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Multiplicity of Positive Solutions for Fractional Differential Equations in Banach Spaces

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Abstract In this paper, existence of multiple positive solutions for fractional differential equations in Banach spaces is obtained by utilizing the fixed point index theory of completely continuous operators.

Keywords fractional differential equations; multiple positive solutions; cone; fixed point index; measure of non-compactness; Banach space.

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1. Introduction

Fractional calculus deals with the generalization of integrals and derivatives of noninteger order. It involves a wide area of applications by bringing into a broader paradigm concepts of physics, mathematics and engineering. This is the main advantage of fractional differential equations in comparison with classical integer-order models. For an extensive collection of such results, we refer the readers to the monographs by Samko et al. [1], Podlubny [2] and Kilbas et al. [3]. For the basic theory and recent developments on the subject, we refer to a text by Lakshmikantham et al. [4]. Recently, there are some papers dealing with the existence of solutions (or positive solutions) of nonlinear fractional differential equation by means of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, adomian decomposition method, lower and upper solution method, etc.), see [5–14].

In a recent paper, by utilizing Guo-Krasnosel'skii fixed point theorem on cones, Zhang [13] investigated the existence and multiplicity of positive solutions for the nonlinear fractional differential equation boundary-value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t)), \quad 0<t<1,\\ u(0)+u'(0)=0, \quad u(1)+u'(1)=0, \end{cases}$$

where $1<\alpha\le 2$ is a real number, D_{0+}^{α} is the Caputo fractional derivative, and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous. In [14], Zhao et al. studied the following fractional eigenvalue

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problems

$$\begin{cases} D_{0+}^{\alpha}u(t) = \lambda f(u(t)), & 0 < t < 1, \\ u(0) + u'(0) = 0, & u(1) + u'(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the Caputo fractional derivative, and $f : [0, +\infty) \to [0, +\infty)$ is continuous. The eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established.

In recent years, the theory of ordinary differential equations in Banach space has become a new important branch of investigation (see, for example, [15–18] and references therein). Byszewski [19] initiated the study of nonlocal Cauchy problems for abstract evolution differential equations. Subsequently several authors discussed the problem for different kinds of nonlinear differential equations and integro-differential equations including functional differential equations in Banach spaces [20–25].

Let E be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P (it is clear, $N \geq 1$). P is called solid if its interior $\overset{\circ}{P}$ is nonempty. If $x \leq y$ and $x \neq y$, we write $x < y$. If P is solid and $y - x \in \overset{\circ}{P}$, we write $x \ll y$. We refer to [16] for details on cone theory.

Motivated by above papers, we are concerned with the existence of multiple solutions for the following fractional differential equations with more general boundary conditions in a Banach space E

$$\begin{cases} D_{0+}^{\alpha}u(t) - f(t, u(t)) = 0, & t \in J', \\ m_1 u(0) + m_2 u'(0) = 0, & n_1 u(1) + n_2 u'(1) = 0, \end{cases} \tag{1}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the Caputo fractional derivative, $J = [0, 1], J' = (0, 1), f \in C[J' \times P, P], m_1, m_2, n_1, n_2$ with $m_1 \leq m_2 < (1 + \frac{n_2}{n_1})m_1$ are nonnegative constants. As far as we know, there are fewer papers considering the multiplicity of positive solutions for BVP (1) in a Banach space. First in this paper, we get the Green function for BVP (1) and discuss its properties. Then, by utilizing the fixed point index theory of completely continuous operators, we obtain the existence results for multiple positive solutions for BVP (1).

2. Several lemmas

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of problem (1). These materials can be found in the recent literature [2, 3, 13].

Definition 1 ([3]) *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to R$ is given by*

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where n is the smallest integer greater than or equal to α, provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 2 ([3]) *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to R$ is given by*

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0,+\infty)$.

Remark 1 ([2]) By Definition 1, under natural conditions on the function $f(t)$, for $\alpha \to n$ Caputo's derivative becomes a conventional n-th derivative of the function $f(t)$.

Lemma 1 ([13]) *Let $\alpha > 0$. Then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in R$, $i = 0,1,2,\ldots,n-1$, where n is the smallest integer greater than or equal to α.

Lemma 2 ([13]) *Let $\alpha > 0$. Assume that $u \in C^n[0,1]$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \text{ for some } c_i \in R,\ i = 0,1,2,\ldots,n-1$$

where n is the smallest integer greater than or equal to α.

