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Complete Convergence of Weighted Sums for ρ^*-Mixing Sequence of Random Variables

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Abstract In this paper, the complete convergence of weighted sums for ρ^*-mixing sequence of random variables is investigated. By applying moment inequality and truncation methods, the equivalent conditions of complete convergence of weighted sums for ρ^*-mixing sequence of random variables are established. We not only promote and improve the results of Li et al. (J. Theoret. Probab., 1995, **8**(1): 49–76) from i.i.d. to ρ^*-mixing setting but also obtain their necessities and relax their conditions.

Keywords ρ^*-mixing sequence of random variables; weighted sums; complete convergence; moment inequality.

MR(2010) Subject Classification 60F15

1. Introduction

Let $\{X_n, n \ge 1\}$ be a sequence of random variables defined on probability space $(\Omega, \mathscr{F}, P)$. Write $\mathscr{F}_S = \sigma(X_k, k \in S) \subset \mathscr{F}$,

$$\rho^*(k) = \sup_{S,T} \Big(\sup_{X \in L^2(\mathscr{F}_S), Y \in L^2(\mathscr{F}_T)} \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X) \cdot \mathrm{Var}(Y)}} \Big)$$

where S, T are the finite subsets of positive integers such that $\mathrm{dist}(S, T) \ge k$.

We call $\{X_n, n \ge 1\}$ a ρ^*-mixing sequence if there exists $k \ge 0$ such that $\rho^*(k) < 1$.

Without loss of generality we may assume that a ρ^*-mixing sequence $\{X_n, n \ge 1\}$ is such that $\rho^*(1) < 1$ (see [1]). The ρ^*-mixing conception is similar to ρ-mixing, but they are quite different from each other. Bryc and Smolenski [1] and Peligrad [2] pointed out the importance of the condition $\rho^*(1) < 1$ in estimating the moments of partial sums or maximum of partial sums. Various limit properties under the condition $\rho^*(1) < 1$ were studied. We refer to Bradley [3] for the central limit theorem, Bryc and Smolenski [1] for moment inequalities and almost sure convergence, An and Yuan [4] for complete convergence of weighted sums for ρ^*-mixing sequence of random variables, and Peligrad and Gut [5] for the Rosenthal-type maximal inequality.

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When $\{X_n, n \geq 1\}$ are independent and identically distributed (i.i.d.), Baum and Katz [6] proved the following remarkable result concerning the convergence rate of the tail probabilities $P(|S_n| > \epsilon n^{1/p})$ for any $\epsilon > 0$, where $S_n = \sum_{i=1}^n X_i$.

Theorem A *Let $0 < p < 2$ and $r \geq p$. Then*

$$\sum_{n=1}^{\infty} n^{\frac{r}{p}-2} P(|S_n| > \epsilon n^{1/p}) < \infty \quad \text{for all} \quad \epsilon > 0,$$

if and only if $E|X_1|^r < \infty$, where $EX_1 = 0$ whenever $1 \leq p < 2$.

There is an interesting and substantial literature of investigation apropos of extending the Baum-Katz Theorem along a variety of different paths. Since partial sums are a particular case of weighted sums and the weighted sums are often encountered in some actual questions, the complete convergence for the weighted sums seems more important. Li et al. [7] discussed the complete convergence for independent weighted sums and obtained the following results.

Theorem B *Let $\{X, X_k, k \in Z\}$ be a sequence of zero mean i.i.d. real random variables and $\{a_{ni}, i \in Z, n \geq 1\}$ be an array of real numbers.*

(i) Let $p > 2$. If $E|X|^p < \infty$, and for some $0 < \delta < \frac{2}{p}$, $2 \leq q < p$,

$$\sum_{k \in Z} |a_{nk}|^2 = O(n^{\delta}) \text{ as } n \to \infty, \text{ and } \sum_{k \in Z} |a_{nk}|^q = o(1) \text{ as } n \to \infty, \tag{1}$$

then, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i \in Z} a_{ni} X_i\Big| > \epsilon n^{1/p}\Big) < \infty. \tag{2}$$

(ii) If

$$\sum_{k \in Z} |a_{nk}|^2 = o(1) \quad \text{as} \quad n \to \infty, \tag{3}$$

and

$$E|X|^2 \log(1 + |X|) < \infty, \tag{4}$$

then, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\Big(\Big|\sum_{i \in Z} a_{ni} X_i\Big| > \epsilon n^{1/2}\Big) < \infty. \tag{5}$$

Wang et al. [8] improved Theorem B and established the necessary and sufficient conditions of complete convergence for weighted sums of i.i.d. random variables. Liang et al. [9] obtained the equivalent conditions of complete convergence of weighted sums of negatively associated random variables.

