0\tag{2.13}$$

where each coefficient C_j is a polynomial in the logarithmic derivatives $\frac{G^{(m)}}{G}$ for $m = 1, 2, \ldots, k$ and in $B_0, \ldots, B_{k-2}$. So by (2.10),

$$m(r, C_j) \leq o(r^d) + O\{\log T(r, G)\}, \quad \text{n.e..}$$

By Clunie Lemma (Lemma 2), (2.13) yields

$$m(r, G) \leq o(r^d) + O\{\log T(r, G)\}, \quad \text{n.e..}$$

Using the fact that $\log N(r, G) \leq \log N(r, 1/g) = o(r^d)$, we obtain

$$m(r, G) = o(r^d), \quad \text{n.e..} \tag{2.14}$$

Since for $j = 1, \ldots, k$,

$$m(r, G^{(j)}) \leq m(r, G) + O\{\log T(r, G) + \log r\}, \quad \text{n.e..} \tag{2.15}$$

It follows from (2.6), (2.14) and (2.15) that

$$m(r, \frac{g^{(j)}}{g}) = o(r^d), \quad \text{n.e.,}$$

for $j = 1, \ldots, k$. Substituting it into (1.4) yields

$$\begin{aligned} m(r, A_1) &\leq m(r, A_1 \frac{g'}{g}) + m(r, \frac{g}{g'}) = m(r, \frac{g^{(k)}}{g} + A_{k-2} \frac{g^{(k-2)}}{g} + \cdots, A_2 \frac{g''}{g} + A_0) + m(r, \frac{g}{g'}) \\ &\leq o(r^d) + T(r, \frac{g'}{g}) + o(1) = o(r^d) + m(r, \frac{g'}{g}) + N(r, \frac{g'}{g}) \\ &= o(r^d), \quad \text{n.e.,} \end{aligned}$$

which gives $\sigma(A_1) \leq d < 1$, a contradiction to the condition $\sigma(A_1) = 1$.

(4) By setting $H = \frac{f'}{f}$, (2.5) yields,

$$H^k + D_{k-1} H^{k-1} + \cdots + D_1 H + D_0 = 0, \tag{2.16}$$

where D_j is a polynomial in the logarithmic derivatives $\frac{H^{(m)}}{H}$ for $m = 1, \ldots, k$ and in A_1. It follows from (2.16) that

$$\begin{aligned} m(r, H) &\leq O\{\log T(r, H) + \log r\} + \sum_{m=1}^{k} m(r, \frac{H^{(m)}}{H}) + m(r, A_1) \\ &= O\{\log T(r, H) + \log r\} + m(r, A_1), \quad \text{n.e..} \end{aligned}$$

So

$$\begin{aligned} \log T(r, H) &\leq \log m(r, A_1) + \log\log r + \log N(r, H) + \log M \\ &\leq M(1 + \varepsilon) \log r, \quad \text{n.e..} \end{aligned}$$

This gives $\log T(r, \frac{f'}{f}) = o(r^d)$, n.e. as $r \to +\infty$ as required.

(5) It follows from $A_{k-1} = 0$ that the Wronskian $W(f_1 \cdots f_k)$ is a non-zero constant, say c, we can write

$$\frac{c}{E} = \frac{W(f_1 \cdots f_k)}{f_1 \cdots f_k},$$

so $\frac{c}{E}$ is represented as a determinant in the functions $f_j^{(m)}/f_j$ for $j = 1, \ldots, k$ and $m = 1, \ldots, k-1$. But each of these functions satisfies, by (4)

$$\log T(r, \frac{f_j^{(m)}}{f_j}) = o(r^d) \quad \text{as} \quad r \to +\infty, \quad \text{n.e..}$$

So $\log T(r, E) = o(r^d)$, n.e. as $r \to +\infty$. This completes the proof of Lemma 4. □

Lemma 5 ([2]) *Let $k \geq 2$, L be a subset of $(1, +\infty)$ having infinite linear measure, and $\phi(r)$ be a positive increasing function on $(1, +\infty)$ such that $\phi(r)/\log r \to +\infty$ as $r \to +\infty$. Suppose that $f_1, \ldots, f_k$ are meromorphic in the plane, such that the following hold, as $r \to +\infty$ in L:*

(i) For each j, $\log T(r, f_j) \neq o(\phi(r))$;

(ii) For $i \neq j$, $\log T(r, f_i/f_j) \neq o(\phi(r))$;

(iii) For each j, $\log T(r, f_j'/f_j) = o(\phi(r))$.

Then $f_1, \ldots, f_k$ are linearly independent.

Remark 3 If $f(z)$ satisfies a homogeneous linear differential equation with rational coefficients, and if $f(z)$ has an essential singularity at infinity, then the order of $f(z)$ is a positive rational number. This follows from the Wiman-Valiron theory [1].