Lemma 3 *Suppose that $y \in C[0,1]$. Then the following linear boundary value problem:*

$$\begin{cases} D_{0+}^{\alpha} u(t) = y(t), \quad t \in J', \\ m_1 u(0) + m_2 u'(0) = 0, \quad n_1 u(1) + n_2 u'(1) = 0, \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s)\mathrm{d}s, \tag{3}$$

where

$$G(t,s) = \begin{cases} \dfrac{1}{\Delta\Gamma(\alpha)}[(m_2 - m_1 t)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2} + \Delta(t-s)^{\alpha-1}], \\ \hfill 0 \le s \le t \le 1, \\ \dfrac{1}{\Delta\Gamma(\alpha)}(m_2 - m_1 t)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2}, \quad 0 \le t \le s \le 1, \end{cases} \tag{4}$$

$$\Delta = m_1(n_1 + n_2) - n_1 m_2. \tag{5}$$

Proof By Lemma 2, we can deduce the equation of problem (2) to an equivalent integral equation

$$u(t) = I_{0+}^{\alpha} y(t) - c_0 - c_1 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s - c_0 - c_1 t. \tag{6}$$

According to the properties of the Caputo derivative, we have

$$u'(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} y(s)\mathrm{d}s - c_1. \tag{7}$$

Then, we get

$$u(0) = -c_0, \quad u'(0) = -c_1,$$

$$u(1) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) \mathrm{d}s + u(0) + u'(0),$$

$$u'(1) = \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) \mathrm{d}s + u'(0),$$

which together with the boundary conditions imply that

$$\begin{cases} m_1 c_0 + m_2 c_1 = 0, \\ n_1 c_0 + n_1 c_1 + n_2 c_1 = n_1 I_{0+}^{\alpha} y(1) + n_2 I_{0+}^{\alpha-1} y(1). \end{cases}$$

Therefore,

$$c_0 = \frac{-m_2(n_1 I_{0+}^{\alpha} y(1) + n_2 I_{0+}^{\alpha-1} y(1))}{m_1(n_1+n_2) - n_1 m_2},$$

$$c_1 = \frac{m_1(n_1 I_{0+}^{\alpha} y(1) + n_2 I_{0+}^{\alpha-1} y(1))}{m_1(n_1+n_2) - n_1 m_2}.$$

We can easily get that

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) \mathrm{d}s + \frac{m_2 - m_1 t}{\Delta} \Big[\frac{n_1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) \mathrm{d}s + \\ & \frac{n_2}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) \mathrm{d}s \Big] \\ =& \frac{1}{\Delta\Gamma(\alpha)} \int_0^t [(m_2 - m_1 t)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2} + \Delta(t-s)^{\alpha-1}] y(s) \mathrm{d}s + \\ & \frac{1}{\Delta\Gamma(\alpha)} \int_t^1 (m_2 - m_1 t)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2} y(s) \mathrm{d}s \\ =& \int_0^1 G(t,s) y(s) \mathrm{d}s \quad \Box \end{aligned} \tag{8}$$

Lemma 4 *$G(t,s)$ defined in (4) satisfies the following conditions*

(i) $G(t,s) \in C([0,1] \times [0,1))$ and $G(t,s) > 0$ for $t, s \in (0,1)$,

(ii)

$$\begin{aligned} \max_{t \in [0,1]} G(t,s) \le & \frac{1}{\Delta\Gamma(\alpha)} [m_2(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2} + \Delta(1-s)^{\alpha-1}] \\ \le & \frac{1}{\Delta\Gamma(\alpha)} [(m_2 n_1 + \Delta)(1-s)^{\alpha-1} + m_2 n_2(\alpha-1)(1-s)^{\alpha-2}]. \end{aligned}$$

Proof It follows from the expression of $G(t,s)$ that $G(t,s) \in C([0,1] \times [0,1))$. Notice that $m_1 \le m_2 < (1 + \frac{n_2}{n_1}) m_1$, we can easily get that $G(t,s) \ge 0$ for $t, s \in (0,1)$. In the following, we are in position to show (ii). Let

$$g_1(t,s) = \frac{1}{\Delta\Gamma(\alpha)} [(m_2 - m_1 t)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2} + \Delta(t-s)^{\alpha-1}], \quad s \le t,$$

$$g_2(t,s) = \frac{1}{\Delta\Gamma(\alpha)} (m_2 - m_1 t)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2}, \quad s \ge t.$$

Obviously,

$$\begin{aligned}
\max_{t\in[0,1]} g_1(t,s) \leq& \frac{1}{\Delta\Gamma(\alpha)}[m_2(n_1(1-s)+n_2(\alpha-1))(1-s)^{\alpha-2}+\Delta(1-s)^{\alpha-1}],\ s\in[0,1],\\
\leq& \frac{1}{\Delta\Gamma(\alpha)}[(m_2n_1+\Delta)(1-s)^{\alpha-1}+m_2n_2(\alpha-1)(1-s)^{\alpha-2}].\\
\max_{t\in[0,1]} g_2(t,s) \leq& \frac{1}{\Delta\Gamma(\alpha)}m_2(n_1(1-s)+n_2(\alpha-1))(1-s)^{\alpha-2}\\
\leq& \frac{1}{\Delta\Gamma(\alpha)}[m_2(n_1(1-s)+n_2(\alpha-1))(1-s)^{\alpha-2}+\Delta(1-s)^{\alpha-1}]\\
\leq& \frac{1}{\Delta\Gamma(\alpha)}[(m_2n_1+\Delta)(1-s)^{\alpha-1}+m_2n_2(\alpha-1)(1-s)^{\alpha-2}].\ \square
\end{aligned}$$