The main purpose of this paper is to discuss again the above results for ρ^*-mixing sequence of random variables. By applying moment inequality and truncation methods, the equivalent conditions of complete convergence of weighted sums for ρ^*-mixing sequence of random variables are established. We not only promote and improve the results of Li et al. [7] from i.i.d. to ρ^*-mixing setting but also obtain their necessities and relax their conditions.

For the proofs of the main results, we need to restate a few lemmas for easy reference. Throughout this paper, C will represent positive constants, the value of which may change from one place to another. The symbol $I(A)$ denotes the indicator function of A, $[x]$ indicates the maximum integer not larger than x. For a finite set B, the symbol $\sharp B$ denotes the number of elements in the set B. Let $a_n \ll b_n$ denote that there exists a constant $C > 0$ such that $a_n \le C b_n$ for sufficiently large n, and let $a_n \approx b_n$ mean $a_n \ll b_n$ and $b_n \ll a_n$.

The following lemma will play an important role in the proof of our main results. The proof is due to Peligrad and Gut [5].

Lemma 1 *Let $\{X_i, 1 \le i \le n\}$ be a ρ^*-mixing sequence of random variables, $Y_i \in \sigma(X_i)$, $EY_i = 0$, $E|Y_i|^M < \infty$, $i \ge 1$, $M \ge 2$. Then there exists a positive constant C such that*

$$E\Big|\sum_{i=1}^{n} Y_i\Big|^M \le C\Big[\sum_{i=1}^{n} E|Y_i|^M + \Big(\sum_{i=1}^{n} EY_i^2\Big)^{M/2}\Big], \tag{6}$$

$$E\max_{1\le j\le n}\Big|\sum_{i=1}^{j} Y_i\Big|^M \le C\Big[\sum_{i=1}^{n} E|Y_i|^M + (\log_2 n)^M\Big(\sum_{i=1}^{n} EY_i^2\Big)^{M/2}\Big]. \tag{7}$$

Lemma 2 *Let $\{X_n, n \ge 1\}$ be a ρ^*-mixing sequence of random variables, and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of real numbers. Then there exists a positive constant C such that, for any $x \ge 0$ and all $n \ge 1$,*

$$\Big(\frac{1}{2} - P(\max_{1\le i\le n}|a_{ni}X_i| > x)\Big)\sum_{i=1}^{n} P(|a_{ni}X_i| > x) \le \Big(1 + \frac{C}{2}\Big)P(\max_{1\le i\le n}|a_{ni}X_i| > x). \tag{8}$$

Proof Since $\{\max_{1\le i\le n}|a_{ni}X_i| > x\} = \bigcup_{i=1}^{n}\{|a_{ni}X_i| > x, \max_{1\le j\le i-1}|a_{nj}X_j| \le x\}$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} P(|a_{ni}X_i| > x)\\
&= \sum_{i=1}^{n} P\big(|a_{ni}X_i| > x, \max_{1\le j\le i-1}|a_{nj}X_j| \le x\big) + \sum_{i=1}^{n} P\big(|a_{ni}X_i| > x, \max_{1\le j\le i-1}|a_{nj}X_j| > x\big)\\
&= P\big(\max_{1\le i\le n}|a_{ni}X_i| > x\big) + \sum_{i=1}^{n} P\big(|a_{ni}X_i| > x, \max_{1\le j\le i-1}|a_{nj}X_j| > x\big).
\end{aligned} \tag{9}$$

Note that

$$\begin{aligned}
&\sum_{i=1}^{n} P\big(|a_{ni}X_i| > x, \max_{1\le j\le i-1}|a_{nj}X_j| > x\big)\\
&\le E\Big(\sum_{i=1}^{n}(I(|a_{ni}X_i| > x) - EI(|a_{ni}X_i| > x))\Big)I\big(\max_{1\le j\le n}|a_{nj}X_j| > x\big) +\\
&\quad \sum_{i=1}^{n} P(|a_{ni}X_i| > x)P\big(\max_{1\le j\le n}|a_{nj}X_j| > x\big).
\end{aligned} \tag{10}$$

