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# Convergence of a Multistep Ishikawa Iteration Algorithm for a Finite Family of Lipschitz Mappings and Its Applications

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**Abstract** The purpose of this paper is to investigate the problem of finding a common fixed point of Lipschitz mappings. We introduce a multistep Ishikawa iteration approximation method which is based upon the Ishikawa iteration method and the Noor iteration method, and we prove some necessary and sufficient conditions for the strong convergence of the iteration scheme to a common fixed point of a finite family of quasi-Lipschitz mappings and pseudocontractive mappings, respectively. In particular, we establish a strong convergence theorem of the sequence generated by the multistep Ishikawa scheme to a common fixed point of nonexpansive mappings. As applications, some numerical experiments of the multistep Ishikawa iteration algorithm are given to demonstrate the convergence results.

**Keywords** convex feasibility problem; common fixed point problem; Lipschitz mappings.

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## 1. Introduction

Recall that the convex feasibility problem (CFP) is formulated as follows: If $\bigcap_{i=1}^k C_i \neq \emptyset$,

$$\text{Find a point } x^* \in \bigcap_{i=1}^k C_i, \tag{1}$$

where $k \geq 1$ is an integer, and each $C_i$ is a nonempty closed convex subset of a Hilbert space $H$. It is a central problem for many areas of applied mathematics and the physical sciences. It has been used to model significant real-world problems in image reconstruction from projections [1], in inverse problems in radiation therapy treatment planning [2, 3], in fractal image coding [4], in compressed sensing [5, 6], in image processing [7, 8], in electron microscopy and signal processing [9], etc. A complete and exhaustive study on algorithms and applications for solving the convex feasibility problem can be found in [10]. A common approach to the (CFP) is to use projection algorithms, which employ orthogonal projections onto the individual sets $C_i$. The orthogonal

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projection $P_\Omega(x)$ of a point $x \in H$ onto a closed convex set $\Omega \subseteq H$ is defined by

$$P_\Omega(x) := \arg\min\{\|x - z\| \mid z \in \Omega\}, \tag{2}$$

where $\|\cdot\|$ denotes the norm in $H$. There are two basic algorithmic structures of projection algorithms, one is the sequential projection algorithm; the other is the simultaneous projection algorithm.

The well-known "Projections Onto Convex Sets" (POCS) algorithm for the (CFP) is a sequential projection algorithm. The POCS algorithm is defined by the iteration process

$$\begin{cases} x_1 \in H, \\ x_{n+1} = x_n + \lambda_n \left(P_{C_{i(n)}}(x_n) - x_n\right), \quad n \geq 1, \end{cases} \tag{3}$$

where $\{\lambda_n\}$ are relaxation parameters and the control sequence $\{i(n)\}$ is periodic, i.e., $i(n) = n(\operatorname{mod} k) + 1$.

The simultaneous projection algorithm is defined by the following

$$\begin{cases} x_1 \in H, \\ x_{n+1} = x_n + \lambda_n \Big(\sum_{i=1}^{k} \omega_i P_i(x_n) - x_n\Big), \quad n \geq 1, \end{cases} \tag{4}$$

where $\{\omega_i\}_{i=1}^k \subseteq (0,1)$ and $\sum_{i=1}^k \omega_i = 1$.

Since the projection algorithm (3) and (4) perform projections onto the individual sets, otherwise the whole family of sets, which make them successful in real-world applications. The sequential algorithmic structures cater for the row action approach while simultaneous algorithmic structures favor parallel computing platforms.

It is observed that the (CFP) can be formulated as finding a common fixed point problem (CFPP) for nonlinear mappings:

$$x^* \in \bigcap_{i=1}^{k} \operatorname{Fix}(T_i), \tag{5}$$

where each $T_i : H \to H$ is a (nonlinear) mapping, $\operatorname{Fix}(T_i)$ denotes the fixed point set of $T_i$. If we take $T_i = P_{C_i}$, then the common fixed point problem (CFPP) is reduced to (CFP).