Let $C[J,E]=\{u|u:J\to E$ is continuous on $J\}$. Then $C[J,E]$ is a Banach space with the norm $\|\cdot\|_c$, where $\|u\|_c=\max_{t\in J}\{\|u(t)\|\}$. Let

$$C[J,P]=\{u\in C[J,E]: u(t)\geq\theta,\ \forall\ t\in J\}.$$

It is clear, $C[J,P]$ is a cone in space $C[J,E]$. A map $u\in C[J,E]\cap C^{\alpha}[J',E]$ is called a nonnegative solution of BVP (1) if $u(t)\geq\theta$ for $t\in J$ and $u(t)$ satisfies (1). A map $u\in C[J,E]\cap C^{\alpha}[J',E]$ is called a positive solution of BVP (1) if it is a nonnegative solution and $u(t)\not\equiv\theta$.

Let α and α_c denote the Kuratowski measure of non-compactness in E and $C[J,E]$, respectively. For details on the definition and properties of the measure of non-compactness, the reader is referred to references [15, 18].

Lemma 5 ([15]) *Let $H\subset C[J,E]$ be bounded and equicontinuous. Then*

$$\alpha_c(H)=\max_{t\in J}\alpha(H(t)).$$

Lemma 6 ([16]) *Let H be a countable set of strongly measurable function $u:J\to E$ such that there exists a $M\in L[J,R_+]$ such that $\|u(t)\|\leq M(t)$ a.e. $t\in J$ for all $u\in H$. Then $\alpha(H(t))\in L[J,R_+]$ and*

$$\alpha\Big(\Big\{\int_J u(t)\mathrm{d}t: u\in H\Big\}\Big)\leq 2\int_J\alpha(H(t))\mathrm{d}t.$$

Let us list some conditions for convenience.

(H_1) There exist $a,b\in L[J',R_+]$ and $g\in C[R_+\times R_+,R_+]$ such that

$$\|f(t,u_0)\|\leq a(t)+b(t)g(\|u_0\|),\quad \forall t\in R_+, u_0\in P,$$

and

$$\begin{aligned}
a^*=\int_0^1 a(s)(1-s)^{\alpha-1}\mathrm{d}s<+\infty,\quad \overline{a}^*=\int_0^1 a(s)(1-s)^{\alpha-2}\mathrm{d}s<+\infty,\\
b^*=\int_0^1 b(s)(1-s)^{\alpha-1}\mathrm{d}s<+\infty,\quad \overline{b}^*=\int_0^1 b(s)(1-s)^{\alpha-2}\mathrm{d}s<+\infty;
\end{aligned}$$

(H_2) There exists $c\in L[J',R_+]$ such that

$$\frac{\|f(t,u_0)\|}{c(t)\|u_0\|}\to 0,\quad \text{as } u_0\in P, \|u_0\|\to\infty$$

uniformly for $t \in J'$, and

$$c^* = \int_0^1 c(s)(1-s)^{\alpha-1}\mathrm{d}s < +\infty, \quad \overline{c}^* = \int_0^1 c(s)(1-s)^{\alpha-2}\mathrm{d}s < +\infty;$$

(H_3) There exists $d \in L[J', R_+]$ such that

$$\frac{\|f(t,u_0)\|}{d(t)\|u_0\|} \to 0, \quad \text{as } u_0 \in P, \|u_0\| \to 0$$

uniformly for $t \in J'$, and

$$d^* = \int_0^1 d(s)(1-s)^{\alpha-1}\mathrm{d}s < +\infty, \quad \overline{d}^* = \int_0^1 d(s)(1-s)^{\alpha-2}\mathrm{d}s < +\infty;$$

(H_4) For any $t \in J'$ and $r > 0, f(t, P_r) = \{f(t,u_0) : u_0 \in P_r\}$ are relatively compact in E, where $P_r = \{u \in P : \|u\| \le r\}$.

Remark 2 It is clear, (H_4) holds automatically when E is finite dimensional.

Define an operator A as follows

$$(Au)(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s. \tag{9}$$

Lemma 7 *If (H_1) and (H_2) are satisfied, then the operator A defined by (9) is a continuous operator from $C[J,P]$ into $C[J,P]$. If in addition, condition (H_4) is satisfied, then A is also compact.*

Proof Let

$$\varepsilon_0 = \frac{1}{2}\widetilde{\alpha}_1^{-1}, \tag{10}$$

where

$$\widetilde{\alpha}_1 = \frac{1}{\Delta\Gamma(\alpha)}[c^*(m_2n_1 + \Delta) + m_2n_2(\alpha-1)\overline{c}^*].$$