Combining with the Cauchy-Schwarz inequality and (6), we obtain

$$E\Big(\sum_{i=1}^{n}(I(|a_{ni}X_i| > x) - EI(|a_{ni}X_i| > x))\Big)I\big(\max_{1\le j\le n}|a_{nj}X_j| > x\big)$$

$$\leq \sqrt{E\Big(\sum_{i=1}^{n}(I(|a_{ni}X_i| > x) - EI(|a_{ni}X_i| > x))\Big)^2 P\big(\max_{1\leq j\leq n}|a_{nj}X_j| > x\big)}$$
$$\leq \sqrt{C\sum_{i=1}^{n}P(|a_{ni}X_i| > x)P\big(\max_{1\leq j\leq n}|a_{nj}X_j| > x\big)}$$
$$\leq \frac{1}{2}\sum_{i=1}^{n}P(|a_{ni}X_i| > x) + \frac{C}{2}P\big(\max_{1\leq i\leq n}|a_{ni}X_i| > x\big). \tag{11}$$

Now we substitute (11) into (10) and then into (9) and obtain (8). □

Lemma 3 *Let $\{X_n, n \geq 1\}$ be a ρ^*-mixing sequence of random variables, and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. Let $\{b_n, n \geq 1\}$ be a sequence of positive real numbers. If for some $M \geq 2$, $\alpha > 0$ the following conditions are fulfilled*

(a) $\sum_{n=1}^{\infty} b_n \sum_{i=1}^{n} P(|a_{ni}X_i| > n^\alpha) < \infty$,

(b) $\sum_{n=1}^{\infty} b_n n^{-M\alpha} \sum_{i=1}^{n} E|a_{ni}X_i|^M I(|a_{ni}X_i| \leq n^\alpha) < \infty$,

(c) $\sum_{n=1}^{\infty} b_n n^{-M\alpha} (\log_2 n)^M (\sum_{i=1}^{n} E|a_{ni}X_{ni}|^2 I(|a_{ni}X_{ni}| \leq n^\alpha))^{M/2} < \infty$,

then for any $\epsilon > 0$

$$\sum_{n=1}^{\infty} b_n P\Big(\max_{1\leq k\leq n}\big|\sum_{i=1}^{k}(a_{ni}X_i - Ea_{ni}X_iI(|a_{ni}X_i| \leq n^\alpha))\big| > \epsilon n^\alpha\Big) < \infty. \tag{12}$$

Proof Similarly to the proof of Theorem 2.3 in [10], we assume $X_{ni} = a_{ni}X_iI(|a_{ni}X_i| \leq n^\alpha)$. Using Lemma 1, Markov's inequality and C_r inequality, we obtain

$$P\Big(\max_{1\leq k\leq n}\big|\sum_{i=1}^{k}(X_{ni} - EX_{ni})\big| > \epsilon n^\alpha\Big)$$
$$\leq \epsilon^{-M} n^{-M\alpha} E\max_{1\leq k\leq n}\big|\sum_{i=1}^{k}(X_{ni} - EX_{ni})\big|^M$$
$$\leq C\epsilon^{-M} n^{-M\alpha}\Big[\sum_{i=1}^{n} E|X_{ni} - EX_{ni}|^M + (\log_2 n)^M\big(\sum_{i=1}^{n} E(X_{ni} - EX_{ni})^2\big)^{M/2}\Big]$$
$$\leq Cn^{-M\alpha}\Big[\sum_{i=1}^{n} E|X_{ni}|^M + (\log_2 n)^M\big(\sum_{i=1}^{n} EX_{ni}^2\big)^{M/2}\Big]. \tag{13}$$

Moreover, we see that

$$P\Big(\max_{1\leq k\leq n}\big|\sum_{i=1}^{k}(a_{ni}X_i - Ea_{ni}X_iI(|a_{ni}X_i| \leq n^\alpha))\big| > \epsilon n^\alpha\Big)$$
$$\leq P\Big(\max_{1\leq k\leq n}\big|\sum_{i=1}^{k}(X_{ni} - EX_{ni})\big| > \epsilon n^\alpha\Big) + \sum_{i=1}^{n} P(|a_{ni}X_i| > n^\alpha). \tag{14}$$

Therefore, by (13), (14), (a), (b) and (c) we see that (12) holds. □

2. Main results

Now we state our main results. The proofs will be given in Section 3.