It is an interesting problem to find out for what kind of mappings $T_i$ one can solve (CFPP) iteratively (assuming the existence of solutions). The iterative methods for solving (CFPP) can be obtained by replacing the projection operators $P_{C_i}$ of (3) and (4) by operators $T_i$. Hence, the iterative algorithms are defined by the recursion:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = x_n + \lambda_n (T_{[n]}(x_n) - x_n), \quad n \geq 1, \end{cases} \tag{6}$$

where $[n] = n(\operatorname{mod} k) + 1$, and

$$\begin{cases} x_1 \in H, \\ x_{n+1} = x_n + \lambda_n \Big(\sum_{i=1}^{k} \omega_i T_i(x_n) - x_n\Big), \quad n \geq 1. \end{cases} \tag{7}$$

The iterative algorithms (6) and (7) maintain the sequential and simultaneous algorithmic structure of (3) and (4), respectively. In the literature, there exists a lot of work for solving (CFPP) by virtue of iterative schemes (6) and (7). Crombez [11] studied the iterative algorithm (7) for finding common fixed point of a finite set of paracontractions. The Halpern type iterative schemes could be seen as a special case of (6) which is defined by:

$$\begin{cases} x_1 = x \in H, \\ x_{n+1} = (1-\lambda_n)x + \lambda_n T_{[n]} x_n, \quad n \geq 1. \end{cases} \tag{8}$$

Bauschke [12] first used the Halpern type iterative sequence (8) to find a common fixed point of a finite family of nonexpansive mappings. Wang [13] constructed an explicit cyclic iteration scheme which is based on the algorithmic structure (6) to approximate a common fixed point of a finite family of nonself asymptotically nonexpansive mappings and proved some strong and weak convergence theorems for such mappings. Osilike and Shehu [14], Qin et al. [15] independently proved that the explicit cyclic iteration scheme converges weakly to a common fixed point of a finite family of asymptotically strictly pseudocontractive mappings. It is recommended for the interested readers to [16] and [17] for an extensive study of the theory about iterative fixed point theory.

Besides the iterative algorithms (6) and (7), there exists other important algorithms to solve (CFPP) which has a different algorithmic structure from (6) and (7). It is usually called the multistep Ishikawa iterative sequence which is defined by the following form:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_{1n})x_n + \alpha_{1n} T_1 y_{1n}, \\ y_{1n} = (1-\alpha_{2n})x_n + \alpha_{2n} T_2 y_{2n}, \\ \qquad \vdots \\ y_{(k-1)n} = (1-\alpha_{kn})x_n + \alpha_{kn} T_k x_n, \quad n \geq 1, \end{cases} \tag{9}$$

where $\{\alpha_{in}\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ for each $i = 1, 2, \ldots, k$. The multistep Ishikawa iterative sequence (9) can be rewritten more compactly as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_{1n})x_n + \alpha_{1n} T_1 y_{1n}, \\ y_{jn} = (1-\alpha_{(j+1)n})x_n + \alpha_{(j+1)n} T_{j+1} y_{(j+1)n}, \quad n \geq 1, \end{cases} \tag{10}$$

where $j = 1, 2, \ldots, k-1$. It is worth mentioning that the multistep Ishikawa iterative sequence includes many well-known iteration schemes, such as the Ishikawa iteration scheme [18], the Krasnoselskij-Mann iteration scheme [19], the Noor iteration scheme [20], etc. We demonstrate the algorithmic structure of (6), (7) and (9) with the aid of Figure 1. For simplicity, we take the convex sets to be hyper planes, denoted by $C_1$ and $C_2$, and assume all operators $T_i$ to be orthogonal projections onto the hyper planes.

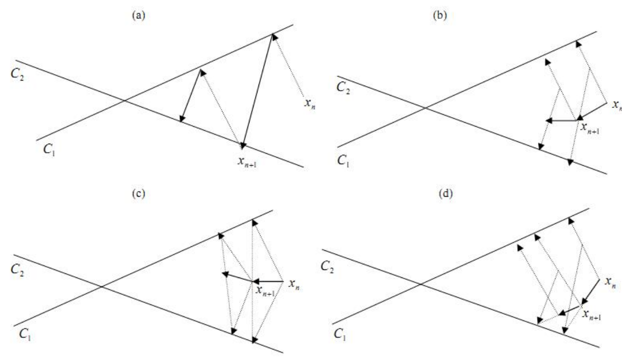


Figure 1 (a) Kaczmarz projections algorithm. (b) Sequential projections algorithm. (c) Simultaneous projections algorithm. (d) Multistep Ishikawa projections algorithm.