By virtue of conditions (H_1) and (H_2), for $\varepsilon_0 > 0$, there exists $r > 0$ such that

$$\|f(t,u_0)\| \le \varepsilon_0 c(t)\|u_0\|, \quad \forall t \in J', \ u_0 \in P, \ \|u_0\| > r,$$

and

$$\|f(t,u_0)\| \le a(t) + Mb(t), \quad \forall\, t \in J', \ u_0 \in P, \ \|u_0\| \le r,$$

where

$$M = \max\{g(x_0) : 0 \le x_0 \le r\}. \tag{11}$$

Hence

$$\|f(t,u_0)\| \le \varepsilon_0 c(t)\|u_0\| + a(t) + Mb(t), \quad \forall t \in J', \ u_0 \in P. \tag{12}$$

Let $u \in C[J,P]$. We have by (12)

$$\begin{aligned}\|f(t,u(t))\| &\le \varepsilon_0 c(t)\|u(t)\| + a(t) + Mb(t)\\ &\le \varepsilon_0 c(t)\|u\|_c + a(t) + Mb(t), \quad \forall t \in J',\end{aligned} \tag{13}$$

which together with (H_1) and (H_2) shows the convergence of the integrals $\int_0^1(1-s)^{\alpha-1}f(s,u(s))\mathrm{d}s$ and $\int_0^1(1-s)^{\alpha-2}f(s,u(s))\mathrm{d}s$. We can easily know from Lemma 4 that A is well defined. It follows from (9) that $Au \in C[J,E]$ and $(Au)(t) \geq \theta$ for $t \in J$. To summarize, A maps $C[J,P]$ into $C[J,P]$. By (9) and Lemma 4, we get

$$\begin{aligned}
\|(Au)(t)\| =&\Big\| \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s\Big\| \\
\leq&\frac{1}{\Delta\Gamma(\alpha)}\int_0^1 [(m_2n_1+\Delta)(1-s)^{\alpha-1}+m_2n_2(\alpha-1)(1-s)^{\alpha-2}]\cdot \\
&\|f(s,u(s))\|\mathrm{d}s \\
\leq&\frac{1}{\Delta\Gamma(\alpha)}\int_0^1 [(m_2n_1+\Delta)(1-s)^{\alpha-1}+m_2n_2(\alpha-1)(1-s)^{\alpha-2}] \\
&(\varepsilon_0 c(s)\|u\|_c+a(s)+Mb(s))\mathrm{d}s \\
\leq&\frac{\varepsilon_0}{\Delta\Gamma(\alpha)}[c^*(m_2n_1+\Delta)+m_2n_2(\alpha-1)\overline{c}^*]\|u\|_c+\frac{1}{\Delta\Gamma(\alpha)}[(m_2n_1+\Delta) \\
&(a^*+Mb^*)+m_2n_2(\alpha-1)(\overline{a}^*+M\overline{b}^*)] \\
=&\varepsilon_0\widetilde{\alpha}_1\|u\|_c+\widetilde{\beta}_1, \qquad (14)
\end{aligned}$$

where

$$\widetilde{\alpha}_1=\frac{1}{\Delta\Gamma(\alpha)}[c^*(m_2n_1+\Delta)+m_2n_2(\alpha-1)\overline{c}^*],$$

$$\widetilde{\beta}_1=\frac{1}{\Delta\Gamma(\alpha)}[(m_2n_1+\Delta)(a^*+Mb^*)+m_2n_2(\alpha-1)(\overline{a}^*+M\overline{b}^*)].$$

It follows from (14) that $Au \in C[J,P]$ and

$$\|Au\|_c \leq \frac{1}{2}\|u\|_c+\widetilde{\beta}_1, \quad \forall u \in C[J,P]. \qquad (15)$$

Next, we are in position to show that A is continuous. Let $u_m, \overline{u} \in C[J,P], \|u_m-\overline{u}\|_c \to 0$ $(m\to\infty)$. Then $r=\sup_m \|u_m\|_c < +\infty$ and $\|\overline{u}\|_c \leq r$. By (9), we get

$$(Au_m)(t)=\int_0^1 G(t,s)f(s,u_m(s))\mathrm{d}s. \qquad (16)$$

It is clear,

$$f(t,u_m(t)) \to f(t,u(t)) \text{ as } m\to\infty, \quad \forall t \in J', \qquad (17)$$

and by (13), we get

$$\begin{aligned}
\|f(t,u_m(t))-f(t,u(t))\| \leq& 2\varepsilon_0 c(t)r+2a(t)+2Mb(t)=\lambda(t), \\
&\forall\ t\in J'\ (m=1,2,3,\ldots,),\ \lambda \in L[J,R_+], \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
(1-t)^{\alpha-2}\|f(t,u_m(t))-f(t,u(t))\| \leq&(1-t)^{\alpha-2}[2\varepsilon_0 c(t)r+2a(t)+2Mb(t)]=\sigma(t), \\
&\forall\ t\in J'\ (m=1,2,3,\ldots,),\sigma\in L[J,R_+]. \qquad (19)
\end{aligned}$$