Theorem 1 *Let $\{X, X_n, n \geq 1\}$ be a ρ^*-mixing sequence of identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers. Let $r > 1$, $p > 2$. If, for some $2 \leq q < p$,*

$$N(n, m+1) \hat{=} \sharp\{k \geq 1, |a_{nk}| \geq (m+1)^{-1/p}\} \approx m^{q(r-1)/p}, \quad n, m \geq 1; \tag{15}$$

$$EX = 0, \quad \text{when} \ \ q(r-1) \geq 1; \tag{16}$$

$$\sum_{k=1}^{n} |a_{nk}|^2 \ll n^{\delta} \quad \text{when} \ \ q(r-1) \geq 2, \quad \text{where} \ \ 0 < \delta < \frac{2}{p}, \tag{17}$$

then, for $r \geq 2$,

$$E|X|^{p(r-1)} < \infty \tag{18}$$

if and only if

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1 \leq k \leq n} \big| \sum_{i=1}^{k} a_{ni} X_i \big| > \epsilon n^{1/p} \Big) < \infty, \quad \forall \epsilon > 0. \tag{19}$$

For $1 < r < 2$, (18) implies (19). Conversely, if $\lim_{n\to\infty} P(\max_{1\leq i\leq n} |a_{ni}X_i| > \epsilon n^{1/p}) = 0$, then (19) implies (18).

For $p = 2, q = 2$, we have the following theorem.

Theorem 2 *Let $\{X, X_n, n \geq 1\}$ be a ρ^*-mixing sequence of identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of real numbers, and let $r > 1$. If*

$$N(n, m+1) \hat{=} \sharp\{k \geq 1, |a_{nk}| \geq (m+1)^{-1/2}\} \approx m^{r-1}, \quad n, m \geq 1; \tag{20}$$

$$EX = 0, \quad \text{when} \ \ 2(r-1) \geq 1; \tag{21}$$

$$\sum_{k=1}^{n} |a_{nk}|^{2(r-1)} = O(1), \tag{22}$$

then, for $r \geq 2$,

$$E|X|^{2(r-1)} \log(1 + |X|) < \infty \tag{23}$$

if and only if

$$\sum_{n=1}^{\infty} n^{r-2} P\Big(\max_{1 \leq k \leq n} \big| \sum_{i=1}^{k} a_{ni} X_i \big| > \epsilon n^{1/2} \Big) < \infty, \quad \forall \epsilon > 0. \tag{24}$$

For $1 < r < 2$, (23) implies (24). Conversely, if $\lim_{n\to\infty} P(\max_{1\leq i\leq n} |a_{ni}X_i| > \epsilon n^{1/2}) = 0$, then (24) implies (23).

Remark 1 Since independent random variables are a special case of ρ^*-mixing random variables, Theorems 1 and 2 extend the results of Wang et al. [8].

Remark 2 Note that $\sum_{k=1}^{n} |a_{nk}|^{q(r-1)} \ll 1$ as $n \to \infty, 2 \leq q < p$ implies

$$\sharp\{k, |a_{nk}| \geq (m+1)^{-1/p}\} \ll m^{q(r-1)/p} \quad \text{as} \quad n \to \infty.$$

Taking $r = 2$, then conditions (15) and (20) are weaker than conditions (1) and (3) in Li et al. [7]. Therefore, Theorems 1 and 2 not only promote and improve the results of Li et al. [7] from i.i.d. to ρ^*-mixing setting but also obtain their necessities and relax the range of r.

