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The Unstabilized Amalgamation of Heegaard Splittings along Disconnected Surfaces

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Abstract Let M be a 3-manifold, $\mathcal{F}=\{F_1,F_2,\ldots,F_n\}$ be a collection of essential closed surfaces in M (for any $i,j\in\{1,...,n\}$, if $i\neq j$, F_i is not parallel to F_j and $F_i\cap F_j=\emptyset$) and $\partial_0 M$ be a collection of components of ∂M. Suppose $M-\bigcup_{F_i\in\mathcal{F}}F_i\times(-1,1)$ contains k components $M_1,M_2,\ldots,M_k$. If each M_i has a Heegaard splitting $V_i\bigcup_{S_i}W_i$ with $d(S_i)>4(g(M_1)+\cdots+g(M_k))$, then any minimal Heegaard splitting of M relative to ∂_0M is obtained by doing amalgamations and self-amalgamations from minimal Heegaard splittings or ∂-stabilization of minimal Heegaard splittings of $M_1,M_2,\ldots,M_k$.

Keywords unstabilized; distance; amalgamation; Heeaggard splitting.

MR(2010) Subject Classification 57M25

1. Introduction

All surfaces and 3-manifolds in this paper are assumed to be compact and orientable.

Let M be a 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $\partial_+V=\partial_+W=S$, then we say that $V\bigcup_S W$ is a Heegaard splitting of M, and S is called a Heegaard surface of M. Moreover, if the genus $g(S)$ of S is minimal among all the Heegaard splittings of M, then $g(S)$ is called the genus of M, denoted by $g(M)$. More generally, let M be a 3-manifold with boundary, and ∂_0M be a collection of boundary components of M. If $M=V\bigcup_S W$ is a Heegaard splitting such that $\partial_0M=\partial_-V$ or $\partial_0M=\partial_-W$, then $M=V\bigcup_S W$ is called a Heegaard splitting relative to ∂_0M. The Heegaard genus of M relative to ∂_0M is the smallest possible genus of a Heegaard splitting of M relative to ∂_0M, denoted by $g(M,\partial_0M)$.

If there are two essential disks $B\subset V$ and $D\subset W$ such that $\partial B=\partial D$ (resp., $\partial B\bigcap\partial D=\varnothing$), then $V\bigcup_S W$ is said to be reducible (resp., weakly reducible). Otherwise, it is irreducible (resp., strongly irreducible). If there are two essential disks $B\subset V$ and $D\subset W$ such that $\partial B\bigcap\partial D$ consists of a single point in S, then $V\bigcup_S W$ is said to be stabilized. Otherwise, it is unstabilized.

If a properly embedded surface F in a 3-manifold M is incompressible and not parallel to ∂M, then F is said to be essential.

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The distance between two isotopy classes of essential simple closed curves α and β on S, denoted by $d(\alpha,\beta)$, is the smallest integer $n\geq 0$ so that there is a sequence of essential simple closed curves $\alpha=\alpha_0,\ldots,\alpha_n=\beta$ on S such that α_{i-1} is disjoint from α_i for $1\leq i\leq n$. The distance of the Heegaard splitting $V\bigcup_S W$ is defined to be $\min\{d(\alpha,\beta)|\alpha$ bounds a disk in V and β bounds a disk in $W\}$ (see [1]).

Let M be a 3-manifold, and F be a connected closed surface in M which cuts M into two 3-manifolds M_1 and M_2. If $M_i=V_i\bigcup_{S_i}W_i$ is a Heegaard splitting of M_i $(i=1,2)$, then M has a natural Heegaard splitting called the amalgamation of $V_1\bigcup_{S_1}W_1$ and $V_2\bigcup_{S_2}W_2$ (see [2]). It follows from the construction that $g(M)\leq g(M_1)+g(M_2)-g(F)$.