It follows from (18), (19), (H_1), (H_2) and the Lebesgue's dominated convergence theorem that

$$\lim_{m\to\infty}\int_0^1 \|f(t,u_m(t))-f(t,u(t))\|\mathrm{d}t=0, \qquad (20)$$

and

$$\lim_{m\to\infty}\int_0^1 (1-t)^{\alpha-2}\|f(t,u_m(t))-f(t,u(t))\|\mathrm{d}t=0. \tag{21}$$

By (9), (16), (20), (21) and Lemma 4, we get

$$\begin{aligned}\|(Au_m)(t)-(Au)(t)\| &\le \int_0^1 G(t,s)\|f(s,u_m(s))-f(s,u(s))\|\mathrm{d}s\\ &\le \int_0^1 \frac{1}{\Delta\Gamma(\alpha)}[(m_2n_1+\Delta)(1-s)^{\alpha-1}+m_2n_2(\alpha-1)(1-s)^{\alpha-2}]\cdot\\ &\quad \|f(s,u_m(s))-f(s,u(s))\|\mathrm{d}s \end{aligned}\tag{22}$$

which means that $\|Au_m-Au\|_c\to 0$ as $m\to\infty$. Thus, the continuity of A is proved.

Finally, assume that condition (H_4) is satisfied, and we are going to show that A is compact. Let $V=\{u_m: m=1,2,3,\ldots\}\subset C[J,P]$ be bounded and $\|u_m\|_c\le\gamma$ $(m=1,2,3,\ldots)$. It is easy to see from (H_1) and (H_2) that

$$\int_0^1 \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(1-s)^{\alpha-1}+m_1n_2(\alpha-1)(1-s)^{\alpha-2})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s<+\infty, \tag{23}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^1 (t_2-s)^{\alpha-1}\cdot(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s<+\infty, \tag{24}$$

and

$$\int_0^1 \frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s<+\infty. \tag{25}$$

By Lebesgue's dominated convergence theorem, we get that

$$\lim_{t_1\to t_2}\int_0^1 \frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s=0. \tag{26}$$

It follows from the absolute continuity of Lebesgue's integral and (24), for any $\varepsilon>0$, there exists $\delta_1>0$ such that for $t_1,t_2\in[0,1], t_1<t_2, t_2-t_1<\delta_1$

$$\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1}\cdot(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s<\frac{\varepsilon}{5}. \tag{27}$$

By (26), we know that there there exists $\delta_2>0$ such that for $t_1,t_2\in[0,1], t_1<t_2, t_2-t_1<\delta_2$

$$\int_0^1 \frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s<\frac{\varepsilon}{5}. \tag{28}$$

Let

$$\delta_3=\frac{\varepsilon}{5}\cdot\Big(\int_0^1 \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(1-s)^{\alpha-1}+m_1n_2(\alpha-1)(1-s)^{\alpha-2})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s\Big)^{-1}.$$

Take $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. Then, for any $u_m\in V, t_1,t_2\in[0,1], t_1<t_2, t_2-t_1<\delta$, we have by (13), (23), (27) and (28) that

$$\begin{aligned}&\|(Au_m)(t_2)-(Au_m)(t_1)\|\\ &\quad=\|\int_0^1 G(t_2,s)f(s,u_m(s))\mathrm{d}s-\int_0^1 G(t_1,s)f(s,u_m(s))\mathrm{d}s\|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_1} |G(t_2,s)-G(t_1,s)|\|f(s,u_m(s))\|\mathrm{d}s + \int_{t_2}^{1} |G(t_2,s)-G(t_1,s)|\|f(s,u_m(s))\|\mathrm{d}s+\\
&\quad \int_{t_1}^{t_2} |G(t_2,s)-G(t_1,s)|\|f(s,u_m(s))\|\mathrm{d}s\\
&= \int_0^{t_1} \Big[\frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(t_2-t_1)(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(t_2-t_1)(1-s)^{\alpha-2})+\\
&\quad \frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\Big]\cdot\|f(s,u_m(s))\|\mathrm{d}s+\\
&\quad \int_{t_2}^{1} \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(t_2-t_1)(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(t_2-t_1)(1-s)^{\alpha-2})\cdot\|f(s,u_m(s))\|\mathrm{d}s+\\
&\quad \int_{t_1}^{t_2} \Big[\frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(t_2-t_1)(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(t_2-t_1)(1-s)^{\alpha-2})+\\
&\quad \frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}\Big]\cdot\|f(s,u_m(s))\|\mathrm{d}s\\
&\leq \delta\int_0^{t_1} \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(1-s)^{\alpha-2})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s+\\
&\quad \delta\int_{t_2}^{1} \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(1-s)^{\alpha-2})\cdot(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s+\\
&\quad \delta\int_{t_1}^{t_2} \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(1-s)^{\alpha-2})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s+\\
&\quad \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\cdot(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s + \int_0^{t_1}\frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\\
&\quad (\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s\\
&\leq 3\delta\int_0^{1} \frac{1}{\Delta\Gamma(\alpha)}(m_1n_1(1-s)^{\alpha-1} + m_1n_2(\alpha-1)(1-s)^{\alpha-2})(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s+\\
&\quad \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\cdot(\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s + \int_0^{1}\frac{1}{\Gamma(\alpha)}((t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1})\\
&\quad (\varepsilon_0\gamma c(s)+a(s)+Mb(s))\mathrm{d}s\\
&\leq \frac{3}{5}\varepsilon + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon, \qquad (29)
\end{aligned}
$$