3. Proofs of the main results

Proof of Theorem 1 We firstly prove (18) $\Rightarrow$ (19). Put $b_n = n^{r-2}$, $\alpha = 1/p$ in Lemma 3. For any $q' > q$, we have

$$\begin{aligned}\sum_{i=1}^{n}|a_{ni}|^{q'(r-1)} &= \sum_{m=1}^{\infty}\sum_{(m+1)^{-1}\le|a_{ni}|^p<m^{-1}}|a_{ni}|^{q'(r-1)}\\ &\ll \sum_{m=1}^{\infty}(N(n,m+1)-N(n,m))m^{-q'(r-1)/p}\\ &\ll \sum_{m=1}^{\infty}m^{q(r-1)/p-q'(r-1)/p-1}<\infty. \qquad (25)\end{aligned}$$

Let $Y = X/\epsilon$. By exchanging sum order and (15), we get

$$\begin{aligned}&\sum_{i=1}^{n}P(|a_{ni}X_i|>\epsilon n^{1/p})=\sum_{i=1}^{n}P(|a_{ni}X|>\epsilon n^{1/p})=\sum_{i=1}^{n}P(|a_{ni}Y|>n^{1/p})\\ &=\sum_{j=1}^{\infty}\sum_{(j+1)^{-1}\le|a_{ni}|^p<j^{-1}}P(|a_{ni}Y|>n^{1/p})\approx\sum_{j=1}^{\infty}(N(n,j)-N(n,j-1))P(|Y|>(nj)^{1/p})\\ &=\sum_{j=1}^{\infty}(N(n,j)-N(n,j-1))\sum_{k=nj}^{\infty}P(k<|Y|^p\le k+1)\\ &=\sum_{k=n}^{\infty}P(k<|Y|^p\le k+1)\sum_{j=1}^{[k/n]}(N(n,j)-N(n,j-1))\\ &\approx\sum_{k=n}^{\infty}(k/n)^{q(r-1)/p}P(k<|Y|^p\le k+1). \qquad (26)\end{aligned}$$

Noting that $r-2-q(r-1)/p>-1$, by (26), we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{r-2}\sum_{i=1}^{n}P(|a_{ni}X_i|>\epsilon n^{1/p})\approx\sum_{n=1}^{\infty}n^{r-2}\sum_{k=n}^{\infty}(k/n)^{q(r-1)/p}P(k<|Y|^p\le k+1)\\ &=\sum_{k=1}^{\infty}k^{q(r-1)/p}P(k<|Y|^p\le k+1)\sum_{n=1}^{k}n^{r-2-q(r-1)/p}\\ &\approx\sum_{k=1}^{\infty}k^{r-1}P(k<|Y|^p\le k+1)\approx E|Y|^{p(r-1)}\approx E|X|^{p(r-1)}<\infty. \qquad (27)\end{aligned}$$

Choosing sufficiently large $M > \max\{2, p(r-1)\}$ such that $r-2-M/p<-1, q(r-1)/p-1-M/p<-1$. By exchanging sum order, we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{r-2-M/p}\sum_{i=1}^{n}E|a_{ni}X_i|^M I(|a_{ni}X_i|\le n^{1/p})\\ &\ll\sum_{n=1}^{\infty}n^{r-2-M/p}\sum_{j=1}^{\infty}(N(n,j)-N(n,j-1))j^{-M/p}E|X|^M I(|X|\le(n(j+1))^{1/p})\end{aligned}$$

$$\approx \sum_{n=1}^{\infty} n^{r-2-M/p} \sum_{j=1}^{\infty} j^{q(r-1)/p-1-M/p} E|X|^M I(|X|^p \le 2n-1)+$$
$$\sum_{n=1}^{\infty} n^{r-2-M/p} \sum_{j=1}^{\infty} j^{q(r-1)/p-1-M/p} \sum_{k=2n}^{n(j+1)} E|X|^M I(k-1 < |X|^p \le k)$$
$$=: I_1 + I_2. \tag{28}$$

Noting that $r-2-M/p < -1$, $q(r-1)/p - 1 - M/p < -1$, we have

$$I_1 \le \sum_{n=1}^{\infty} n^{r-2-M/p} E|X|^M I(|X|^p \le 2n-1) \approx E|X|^{p(r-1)} < \infty. \tag{29}$$

By exchanging sum order, we obtain

$$I_2 = \sum_{n=1}^{\infty} n^{r-2-M/p} \sum_{k=2n}^{\infty} E|X|^M I(k-1 < |X|^p \le k) \sum_{j=[k/n]-1}^{\infty} j^{q(r-1)/p-1-M/p}$$
$$\approx \sum_{n=1}^{\infty} n^{r-2-M/p} \sum_{k=2n}^{\infty} (k/n)^{q(r-1)/p-M/p} E|X|^M I(k-1 < |X|^p \le k)$$
$$= \sum_{k=2}^{\infty} k^{q(r-1)/p-M/p} E|X|^M I(k-1 < |X|^p \le k) \sum_{n=1}^{[k/2]} n^{r-2-q(r-1)/p}$$
$$\approx \sum_{k=2}^{\infty} k^{r-1-M/p} E|X|^M I(k-1 < |X|^p \le k) \approx E|X|^{p(r-1)} < \infty. \tag{30}$$