3. Proof of Theorem 1

Proof We define $k+1$ solutions of (1.4) from $f(z)$ by

$$f_j(z) = f(z + j2\pi i) \;\; \text{for} \;\; j = 0, \ldots, k.$$

By Lemma 4 (2), there exists a set $L_j \subset (0, +\infty)$ of infinite linear measure for each f_j such that

$$\log T(r, f_j) \neq o(r^d) \;\; \text{as} \;\; r \to +\infty \;\; \text{in} \;\; L_j.$$

We take $L = \bigcup_{j=0}^{k} L_j$, then $m(L) = +\infty$ and for each j

$$\log T(r, f_j) \neq o(r^d) \;\; \text{as} \;\; r \to +\infty \;\; \text{in} \;\; L$$

holds obviously. By Lemma 4 (4), we have for each j

$$\log T(r, f_j'/f_j) = o(r^d), \;\text{n.e. as}\; r \to +\infty.$$

We can assume for $i \neq j$,

$$\log T(r, f_i/f_j) \neq o(r^d) \;\; \text{as} \;\; r \to +\infty,$$

for otherwise by Lemma 4 (3), the functions f_i and f_j are linearly dependent, and the conclusion of the Theorem 1 holds with $q = |i - j|$. Now we can apply Lemma 5 with $\phi(r) = r^d$ to conclude that $\{f_0, \ldots, f_{k-1}\}$ and $\{f_1, \ldots, f_k\}$ are both fundamental solution sets for (1.4). Now form the product

$$E_1 = f_0 \cdots f_{k-1} \;\; \text{and} \;\; E_2 = f_1 \cdots f_k.$$

By Lemma 4 (5), we have

$$\log T(r, E_2/E_1) \leq \log T(r, E_1) + \log T(r, E_2) + O(1) = o(r^d), \tag{3.1}$$

as r$\to +\infty$, n.e.. But $E_2/E_1 = f_k/f_0 = f(z + k2\pi i)/f(z)$, so that (3.1) and Lemma 4 (3) imply that f_0 and f_k are linearly dependent. This completes the proof of Theorem 1. □

4. Proof of Theorem 2

The proof of Theorem 2 is similar to the proof of Theorem 2 in [2].

Proof From Theorem 1, we know that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent for some integer q with $1 \le q \le k$. We can therefore write

$$f(z) = e^{d_1 z} G(e^{z/q}), \tag{4.1}$$

where d_1 is a constant and $G(\xi)$ is analytic on $0 < |\xi| < \infty$. Now for any $R > 1$ and any zero ξ_1 of G in $R^{-1} < |\xi| < R$, there exists z_1 with $|z_1| < q(\log R + \pi)$ and $\exp(z_1/q) = \xi_1$, such that $f(z_1) = 0$. It follows that counting multiplicity, the number n_R of zeros of G in the annulus $R^{-1} < |\xi| < R$ satisfies $\log n_R = o(\log R)$. We can therefore write

$$G(\xi) = \xi^Q u(\xi) v(1/\xi) \exp(K(\xi)) \tag{4.2}$$

where Q is an integer, K is analytic on $0 < |\xi| < \infty$, and u, v are entire of order zero, formed as follows. The function u is the canonical product formed with the zeros of G in $|\xi| \ge 1$, and so has order zero. Similarly, v is a canonical product formed with the zeros of G in $0 < |\xi| < 1$, for each zero ξ_1 of G which satisfies $0 < |\xi_1| < 1$, v has a zero of the same multiplicity at $1/\xi_1$, so has order zero, too.

We first prove that K is rational. Set $h(z) = K(e^{z/q})$. Then

$$f(z) = W(z) e^{h(z)}, \tag{4.3}$$

where $W(z) = e^{d_1 z} \xi^Q u(\xi) v(1/\xi)$ and $\log T(r, W) = o(r)$. Substituting (4.3) in Eq. (1.4) gives

$$(h')^k + \sum_{j=0}^{k-1} B_j (h')^j + A_0 = 0, \tag{4.4}$$

where each B_j is a polynomial in the logarithmic derivatives $W^{(m)}/W$ and $h^{(m)}/h'$ for $m = 1, \ldots, k$ and in A_1, and hence satisfies

$$m(r, B_j) = O\{\log T(r, h') + o(r)\} + m(r, A_1) = O\{\log T(r, h') + o(r)\} + O(r). \tag{4.5}$$

Thus from Clunie Lemma, (4.4) and (4.5) give $T(r, h') = O(r)$, so that $T(r, h) = O(r)$ and by Lemma 3, h is rational in $e^{z/q}$.

We now set $U(\xi) = u(\xi) v(1/\xi)$. By (4.1), (4.2) and the fact K is rational, U satisfies a linear differential equation with rational coefficients. Suppose U is transcendental, from Remark 3, the order of U is a positive rational number, this is a contradiction, since u and v both have order zero. Set $R(e^{z/q}) = u(e^{z/q}) v(e^{-z/q})$, $S(e^{z/q}) = K(e^{z/q})$, $d = d_1 + Q/q$, then $f(z) = R(e^{z/q}) \exp(\mathrm{d}z + S(e^{z/q}))$. This completes the proof of Theorem 2. □

Acknowledgments The author would like to thank the referees for helpful suggestions to improve this paper.

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