which implies that $\{(Au_m)(t)\}$ $(m = 1,2,3,\ldots)$ is equicontinuous on J. Hence, by Lemma 5

$$\alpha_c(AV) = \max_{t\in J}\{\alpha((AV)(t))\}, \tag{30}$$

where $AV = \{Au_m : m = 1,2,3,\ldots\}$ and $(AV)(t) = \{(Au_m)(t) : m = 1,2,3,\ldots\}$. It follows from Lemma 6 that

$$\alpha((AV)(t)) \leq 2\int_0^1 G(t,s)\alpha(f(s,V(s)))\mathrm{d}s, \quad \forall t\in J. \tag{31}$$

For fixed $s\in J, V(s)\subset P_\gamma$, we have, by condition (H_4),

$$\alpha(f(s,V(s))) = 0, \quad \forall s\in J'. \tag{32}$$

Hence, it follows from (31) and (32) that $\alpha((AV)(t))$ for $t\in J$. Therefore, by (30), $\alpha_c(AV) = 0$, and the compactness of A is proved. □

Lemma 8 ([26]) *A bounded set W of c_0 is relatively compact if and only if*

$$\lim_{n\to\infty}\Big\{\sup_{w\in W}[\max\{|w_m| : m\geq n\}]\Big\}=0,$$

where $c_0=\{u=(u_1,\ldots,u_n) : u_n\to 0\}$ is a Banach space with norm $\|u\|=\sup_n|u_n|$.

3. Main results

Theorem 1 *Let conditions (H_1)–(H_4) be satisfied. Assume that P is normal and solid, and there exist $v\gg\theta$, $0<t_*<t^*<1$ and $\rho\in C[I,R_+]$ ($I=[t_*,t^*]$) such that*

$$f(t,u_0)\geq\rho(t)v,\quad \forall t_*\leq t\leq t^*,\ u_0\geq v, \tag{33}$$

and

$$\frac{1}{\Delta\Gamma(\alpha)}(m_2-m_1t^*)\int_{t_*}^{t^*}(n_1(1-s)+n_2(\alpha-1)(1-s)^{\alpha-2})\rho(s)\mathrm{d}s>1. \tag{34}$$

Then, BVP (1) has at least two positive solutions $u^*,u^{**}\in C[J,P]\cap C^2[J'E]$ such that $u^*(t)\gg v$ for $t_*\leq t\leq t^*$.

Proof By Lemma 7, operator A defined by (9) is completely continuous from $C[J,P]\to C[J,P]$, and we now need only to show that A has two positive fixed points u^* and u^{**} in $C[J,P]$ such that $u^*(t)\gg v$ for $t_*\leq t\leq t^*$. Choose

$$R>\max\big\{2\widetilde{\beta}_1,\frac{2\|v\|}{\cos t^*}\big\}. \tag{35}$$

Let $U_1=\{u\in C[J,P] : \|u\|_c<R\}$. Then $\overline{U}_1=\{u\in C[J,P] : \|u\|_c\leq R\}$ and, by (14) and (27), we get that

$$A(\overline{U}_1)\subset U_1. \tag{36}$$

Let

$$\varepsilon_1=\frac{1}{2}\Delta\Gamma(\alpha)[d^*(m_2n_1+\Delta)+m_2n_2(\alpha-1)\overline{d}^*]^{-1}.$$

By condition (H_3), for ε_1, there exists a $r_1>0$ such that

$$\|f(t,u_0)\|\leq\varepsilon_1 d(t)\|u_0\|,\quad \forall t\in J', u_0\in P, \|u_0\|\leq r_1. \tag{37}$$

Then for $u\in C[J,P]$ with $\|u\|_c\leq r_1$, we have by (37) that

$$\|f(t,u(t))\|\leq\varepsilon_1 d(t)\|u\|_c,\quad \forall t\in J'. \tag{38}$$

It follows from (9) and (38) that $Au\in C[J,P]$ and

$$\|(Au)(t)\|\leq\frac{1}{\Delta\Gamma(\alpha)}[d^*(m_2n_1+\Delta)+m_2n_2(\alpha-1)\overline{d}^*]\|u\|_c. \tag{39}$$

Therefore,

$$\|Au\|_c\leq\frac{1}{2}\|u\|_c,\quad \forall u\in C[J,P], \|u\|_c\leq r_1. \tag{40}$$