Figure 1(a) depicts the Kaczmarz projections' algorithm, which is a special case of sequential projections' algorithm. It is also called POCS (Projections Onto Convex Sets) algorithm. Figure 1(b) exhibits the sequential algorithmic structure with relaxation parameters belonging to $(0, 1)$. In Figure 1(c) we show the fully simultaneous algorithmic structure. Cimmino first proposed such an algorithm for solving linear equations, which involved orthogonal reflections instead of orthogonal projections. Figure 1(d) gives the examples of multistep Ishikawa iterative algorithm.

Since the iterative schemes (6) and (7) belong to the sequential algorithm and the simultaneous algorithm, respectively. They have been used to solve the (CFPP) successfully. Nevertheless, the multistep Ishikawa iterative scheme (9) neither belongs to the sequential algorithm nor the simultaneous algorithm. We are motivated to use the multistep Ishikawa iterative scheme (9) to solve the (CFPP).

The paper is organized as follows. In the next section, we present some preliminaries. In Section 3, we prove necessary and sufficient condition for the strong convergence of the multistep Ishikawa iterative sequence to a common fixed point of quasi-Lipschitz mappings. In particular, we prove that the sequence generated by this iterative scheme converges strongly to a common fixed point of nonexpansive mappings. In Section 4, we establish some necessary and sufficient conditions for the strong convergence of the multistep Ishikawa iterative sequence to fixed point of a finite family of pseudocontractive mappings. In Section 5, we give numerical examples to demonstrate the convergence results. Lastly, we make conclusion and give some recommendation for future work.

## 2. Preliminaries

Throughout this paper, $E$ is a real Banach space and $E^*$ is the dual space of $E$. $\langle \cdot, \cdot \rangle$ denotes the duality pairing of $E$ and $E^*$. $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad x \in E.$$

We shall use the notations $\rightharpoonup$ and $\to$ for weak convergence and strong convergence, respectively.

**Definition 2.1** *Let $C$ be a nonempty closed convex subset of $E$, $T : C \to C$ be a mapping.*

*(i) $T$ is called pseudocontractive mapping, if there exists $j(x-y) \in J(x-y)$ such that*

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2, \quad \text{for all } \ x, y \in C.$$

*(ii) $T$ is said to be L-Lipschitzian, where $L$ is a positive constant, if*

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \text{for all } \ x, y \in C.$$

*(iii) $T$ is called quasi-Lipschitzian, if* $\mathrm{Fix}(T) \neq \emptyset$ *and there exists a positive constant $L$, such that*

$$\|Tx - p\| \leq L\|x-p\|, \quad \text{for all } \ x \in C, \ p \in \mathrm{Fix}(T).$$

**Remark 2.1** If $T$ is pseudocontractive mapping, then the following inequality holds:

$$\|x-y\| \leq \|x-y+r[(I-T)x-(I-T)y]\|,$$

for all $x, y \in C$ and $r > 0$. If $T$ is a nonexpansive mapping, then it is obvious that $T$ is pseudocontractive mapping. If $L = 1$ in (ii) and (iii), then $T$ is called nonexpansive mapping and quasi-nonexpansive mapping, respectively.

In the sequel we shall need the following lemma.

**Lemma 2.1** ([21]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+c_n)a_n + b_n, \quad n \geq 1.$$

*If $\sum_{n=1}^{\infty} c_n < +\infty$ and $\sum_{n=1}^{\infty} b_n < +\infty$, then (i) $\lim_{n\to\infty} a_n$ exists; (ii) Further, if $\liminf_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} a_n = 0$.*

## 3. Common fixed point of quasi-Lipschitz mappings

In this section, we first prove a necessary and sufficient condition for a finite family of quasi-Lipschitz mappings. Then, we prove a strong convergence theorem for a finite family of nonexpansive mappings, which plays a central role from the practical point of view.