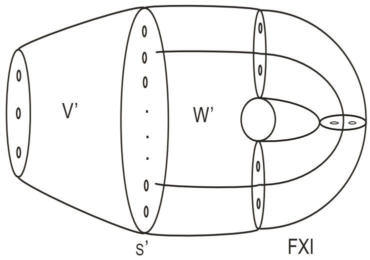

Figure 1 Amalgamation of Heegaard splittings

Suppose now F is an essential non-separating closed surface in M. Let $M'=M-F\times(0,1)$, $F_1=F\times\{0\}$ and $F_2=F\times\{1\}$. If $V'\bigcup_{S'}W'$ is a Heegaard splitting of M' such that F_1 and F_2 lie in the same side of S', say in W', then there is a natural Heegarrd splitting of M as follows. See Figure 1.

Since W' is obtained by attaching some 1-handles to $\partial_-W'\times I$, we can take two unknotted arcs $a=\{a_0\}\times I$ and $b=\{b_0\}\times I$ in $\partial_-W'\times I$, where a_0 and b_0 lie in F, such that they are disjoint from all 1-handles in W'. Let c be another unknotted arc in $F\times[0,1]$, such that $r=a\bigcup b\bigcup c$ is a properly embedded arc in $W'\bigcup F\times[0,1]$. Let $V=V'\bigcup\overline{N(r)}$, $W=\overline{(M-V)}$. It is easy to see that V and W are compression bodies. The Heegaard splitting $V\bigcup_S W$ is said to be the self-amalgamation of $V'\bigcup_{S'}W'$. From this construction, it is easy to see that $g(M)\leq g(M',F_1\bigcup F_2)+1$.

Suppose $M=V\bigcup_S W$ is a Heegaard splitting for M and F is a boundary component of M lying in V. Since V is a compression body, we can take an arc $r=\{r_0\}\times I$ in $V-F\times[0,\frac{1}{2}]$ where $\{r_0\}\times 0\subset F\times\{\frac{1}{2}\}$ and $\{r_0\}\times 1\subset S$. See Figure 2. Let $W'=W\bigcup N(r)\bigcup F\times[0,\frac{1}{2}]$, $V'=\mathrm{cl}\,(M-W')$. It is easy to see that $V'\bigcup_{S'}W'$ is a Heegarrd splitting of M (see [3]). The Heegaard splitting $V'\bigcup_{S'}W'$ is said to be the ∂-stabilization of $V\bigcup_S W$ along F.

An important problem on the amalgamation of Heegaard splitting is when $g(M)<g(M_1)+g(M_2)-g(F)$ and when $g(M)=g(M_1)+g(M_2)-g(F)$. In [4] and [5], the authors constructed their examples of $g(M)<g(M_1)+g(M_2)-g(F)$.

In [6], Lackenby proved that if M is obtained by gluing two simple manifolds M_1 and M_2 via a sufficiently complicated mapping $\phi:\partial M_1\to\partial M_2$, then $g(M)=g(M_1)+g(M_2)-g(F)$. Souto and Li also obtained two different versions of Lackenby's result [7, 8].

From another perspective, Kobayashi and Qiu in [9] proved that if M_1 and M_2 have high distance Heegaard splittings, then the minimal Heegaard splitting of the amalgamated 3-manifold of M_1 and M_2 along F is unique. Yang and Lei in [10] extended the result in [9]. Du in [11] proved that if F is an essential non-separating closed surface in an irreducible 3-manifold M and $M - F \times (-1, +1)$ has a high distance Heegaard splitting, then the minimal Heegaard splitting of M is unique up to isotopy.

In [15], Kobayashi and Rieck defined the amalgamation of two Heegaard splittings along disconnected surfaces. In this paper, we prove that:

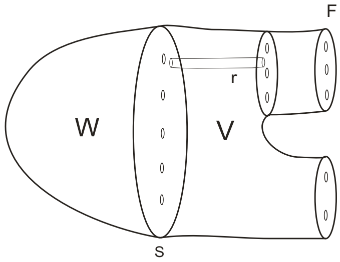

Figure 2 ∂-stabilization of Heegaard splitting

Theorem 1.1 *Let M be a 3-manifold, $\mathcal{F}= \{F_1, F_2, \ldots, F_n\}$ be a collection of essential closed surfaces in M (for any $i, j \in \{1, \ldots, n\}$, if $i \neq j$,F_i is not parallel to F_j and $F_i \cap F_j = \emptyset$) and $\partial_0 M$ be a collection of components of ∂M. Suppose $M - \bigcup_{F_i \in \mathcal{F}} F_i \times (-1, 1)$ consists of k components $M_1, M_2, \ldots, M_k$. If each M_i has a Heegaard splitting $V_i \bigcup_{S_i} W_i$ with $d(S_i) > 4(g(M_1)+\cdots+g(M_k))$, then any minimal Heegaard splitting of M relative to $\partial_0 M$ is obtained by doing amalgamations and self-amalgamations from minimal Heegaard splittings or ∂-stabilization of minimal Heegaard splittings of $M_1, M_2, \ldots, M_k$.*

2. Premilinary

Definition 2.1 *Let M be a 3-manifold. A good separating system $\mathcal{H}$ in M is a collection of closed surfaces $H_1, H_2, \ldots, H_l$, such that*

(1) $M - \bigcup_{i=1}^{l} H_i \times (-1, 1)$ consists of two components, and

(2) for any proper subset $\mathcal{H}'$ of $\mathcal{H}$, $M - \bigcup_{H \in \mathcal{H}'} H \times (-1, 1)$ is connected.

Lemma 2.1 *Let $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ be a collection of closed surfaces in M. Suppose $M - \bigcup_{i=1}^{n} F_i \times (-1, 1)$ has k components $M_1, M_2, \ldots, M_k$. Then there exits a unique subset $\mathcal{F}_0$ of $\mathcal{F}$, such that*

(1) $M - \bigcup_{i=1}^{n} F_i \times (-1, 1)$ consists of k components $\overline{M_1}, \overline{M_2}, \ldots, \overline{M_k}$, and $M_i \subset \overline{M_i}$ for each i;

(2) $\mathcal{F}_0$ is minimal among all the subsets of $\mathcal{F}$ satisfying (1).

Proof We construct a graph with respect to $(\mathcal{M}, \mathcal{F})$ as follows:

(1) The set of vertices is $\{M_1, M_2, \ldots, M_k\}$ and the set of edges is $\{F_1, F_2, \ldots, F_n\}$;

(2) If $F_i \times \{-1\} \subset M_{i_1}$ and $F_i \times \{+1\} \subset M_{i_2}$, then the edge F_i connects M_{i_1} and M_{i_2}(it is possible that $i_1 = i_2$ for some F_i). Let $\mathcal{F}_0 = \{F_i : F_i$ connects distinct vertices M_{i_1} and $M_{i_2}\}$. It is easy to see that $\mathcal{F}_0$ meets the requirement.

Lemma 2.2 ([12, 13]) *Let $M = V \bigcup_S W$ be a Heegaard splitting, and F be an incompressible surface in M. Then either F can be isotoped to be disjoint from S or $d(S) \leq 2 - \chi(F)$.*

Lemma 2.3 ([3]) *Suppose P and Q are two Heegaard surfaces for a compact orientable 3-manifold M. Then either $d(P) \leq 2g(Q)$ or Q is isotopic to P or to a stabilization or ∂-stabilization to P.*

Lemma 2.4 ([3]) *Let $V \bigcup_S W$ be a Heegaard splitting such that $d(S) > 2g(M)$. Then $V \bigcup_S W$ is the unique minimal Heegaard splitting of M up to isotopy.*

Lemma 2.5 ([9]) *Let $M = V \bigcup_S W$ be a strongly irreducible Heegaard splitting, and F be an essential closed surface which cuts M into M_1 and M_2. Then S can be isotoped so that*

(1) Each component of $S \bigcap F$ is an essential simple closed curve on both S and F, and

(2) one of $S \bigcap M_1$ and $S \bigcap M_2$ is incompressible.

In a good separating system, it is the same.