Choose

$$0<r<\min\big\{r_1,R,\frac{\|v\|}{N}\big\}, \tag{41}$$

where N denotes the normal constant of P and let $U_2 = \{u \in C[J,P] : \|u\|_c < r\}$. Then $\overline{U}_2 = \{u \in C[J,P] : \|u\|_c \leq r\}$, and by (39) and (40),

$$A(\overline{U}_2) \subset U_2. \tag{42}$$

Let $U_3 = \{u \in C[J,P] : \|u\|_c < R, u(t) \gg v, \ \forall\ t \in I\}$. As in the proof of Theorem 1, in [27, p244], we can show that U_3 is open in $C[J,P]$. Let

$$w(t) = 2\frac{\cos t}{\cos t^*}v. \tag{43}$$

It is easy to see that $w \in C[J,P], \|w\| \leq \frac{2}{\cos t^*}\|v\|, w(t) \geq 2v \gg v, \forall\ t \in I$. Hence, $w \in U_3$ and so, $U_3 \neq \emptyset$. Obviously,

$$\overline{U}_3 = \{u \in C[J,P] : \|u\|_c \leq R, u(t) \geq v, \forall\ t \in I\}.$$

Let $u \in U_3$. By (36), we know that $\|Au\|_c \leq R$. On the other hand,

$$\begin{aligned}(Au)(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \geq \int_{t_*}^{t^*} G(t,s)f(s,u(s))\mathrm{d}s \\ &\geq \int_{t_*}^{t^*} \frac{1}{\Delta\Gamma(\alpha)}(m_2 - m_1t^*)(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2}f(s,u(s))\mathrm{d}s \\ &\geq \frac{1}{\Delta\Gamma(\alpha)}(m_2 - m_1t^*)\int_{t_*}^{t^*}(n_1(1-s) + n_2(\alpha-1))(1-s)^{\alpha-2}\rho(s)\mathrm{d}s \cdot v \\ &\gg v. \end{aligned} \tag{44}$$

Hence,

$$A(\overline{U}_3) \subset U_3. \tag{45}$$

It follows from (36), (42), (45) and Lemma 17 that

$$i(A, U_i, C[J,P]) = 1, \quad i = 1,2,3. \tag{46}$$

On the other hand, for any $u \in U_3$, we have $u(t_*) \gg v$, and so,

$$\|u\|_c \geq \|u(t_*)\| \geq \frac{1}{N}\|v\|.$$

As a consequence,

$$U_2 \subset U_1 \subset C[J,P], U_3 \subset U_1 \subset C[J,P], U_3 \cap U_2 = \emptyset. \tag{47}$$

By (46) and (47), we can obtain

$$i(A, U_1\backslash(\overline{U_2 \cup U_3}), C[J,P]) = 1 - 1 - 1 = -1. \tag{48}$$

Therefore, the operator A has at least two fixed points $u^* \in U_3$ and $u^{**} \in U_1\backslash(\overline{U_2 \cup U_3})$, and $\|u^*\|_c > r, \|u^{**}\|_c > r$. Hence, $u^*(t) \not\equiv 0, u^{**}(t) \not\equiv 0$. The proof is completed. $\square$

Remark 3 Condition (H_3) and the continuity of f imply that $f(t,\theta) = \theta$ for $t \in J'$. Hence, under the conditions of Theorem 1, BVP (1) has the trivial solution $u(t) \equiv 0$ besides two positive solutions u^* and u^{**}.

Theorem 2 *Let conditions* (H_1), (H_2) *and* (H_4) *be satisfied. Assume that there exist* $v > \theta, 0 < t_* < t^* < 1$ *and* $\rho \in C[I, R_+](I = [t_*, t^*])$ *such that (33) and (34) hold. Then, BVP (1) has at least one positive solution* $u^* \in C[J,P] \cap C^2[J'E]$ *such that* $u^*(t) \geq v$ *for* $t_* \leq t \leq t^*$.

Proof As in the proof of Theorem 1, we need only to show that A has one positive fixed point $u^* \in C[J,P]$ such that $u^*(t) \geq v$ for $t_* \leq t \leq t^*$. Choose R satisfying (35) and let

$$U_4 = \{u \in C[J,P] : \|u\|_c \leq R, u(t) \geq v, \ \forall\, t \in I\}.$$

Clearly, U_4 is bounded closed convex set in $C[J,P]$ and it is nonempty since $w \in U_4$, where w is defined as in (43). Let $u \in U_4$. By (36), we have that $\|Au\|_c < R$. As in the proof of Theorem 1, we can show that

$$(Au)(t) \geq v, \quad \forall\, t \in I, u \in U_4. \tag{49}$$

Thus, $Au \in U_4$, and therefore

$$A(U_4) \subset U_4. \tag{50}$$

Then, the Schauder fixed point theorem implies that A has at least one fixed point $u^* \in U_4 \subset C[J,P]$ and $u^*(t) \geq v$ for $t \in I$. □