For the rest of the results, $L_i \geq 1$ denotes the Lipschitz constant of $T_i$ and $L = \max_{1\leq i\leq k}\{L_i\}$.

**Lemma 3.1** *Let $E$ be a real Banach space and $C$ be a nonempty closed convex subset of $E$. Let $\{T_i : i = 1, 2, \ldots, k\} : C \to C$ be a finite family of quasi-Lipschitzian mappings with Lipschitz constants $\{L_i, i = 1, 2, \ldots, k\}$. Suppose that the sequence $\{x_n\}$ is defined by (9) satisfying the condition $\sum_{n=1}^{\infty} \alpha_{1n} < +\infty$. If $F := \bigcap_{i=1}^{k} \mathrm{Fix}(T_i) \neq \emptyset$, then*

*(i) There exists a sequence $\{r_n\} \subseteq (0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < +\infty$ and*

$$\|x_{n+1} - p\| \leq (1+r_n)\|x_n - p\|,$$

*for all $p \in F$ and $n \geq 1$. Furthermore, $\lim_{n\to\infty} \|x_n - p\|$ exists.*

*(ii) There exists a constant $M > 1$, for all integer $m \geq 1$ such that*

$$\|x_{n+m} - p\| \leq M\|x_n - p\|,$$

*for all $p \in F$.*

**Proof** (i) Let $p \in F$. By (9), we have

$$\begin{aligned}
\|y_{(k-1)n} - p\| &= \|(1 - \alpha_{kn})(x_n - p) + \alpha_{kn}(T_k x_n - p)\| \\
&\leq (1 - \alpha_{kn})\|x_n - p\| + \alpha_{kn}\|T_k x_n - p\| \\
&\leq [1 + \alpha_{kn}(L - 1)]\|x_n - p\| \\
&\leq L\|x_n - p\|,
\end{aligned}$$

and

$$\begin{aligned}
\|y_{(k-2)n} - p\| &= \|(1 - \alpha_{(k-1)n})(x_n - p) + \alpha_{(k-1)n}(T_{k-1} y_{(k-1)n} - p)\| \\
&\leq (1 - \alpha_{(k-1)n})\|x_n - p\| + \alpha_{(k-1)n} L\|y_{(k-1)n} - p\| \\
&\leq L^2\|x_n - p\|.
\end{aligned}$$

By induction, we obtain

$$\|y_{1n} - p\| \leq (1 - \alpha_{2n})\|x_n - p\| + \alpha_{2n} L^{k-1}\|x_n - p\| \leq L^{k-1}\|x_n - p\|.$$

On the other hand, we get from (9) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_1 y_{1n} - p)\| \\
&\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n} L\|y_{1n} - p\| \\
&\leq (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n} L^k\|x_n - p\| \\
&= [1 + \alpha_{1n}(L^k - 1)]\|x_n - p\| \\
&= (1 + r_n)\|x_n - p\|,
\end{aligned}$$

where $r_n = \alpha_{1n}(L^k - 1)$. Since $\alpha_{1n} < +\infty$, we have $\sum_{n=1}^{\infty} r_n < +\infty$. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Therefore, $\{x_n\}$ is bounded.

(ii) By virtue of the inequality $1 + x \leq e^x$, for all $x \geq 0$. For any integer $m \geq 1$, we have

$$\begin{aligned}
\|x_{n+m} - p\| &\leq (1 + r_{n+m-1})\|x_{n+m-1} - p\| \\
&\leq e^{r_{n+m-1}}\|x_{n+m-1} - p\| \\
&\leq e^{r_{n+m-1}} e^{r_{n+m-2}}\|x_{n+m-2} - p\| \\
&\quad \cdots \\
&\leq e^{\sum_{k=n}^{n+m-1} r_k}\|x_n - p\| \\
&\leq e^{\sum_{n=1}^{\infty} r_n}\|x_n - p\| = M\|x_n - p\|,
\end{aligned}$$