3. Proof of main result

Lemma 3.1 *Suppose that M is a 3-manifold, $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ is a collection of essential closed surfaces (for any $i, j \in \{1, ...n\}$,if $i \neq j$,F_i is not parallel to F_j) and $\partial_0 M$ is a collection of components of ∂M. Suppose that $\mathcal{F}$ is a good separating system of M, and $M - \bigcup_{i=1}^n F_i \times (-1, +1) = M_1 \bigcup M_2$. If each M_i has a Heegaard splitting $V_i \bigcup_{S_i} W_i$ with $d(S_i) > 4(g(M_1) + g(M_2))$, then any minimal Heegaard splitting $V \bigcup_S W$ of M relative to $\partial_0 M$ is obtained by doing amalgamations and self-amalgamations from minimal Heegaard splittings or ∂-stabilization of minimal Heegaard splittings of M_1 and M_2.*

Proof First, we show that S is weakly reducible.

Suppose that S is strongly irreducible. In this case, S can be isotoped so that all components of $S \bigcap F_i$ are essential on both S and F_i $(i = 1, \ldots, n)$, and some $S \bigcap M_i$ are incompressible in M_i, by Lemma 2.5. We note that $\chi(S \bigcap M_i) \geq \chi(S)$. Since $d(S_i) > 4(g(M_1)+g(M_2)) \geq 2g(S) \geq 2 - \chi(S) \geq 2 - \chi(S \bigcap M_i)$, by Lemma 2.2, $S \bigcap M_i$ can be isotoped to disjoint from S_i, hence each component of $S \bigcap M_i$ is parallel into $\bigcup_{i=1}^n F_i$. Then we can isotope S so that $S \bigcap F_i = \varnothing$. This is impossible.

Since S is weakly reducible, by [14], $V \bigcup_S W$ is the amalgamation of strongly irreducible Heegaard splittings, i.e.,

$$V \bigcup_S W = (V_1' \bigcup_{S_1'} W_1') \bigcup_{H_1} (V_2' \bigcup_{S_2'} W_2') \bigcup_{H_2}, \ldots, \bigcup_{H_{m-1}} (V_m' \bigcup_{S_m'} W_m')$$

where each H_i is essential, otherwise $V \bigcup_S W$ is not a minimal Heegarrd splitting of M relative to $\partial_0 M$. It is not hard to see that each component of H_1 is parallel to some F_i.

Now we prove the lemma by induction on $n = |\mathcal{F}|$.

When $n = 1$. Considering H_1, there are two cases:

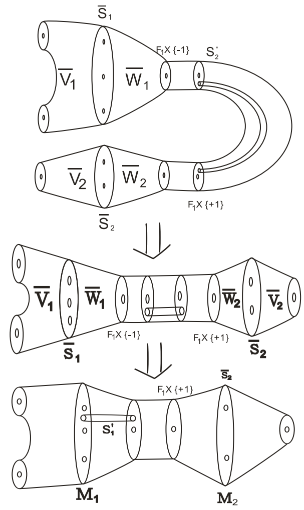

Figure 3 $V\bigcup_S W = ((\overline{V_1}\bigcup_{\overline{S_1}}\overline{W_1})\bigcup(\overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2}))\bigcup_{F_1\times\{\pm 1\}}(V_2''\bigcup_{S_2''}W_2'')$

Case 1 H_1 contains only one copy of F_1, then $V\bigcup_S W = (V_1'\bigcup_{S_1'}W_1')\bigcup_{F_1}(\overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2})$. Without loss of generality, assume $M_1 = V_1'\bigcup_{S_1'}W_1'$, $M_2 = \overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2}$. Notice that $g(S_1') < g(S) = g(M,\partial_0 M) \le 2(g(M_1)+g(M_2)) < \frac{1}{2}d(S_i)$. By Lemma 2.3, S_1' is isotopic to S_1 or to a ∂-stabilization of S_1 (it is easy to see S_1' is not a stabilization of S_1). For the same reason, $\overline{S_2}$ is isotopic to S_2 or to be ∂-stabilization of S_2. So S is as stated.

Case 2 H_1 contains two copies of F_1.