4. An example

Consider the infinite system of scalar fractional singular differential equations

$$\begin{cases} D_{0+}^{\frac{3}{2}} u_n(t) = \dfrac{3}{\sqrt{nt}}(1 + 10u_n(t))^{\frac{1}{2}}, \quad t \in J', \\ u_n(0) + u_n'(0) = 0, \quad u_n(1) + 2u_n'(1) = 0 \ (n = 1, 2, 3, \ldots). \end{cases} \tag{51}$$

Conclusion Infinite system (51) has at least one positive solution $u_n(t)$ satisfying $u_n(t) \geq \frac{1}{n}$ for $t \in [\frac{1}{4}, \frac{1}{2}]$ $(n = 1, 2, 3, \ldots)$.

Proof Let $E = c_0 = \{u = (u_1, \ldots, u_n, \ldots) : u_n \to 0\}$ with norm $\|u\| = \sup_n |u_n|$ and $P = \{u = (u_1, \ldots, u_n, \ldots) \in c_0 : u_n \geq 0, n = 1, 2, 3, \ldots\}$. Then P is a normal cone in E (but P is not solid), and infinite system (51) can be regarded as a BVP of form (1) in E. In this situation, $u = (u_1, \ldots, u_n, \ldots)$, $m_1 = m_2 = n_1 = 1$, $n_2 = 2$, $\alpha = \frac{3}{2}$, $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Delta = 2$ and

$$f_n(t,u) = \frac{3}{\sqrt{nt}}(1 + 10u_n(t))^{\frac{1}{2}}, \quad \forall t \in J', u \in P. \tag{52}$$

It is clear, $f \in C[J' \times P, P]$ and

$$\|f(t,u)\| \leq \frac{3}{\sqrt{t}}(1 + 10\|u\|)^{\frac{1}{2}}, \quad \forall t \in J', u \in P. \tag{53}$$

By direct computation, we have

$$b^* = c^* = \int_0^1 \frac{3}{\sqrt{t}} \cdot \sqrt{1-t}\mathrm{d}t = \frac{3}{2}\pi, \ \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1-t}{\sqrt{t}}\mathrm{d}t = \frac{5}{6}\sqrt{2} - \frac{11}{12}, \tag{54}$$

and

$$\overline{b}^* = \overline{c}^* = \int_0^1 \frac{3}{\sqrt{t}} \cdot (1-t)^{-\frac{1}{2}}\mathrm{d}t = 3\pi, \ \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{1}{\sqrt{t(1-t)}}\mathrm{d}t = \frac{\pi}{6}. \tag{55}$$

It follows from (53)–(55) that (H_1) and (H_2) hold for $a(t) = 0, b(t) = c(t) = \frac{3}{\sqrt{t}}, g(x_0) = (1 + 10x_0)^{\frac{1}{2}}$. Let $t \in J'$ and $r > 0$. For any $w = (w_1, \ldots, w_n, \ldots) \in f(t \times P_r, P_r)$, we have by (52)

$$0 \le w_n \le \frac{3}{\sqrt{nt}}(1 + 10r)^{\frac{1}{2}}, \quad n = 1, 2, 3, \ldots.$$

So, the relative compactness of $f(t, P_r)$ in c_0 follows directly from Lemma 8. Hence, condition (H_4) is satisfied.

Let $v = (1, 1/2, \ldots, 1/n, \ldots)$. Then, $v > \theta$. For $t \in [\frac{1}{4}, \frac{1}{2}], u > v$, we have by (52)

$$f_n(t, u) \ge \frac{3}{\sqrt{nt}}(1 + 10\frac{1}{n})^{\frac{1}{2}} \ge \frac{3\sqrt{10}}{\sqrt{nt}} \cdot \frac{1}{\sqrt{n}} = \frac{1}{n} \cdot \frac{3\sqrt{10}}{\sqrt{t}},$$

which implies that (33) holds for $\rho(t) = \frac{3\sqrt{10}}{\sqrt{t}}$. Thus, by (54) and (55), we get

$$\begin{aligned} &\frac{1}{\Delta\Gamma(\alpha)}(m_2 - m_1 t^*) \int_{t_*}^{t^*} (n_1(1 - s) + n_2(\alpha - 1)(1 - s)^{\alpha-2})\rho(s)\mathrm{d}s \\ &= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \int_{\frac{1}{4}}^{\frac{1}{2}} [(1 - s) + (1 - s)^{-\frac{1}{2}}] \cdot \frac{3\sqrt{10}}{\sqrt{s}}\mathrm{d}s \\ &= 3\sqrt{10} \times \frac{\sqrt{\pi}}{2\pi}(\frac{5}{6}\sqrt{2} - \frac{11}{12} + \frac{\pi}{6}) \approx 2.1019927969 > 1, \end{aligned} \tag{56}$$

which means that (34) holds. Hence, our conclusion follows from Theorem 2. □

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