where $M = \sum_{n=1}^{\infty} r_n$. This completes the proof. □

**Theorem 3.1** *Let $E$ be a real Banach space and $C$ be a nonempty closed convex subset of $E$.*

*Let $\{T_i : i = 1, 2, \dots, k\} : C \to C$ be a finite family of quasi-Lipschitz mappings with Lipschitz constant $\{L_i : i = 1, 2, \dots, k\}$. Suppose that the sequence $\{x_n\}$ is defined by (9) satisfying the condition: $\sum_{n=1}^{\infty} \alpha_{1n} < +\infty$. Let $F := \bigcap_{i=1}^{k} \mathrm{Fix}(T_i) \neq \emptyset$. Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{q\in F} \|x - q\|$.*

**Proof** We follow the line of proof in [22]. First, the necessity of Theorem 3.1 is obvious. We just need to prove the sufficiency. From Lemma 3.1(i), we have

$$d(x_{n+1}, F) \leq (1 + r_n)d(x_n, F).$$

By Lemma 2.1 and notice the condition $\liminf_{n\to\infty} d(x_n, F) = 0$, then $\lim_{n\to\infty} d(x_n, F)=0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. Indeed, for any $\epsilon > 0$, there exists an integer $n_1 > 0$ such that

$$d(x_n, F) \leq \frac{\epsilon}{4M}, \quad \text{for all} \quad n \geq n_1.$$

In particular, there exists $p_1 \in F$ and a constant $n_2 > n_1$ such that

$$\|x_{n_2} - p_1\| < \frac{\epsilon}{2M}.$$

Using Lemma 3.1(ii) and the above inequality, for all $n \geq n_2$ and $m \geq 1$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|p_1 - x_n\| \\ &\leq 2M\|x_{n_2} - p_1\| < 2M\frac{\epsilon}{2M} = \epsilon. \end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. Since $C$ is a nonempty closed convex subset of $E$, there exists a $q \in C$ such that $x_n \to q$ as $n \to \infty$. Finally, we prove that $q \in F$. In fact, note that $d(q, F) = 0$. Therefore, for any $\epsilon_1 > 0$, there exists a $p_2 \in F$ such that $\|p_2 - q\| < \epsilon_1$. Then, we have

$$\begin{aligned} \|T_i q - q\| &\leq \|T_i q - p_2\| + \|p_2 - q\| \\ &\leq (1 + L)\|p_2 - q\| \leq (1 + L)\epsilon_1. \end{aligned}$$

By the arbitrariness of $\epsilon_1$, we know that $\|T_i q - q\| = 0$, for all $i = 1, 2, \dots, k$, i.e., $q \in F$.

**Remark 3.1** The condition $\sum_{n=1}^{\infty} \alpha_{in} < +\infty$, for all $i = 1, 2, \dots, k$ in [23] weakened to $\sum_{n=1}^{\infty} \alpha_{1n} < +\infty$. Moreover, a simple example of sequences $\{\alpha_{in}\}_{n=1}^{\infty}$, for all $i = 1, 2, \dots, k$ are $\alpha_{1n} = \frac{1}{(n+1)^2}$, $\alpha_{jn} = \delta$, for each $j = 2, \dots, k$, where $\delta \in (0, 1)$ is a constant.

Nonexpansive mapping with a nonempty fixed point set is a special case of quasi-Lipschitz mapping with Lipschitz constant $L = 1$. By using iterative sequence (9), Chidume and Ali [24, 25] proved weak convergence theorems for finite family of (asymptotically) nonexpansive mappings in real uniformly convex Banach spaces that satisfy Opial's condition, or have Fréchet differentiable norms or the dual space $E^*$ of $E$ have the Kadec-Klee property. Further, they proved some strong convergence theorems if one member within the family of (asymptotically) nonexpansive mappings $\{T_i\}$ satisfies completely continuous or semicompact or the family $\{T_i\}_{i=1}^{k}$ satisfies condition $(\overline{C})$ in a real uniformly convex Banach space. For application, we build the following theorem and prove it just in finite dimension case.