Since F_1 is separating, $V\bigcup_S W = ((\overline{V_1}\bigcup_{\overline{S_1}}\overline{W_1})\bigcup(\overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2}))\bigcup_{F_1\times\{\pm1\}}(V_2''\bigcup_{S_2''}W_2'')$, where $\overline{V_1}\bigcup_{\overline{S_1}}\overline{W_1}$ is a Heegaard splitting of M_1, $\overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2}$ is a Heegaard splitting of M_2 and $V_2''\bigcup_{S_2''}W_2''$ is the unique minimal Heegaard splitting of $F_1\times I$ relative to $F_1\times\partial I$, then

$$\begin{aligned} V\bigcup_S W &= (\overline{V_1}\bigcup_{\overline{S_1}}\overline{W_1}) \bigcup_{F_1\times\{-1\}} (V_2''\bigcup_{S_2''}W_2'') \bigcup_{F_1\times\{+1\}} (\overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2}) \\ &= (V_1''\bigcup_{S_1''}W_1'') \bigcup_{F_1\times\{+1\}} (\overline{V_2}\bigcup_{\overline{S_2}}\overline{W_2}). \end{aligned}$$

It is easy to see that $V_1''\bigcup_{S_1''}W_1''$ is a ∂-stabilization of $\overline{V_1}\bigcup_{\overline{S_1}}\overline{W_1}$. As in Case 1, S_1' $(\overline{S_2})$ is isotopic to S_1 (S_2) or a ∂-stabilization of S_1 (S_2). So S is as stated. (See Figure 3)

Suppose the lemma is true for $n \le k$.

When $n = k+1$. There are again two cases:

Case 1 H_1 contains a good separating system. Similarly to case 1 when $n = 1$, S is as stated.

Case 2 H_1 contains two copies of some F_j.

In this case, $V\bigcup_S W$ is the amalgamation of Heegaard splitting $\overline{V}\bigcup_{\overline{S}}\overline{W}$ of $\overline{M} = \overline{M - F_j\times I}$ and a unique minimal Heegaard splitting $V_2''\bigcup_{S_2''}W_2''$ of $F_j\times I$ relative to $F_j\times\partial I$.

Let $\overline{F_j} = (\bigcup_{i=1}^n F_i - F_j)$ and $\overline{M} = \overline{M - F_j \times I}$. Then $\overline{V} \bigcup_{\overline{S}} \overline{W}$ is a minimal Heegaard splitting of $\overline{M}$ relative to $\overline{\partial_0 M}$, where $\overline{\partial_0 M}$ is a collection of the components of $\overline{\partial M}$, since $V \bigcup_S W$ is a minimal Heegaard splitting of M relative to $\partial_0 M$. In fact, $\overline{\partial_0 M} = \partial_0 M$ or $\partial_0 M \bigcup F_j \times \partial I$. Since $\overline{M} = M_1 \bigcup_{\overline{F_j}} M_2$, by induction, $\overline{V} \bigcup_{\overline{S}} \overline{W} = (\overline{V}_1 \bigcup_{\overline{S}_1} \overline{W}_1) \bigcup_{\overline{F}_j} (\overline{V}_2 \bigcup_{\overline{S}_2} \overline{W}_2)$ where $M_1 = \overline{V}_1 \bigcup_{\overline{S}_1} \overline{W}_1$ and $M_2 = \overline{V}_2 \bigcup_{\overline{S}_2} \overline{W}_2$. Since $d(S_1) \geq 4(g(M_1) + g(M_2)) > 2g(M) > 2g(\overline{S_1})$, by Lemma 2.3, $\overline{S}_1$ is isotopic to S_1 or a ∂-stabilization to S_1 (Obviously, $\overline{S}_1$ is not a stabilization of S_1). Similarly, $\overline{S}_2$ is isotopic to S_2 or a ∂-stabilization to S_2.

Without loss of generality, as illustrated in Figure 4,

$$V \bigcup_S W = (\overline{V}_1 \bigcup_{\overline{S}_1} \overline{W}_1) \bigcup_{F_j \times \{-1\}} (V_2'' \bigcup_{S_{2''}} W_2'') \bigcup_{F_j \times \{+1\}} (\overline{V}_2 \bigcup_{\overline{S}_2} \overline{W}_2)$$

is the amalgamation of a ∂-stabilization of $\overline{S}_1$ and $\overline{V_2''} \bigcup_{\overline{S_{2''}}} \overline{W_2''}$. Hence S is as stated.