**Theorem 3.2** *Let $\{T_i : i = 1, 2, \ldots, k\}$ be a finite family of nonexpansive mappings on Hilbert space $H$. Assume that $F := \bigcap_{i=1}^k \mathrm{Fix}(T_i) \neq \emptyset$. Let $\{\alpha_{in}\}_{n=1}^\infty$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$, $i = 1, 2, \ldots, k$. Then the sequence $\{x_n\}$ defined iteratively (9) converges weakly to a fixed point of $F$.*

Although the above theorem holds for infinite dimensional $H$ using weak convergence, we restrict the discussion here to the finite dimensional case. Before we prove this theorem, for the finite dimensional case, we shall need the following lemma.

**Lemma 3.2** ([24]) *Let $E$ be a real uniformly convex Banach space and $C$ be a closed convex nonempty subset of $E$. Let $\{T_i : i = 1, 2, \ldots, k\} : C \to C$ be a finite family of nonexpansive mappings. Let $\{\alpha_{in}\}_{n=1}^\infty$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$, $i = 1, 2, \ldots, k$. Let $\{x_n\}$ be a sequence defined iteratively by (9). Then,*

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = \cdots = \lim_{n\to\infty} \|x_n - T_k x_n\| = 0.$$

Now, we give the proof of Theorem 3.2 in finite dimensional case.

**Proof** Let $p \in F$. By Lemma 3.1, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|.$$

Consequently, the sequence $\{x_n\}$ is bounded and the sequence $\{\|x_n - p\|\}$ is decreasing. Let $x^*$ be a cluster point of $\{x_n\}$, i.e., there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x^*$. Then by Lemma 3.2, we know that $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, $i = 1, 2, \ldots, k$. Therefore, we have

$$\begin{aligned}\|x^* - T_i x^*\| &\leq \|x^* - x_{n_i}\| + \|x_{n_i} - T_i x_{n_i}\| + \|T_i x_{n_i} - T_i x^*\| \\ &\leq 2\|x^* - x_{n_i}\| + \|x_{n_i} - T_i x_{n_i}\| \to 0 \ \text{ as } \ i \to \infty.\end{aligned}$$

Then $x^* \in \mathrm{Fix}(T_i)$, $i = 1, 2, \ldots, k$, i.e., $x^* \in F$. Therefore, we may use $x^*$ in place of the arbitrary fixed point of $p$. It follows then that the sequence $\{\|x_n - x^*\|\}$ is decreasing. Since a subsequence converges to zero, the entire sequence converges to zero. The proof is completed. □

## 4. Common fixed point of pseudocontractive mappings

The class of pseudocontractive mapping is strongly connected with the nonlinear accretive operators. It is a classical result that if $T$ is an accretive operator, then the solutions of the equations' $Tx = 0$ correspond to the equilibrium points of some evolution systems. In 1974, Ishikawa [18] introduced the so-called Ishikawa iterative process to approximate fixed points of Lipschitz pseudocontractive mapping in Hilbert space. Recently, one of the authors [22] established a necessary and sufficient condition for the strong convergence of the Ishikawa iterative sequence to a fixed point of pseudocontractive mapping in Banach space, which improved the well-known results of Ishikawa [18]. In [18] and [22], they just considered the single Lipschitz pseudocontractive mapping. In order to solve the (CFPP) involved by pseudocontractive mapping, we establish a necessary and sufficient condition for the multistep Ishikawa iterative sequence (9) that guarantees the strong convergence of $\{x_n\}$ to a common fixed point to the family of $\{T_i\}_{i=1}^k$.

We prove the following theorem.

**Theorem 4.1** *Let $C$ be a nonempty convex subset of Banach space $E$. Let $\{T_i : i = 1, 2, \ldots, k\} : C \to C$ be a finite family of Lipschitzian pseudocontractive mappings with Lipschitz constant $\{L_i, i = 1, 2, \ldots, k\}$. Suppose that $F := \bigcap_{i=1}^{k} \mathrm{Fix}(T_i) \neq \emptyset$. Let the sequence $\{x_n\}$ be defined by (9) satisfying the conditions: $\sum_{n=1}^{\infty} \alpha_{1n}\alpha_{2n} < +\infty$ and $\sum_{n=1}^{\infty} \alpha_{1n}^2 < +\infty$. Then*

*(i) There exists a sequence $\{r_n\} \subseteq (0, \infty)$ such that $\sum_{n=1}^{\infty} r_n < +\infty$ and*

$$\|x_{n+1} - p\| \leq (1 + r_n)\|x_n - p\|,$$

*for all $p \in F$ and $n \geq 1$.*

*(ii) There exists a constant $M > 1$, for all integer $m \geq 1$ such that*

$$\|x_{n+m} - p\| \leq M\|x_n - p\|,$$

*for all $p \in F$.*

**Proof** (i) Let $p \in F$. By (9), we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_{1n}x_n - \alpha_{1n}T_1y_{1n} \\ &= x_{n+1} + \alpha_{1n}(I - T_1)x_{n+1} + \alpha_{1n}^2(x_n - T_1y_{1n}) + \alpha_{1n}(T_1x_{n+1} - T_1y_{1n}). \end{aligned} \tag{11}$$

Observe that

$$p = p + \alpha_n(I - T_1)p. \tag{12}$$

Together with (11) and (12), we obtain

$$\begin{aligned} x_n - p =& x_{n+1} - p + \alpha_{1n}[(I - T_1)x_{n+1} - (I - T_1)p] + \\ & \alpha_{1n}^2(x_n - T_1y_{1n}) + \alpha_{1n}(T_1x_{n+1} - T_1y_{1n}). \end{aligned}$$

It follows from Remark 2.1 and (13) that

$$\begin{aligned} \|x_n - p\| \geq& \|x_{n+1} - p + \alpha_{1n}[(I - T_1)x_{n+1} - (I - T_1)p]\| - \\ & \alpha_{1n}^2\|x_n - T_1y_{1n}\| - \alpha_{1n}\|T_1x_{n+1} - T_1y_{1n}\| \\ \geq& \|x_{n+1} - p\| - \alpha_{1n}^2\|x_n - T_1y_{1n}\| - \alpha_{1n}\|T_1x_{n+1} - T_1y_{1n}\|. \end{aligned}$$

This implies that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \alpha_n^2\|x_n - T_1y_{1n}\| + \alpha_{1n}\|T_1x_{n+1} - T_1y_{1n}\|. \tag{14}$$

Since

$$\begin{aligned} \|x_n - T_1y_{1n}\| &\leq \|x_n - p\| + \|p - T_1y_{1n}\| \leq \|x_n - p\| + L\|y_{1n} - p\| \\ &\leq (1 + L^k)\|x_n - p\|, \end{aligned} \tag{15}$$

we have

$$\begin{aligned} \|T_1x_{n+1} - T_1y_{1n}\| &\leq L\|x_{n+1} - y_{1n}\| \\ &\leq L\|x_n - y_{1n} + \alpha_{1n}(T_1y_{1n} - x_n)\| \end{aligned}$$

$$\leq L\|x_n - y_{1n}\| + L\alpha_{1n}\|T_1 y_{1n} - x_n\|$$
$$\leq \alpha_{2n} L(1 + L^{k-1})\|x_n - p\| + \alpha_{1n} L(1 + L^k)\|x_n - p\|. \tag{16}$$

Substituting (15) and (16) into (14), we get

$$\begin{aligned}\|x_{n+1} - p\| \leq &\|x_n - p\| + \alpha_{1n}^2(1 + L^k)\|x_n - p\| + \\ &\alpha_{1n}\alpha_{2n} L(1 + L^{k-1})\|x_n - p\| + \alpha_{1n}^2 L(1 + L^k)\|x_n - p\| \\ =&(1 + r_n)\|x_n - p\|,\end{aligned}$$

where $r_n = \alpha_{1n}^2(1 + L^k) + \alpha_{1n}\alpha_{2n} L(1 + L^{k-1}) + \alpha_{1n}^2 L(1 + L^k)$. Since $\sum_{n=1}^{\infty} \alpha_{1n}\alpha_{2n} < +\infty$ and $\sum_{n=1}^{\infty} \alpha_{1n}^2 < +\infty$, we have $\sum_{n=1}^{\infty} r_n < +\infty$. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists.