Lemma 3.2 *Let M be a 3-manifold, $\mathcal{F} = \{F_1, F_2, \ldots, F_n\}$ be a collection of essential closed surfaces in M, and $\partial_0 M$ be a collection of components of ∂M. Suppose $M_1 = M - \bigcup_{i=1}^n F_i \times (-1, +1)$ is connected and $V_1 \bigcup_{S_1} W_1$ is a Heegaard splitting of M_1 with $d(S_1) > 4g(M_1)$. Then any minimal Heegaard splitting of M relative to $\partial_0 M$ is obtained by doing self-amalgamations from minimal Heegaard splittings or ∂-stabilization of minimal Heegaard splittings of M_1.*

The proof is essentially the same as that of Theorem 1 in [11], and is omitted.

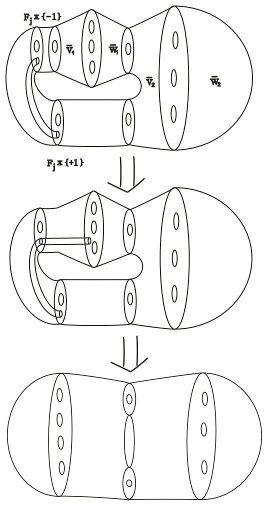

Figure 4 Amalgamation of a ∂-stabilization of $\overline{S}_1$ and $\overline{V_2''} \bigcup_{\overline{S_{2''}}} \overline{W_2''}$

Proof of Theorem 1.1 Let $\mathcal{F}_0$ be the subset of $\mathcal{F}$ as stated in Lemma 2.1, $\mathcal{F}_1 = \mathcal{F} - \mathcal{F}_0$ and let M_i be as in Lemma 2.1.

For any such pair $(M, \mathcal{F})$, define complexity $C(M, \mathcal{F}) = (k, |\mathcal{F}_0|, |\mathcal{F}_1|)$. The proof proceeds by induction on $C(M, \mathcal{F})$. In Lemmas 3.1 and 3.2, we have dealt with the case $k = 2$ and $k = 1$, $|\mathcal{F}_1| = 0$.

Assume $k \geq 2$, suppose that $V \bigcup_S W$ is a minimal Heegaard splitting of M relative to $\partial_0 M$ and $\partial_0 M \subset V$. There are two cases to consider:

Case 1 S is strongly irreducible.

It is easy to see that S can be isotoped so that some $S\bigcap\overline{M_i}$ are incompressible in $\overline{M_i}$ for some i, and each component of $S\bigcap F_i$ is essential on both S and F_i. Hence $S\bigcap M_i$ is incompressible in M_i and $\chi(S\bigcap M_i)\geq\chi(S\bigcap\overline{M_i})\geq\chi(S)$, so $d(S_i)>4(g(M_1)+\cdots+g(M_n))\geq 2g(S)=2-\chi(S)>2-\chi(S\bigcap M_i)$, where S_i is a minimal Heegaard splitting of M_i. As before, $S\bigcap M_i$ can be isotopied to disjoint from S_i, hence disjoint from M_i. That is impossible.

Case 2 S is weakly reducible.

By [14], $V\bigcup_S W$ is the amalgamation of m strongly irreducible Heegaard splittings as

$$(V_1'\bigcup_{S_1'}W_1')\bigcup_{H_1}(V_2'\bigcup_{S_2'}W_2')\bigcup_{H_2}\cdots\bigcup_{H_{m-1}}(V_m'\bigcup_{S_m'}W_m')$$

where each H_i is essential and each component of each H_i is parallel to some F_i. Considering H_1, there are two cases:

Case 2.1 H_1 contains two copies of some F_i. Without loss of generality, let $H_1=F_i\times\{-1,+1\}$, $M_1'=V_1'\bigcup_{S_1'}W_1'=F_i\times[-1,+1]$ and $M_2'=(V_2'\bigcup_{S_2'}W_2')\bigcup_{H_2}\cdots\bigcup_{H_{m-1}}(V_m'\bigcup_{S_m'}W_m')=V_2''\bigcup_{S_2''}W_2''$ (See Figure 5).