(ii) It follows from Lemma 3.1(ii) immediately.

**Theorem 4.2** *Let $C$ be a nonempty closed convex subset of Banach space $E$. Let $\{T_i, i = 1, 2, \ldots, k\} : C \to C$ be a finite family of Lipschitzian pseudocontractive mappings with Lipschitz constant $\{L_i, i = 1, 2, \ldots, k\}$. Suppose that $F := \bigcap_{i=1}^{k} \mathrm{Fix}(T_i) \neq \emptyset$. Let the sequence be defined by (9) satisfying the conditions: $\sum_{n=1}^{\infty} \alpha_{1n}\alpha_{2n} < +\infty$ and $\sum_{n=1}^{\infty} \alpha_{1n}^2 < +\infty$. Then $\{x_n\}$ converges strongly to a fixed point of $F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{q\in F} \|x - q\|$*

**Proof** The proof is the same as that of Theorem 3.1, so it is omitted here.

**Remark 4.1** Theorem 4.2 is significant generalization of the results of [22] from single Lipschitz pseudocontractive mapping to a finite family of Lipschitz pseudocontractive mappings. The real sequence $\{\alpha_{in}\}_{n=1}^{\infty}$ for all $i = 1, 2, \ldots, k$ satisfying the theorem can be chosen by

$$\alpha_{1n} = \alpha_{2n} = \frac{1}{n+1}, \quad \alpha_{jn} = \delta,\ j = 3, \ldots, k,$$

where $\delta \in (0, 1)$.

## 5. Applications

In the following, we provide some numerical experiments for the multistep Ishikawa iteration scheme (9). We consider the linear system equations:

$$Ax = b, \tag{17}$$

where $A_{m\times n}$ is a matrix, $b \in R^m$ and $x$ is the unknown vector. Our simulations are performed in Matlab 7.8 environment.

First, suppose the equation of (17) is consistent (i.e., it has a solution), and the matrix $A$ is a $20 \times 20$ whose entries are independent $N(0, 1)$ random variables. The iterative sequence is generated by (9) with relaxation parameters $\alpha_{in} = 0.6$, for all $i = 1, 2, \ldots, k$ and initial point $x_1$ used here is the zero vector. The numerical results are in Figure 2.

Figure 2 exhibits the error of $\|Ax_n - b\|_2$ with the increasing of iteration's numbers, where

$\|\cdot\|_2$ denotes the Euclidean norm. Secondly, we consider the system of linear equations (17) is corrupted by noise, i.e., $Ax \approx b + r$. The matrix $A$ is the same as in the first experiment with the noise $r$ being Gaussian distribution with mean 0 and variance 0.01. Figure 3 indicates that the multistep Ishikawa iteration algorithm is robust, even in the presence of small noise.

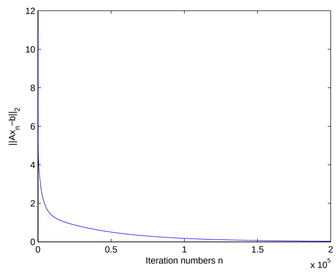


Figure 2 Error of each iteration

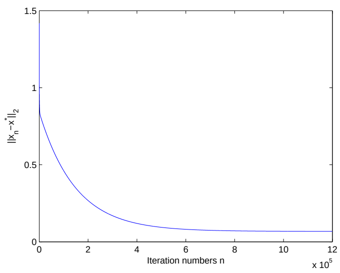


Figure 3 $x^*$ is the true value of system (17) and $x_n$ is generated by the proposed algorithm

## 6. Conclusions

In this paper, we use the multistep Ishikawa iterative process to solve the (CFPP), and hence to solve the (CFP). We found that this iteration scheme is robust when the observed data is corrupted by small noise. The problem concerned with the multistep Ishikawa iterative process is slow convergence. How to accelerate the iteration scheme is the work to be undertaken.

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