Then $V\bigcup_S W$ is an amalgamation of $V_1'\bigcup_{S_1'}W_1'$ and $V_2''\bigcup_{S_2''}W_2''$. Since $V\bigcup_S W$ is a minimal Heegaard splitting of M relative to $\partial_0 M$, $V_1'\bigcup_{S_1'}W_1'$ is a minimal Heegaard splitting of $M_1'=F_i\times[-1,+1]$ relative to $F_i\times\{\pm1\}$ and $M_2'=V_2''\bigcup_{S_2''}W_2''$ is a minimal Heegaard splitting M_2' relative to $\partial_0 M\bigcup F_i\times\{-1,+1\}$.

By Scharlemann-Thompson [16], we see that $V\bigcup_S W$ is obtained by doing self-amalgamation to $V_2''\bigcup_{S_2''}W_2''$. Let $\mathcal{F}'=\mathcal{F}-\{F_i\}$. It is easy to see that $(M_2',\mathcal{F}')$ satisfies the hypothesis of Theorem 1.1 and $C(M_2',\mathcal{F}')<C(M,\mathcal{F})$. By induction, $V_2''\bigcup_{S_2''}W_2''$ is obtained by doing amalgamations and self-amalgamations of minimal Heegaard splittings or ∂-stabilization of minimal Heegaard splittings of $M_2'-\bigcup_{F\in\mathcal{F}'}F\times(-1,+1)=M_1\bigcup M_2\bigcup\cdots\bigcup M_k$. Hence $V\bigcup_S W$ is as stated.

Case 2.2 H_1 does not contain two copies of any F_i.

In this case, H_1 contains a good separating system, each component of which is parallel to some F_i. Without loss of generality, we may assume that $H_1=\{F_1,\ldots,F_j\}$ is a good separating system of M, and S_i in H_1 are not mutually parallel. Assume $M-H_1\times(-1,+1)=M_1'\bigcup M_2'$.

$V\bigcup_S W$ is an amalgamation of Heegaard splittings of M_1' and M_2', either of which is a minimal Heegaard splitting of M_i' relative to some collection of boundary components of $\partial M_i'$, say $\partial_0 M_i'$. Consider M_1' and $\mathcal{F}_1'=\{F:F\in\mathcal{F},F\subset M-H_1\}$. Obviously, $C(M_1',\mathcal{F}_1')<C(M,\mathcal{F})$ and $(M_1',\mathcal{F}_1')$ also satisfies the hypothesis of Theorem 1.1. By induction, $V_1'\bigcup_{S_1'}W_1'$ is obtained by doing amalgamations and self-amalgamations on minimal Heegaard splittings or ∂-stabilization of minimal Heegaard splittings of $M_1'-\bigcup_{F\in\mathcal{F}_1'}F\times(-1,+1)$, so is $V_2''\bigcup_{S_2''}W_2''$ (see Figure 6). Thus $V\bigcup_S W$ is obtained as stated.

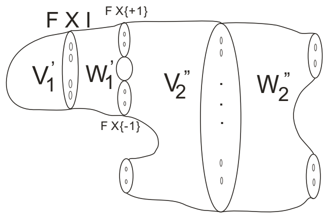

Figure 5 $\bigcup_S W = (F \times I)\bigcup_{F_j\times\{\pm 1\}}(V_2''\bigcup_{S_{2''}} W_2'')$

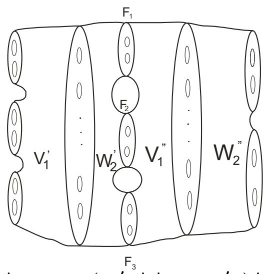

Figure 6 $V\bigcup_S W = (V'_1\bigcup_{S'_1} W'_1)\bigcup(V_2''\bigcup_{S_{2''}} W_2'')$

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