Combining with (28), (29) and (30), we see

$$\sum_{n=1}^{\infty} n^{r-2-M/p} \sum_{i=1}^{n} E|a_{ni}X_i|^M I(|a_{ni}X_i| \le n^{1/p}) < \infty. \tag{31}$$

When $q(r-1) < 2$, take $q < q' < p$ such that $q'(r-1) < 2$. Taking sufficiently large M such that $r-2-Mq'(r-1)/(2p) < -1$, by (25) and $E|X|^{q'(r-1)} < \infty$, we have

$$\sum_{n=1}^{\infty} n^{r-2-M/p} (\log_2 n)^M \big(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le n^{1/p}) \big)^{M/2}$$
$$\le \sum_{n=1}^{\infty} n^{r-2-M/p} n^{M/p-Mq'(r-1)/(2p)} (\log_2 n)^M \big(\sum_{i=1}^{n} E|a_{ni}X_i|^{q'(r-1)} I(|a_{ni}X_i| \le n^{1/p}) \big)^{M/2}$$
$$\ll \sum_{n=1}^{\infty} n^{r-2-Mq'(r-1)/(2p)} (\log_2 n)^M < \infty. \tag{32}$$

For $q(r-1) \ge 2$, since $\delta < 2/p$, we can take sufficiently large M such that $r-2-M/p+M\delta/2 < -1$. Therefore, by (17), we get

$$\sum_{n=1}^{\infty} n^{r-2-M/p} (\log_2 n)^M \big(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le n^{1/p}) \big)^{M/2}$$
$$\ll \sum_{n=1}^{\infty} n^{r-2-M/p} (\log_2 n)^M \big(\sum_{i=1}^{n} |a_{ni}|^2 \big)^{M/2} \ll \sum_{n=1}^{\infty} n^{r-2-M/p+M\delta/2} (\log_2 n)^M < \infty. \tag{33}$$

Thus we have established that all assumptions from Lemma 3 are fulfilled. Therefore, to prove

(19), it suffices to prove that

$$\frac{1}{n^{1/p}}\max_{1\le k\le n}\big|\sum_{i=1}^{k}Ea_{ni}X_iI(|a_{ni}X_i|\le n^{1/p})\big|\to 0 \ \text{ as } \ n\to\infty. \tag{34}$$

For $q(r-1)<1$, taking $q<q'<p$ such that $q'(r-1)<1$, by (25), we get

$$\frac{1}{n^{1/p}}\max_{1\le k\le n}\big|\sum_{i=1}^{k}Ea_{ni}X_iI(|a_{ni}X_i|\le n^{1/p})\big|\le\frac{1}{n^{1/p}}\sum_{i=1}^{n}E|a_{ni}X_i|I(|a_{ni}X_i|\le n^{1/p})$$
$$\le\frac{1}{n^{1/p}}n^{1/p-q'(r-1)/p}\sum_{i=1}^{n}E|a_{ni}X_i|^{q'(r-1)}I(|a_{ni}X_i|\le n^{1/p})\ll n^{-q'(r-1)/p}\to 0 \ \text{ as } \ n\to\infty.$$

For $q(r-1)\ge 1$, noting that $EX=0$, by (25) , we obtain

$$\frac{1}{n^{1/p}}\max_{1\le k\le n}\big|\sum_{i=1}^{k}Ea_{ni}X_iI(|a_{ni}X_i|\le n^{1/p})\big|=\frac{1}{n^{1/p}}\max_{1\le k\le n}\big|\sum_{i=1}^{k}Ea_{ni}X_iI(|a_{ni}X_i|>n^{1/p})\big|$$
$$\le\frac{1}{n^{1/p}}n^{1/p-r+1}\sum_{i=1}^{n}E|a_{ni}X_i|^{p(r-1)}I(|a_{ni}X_i|>n^{1/p})\ll n^{-r+1}\to 0 \ \text{ as } \ n\to\infty.$$

Now we proceed to prove (19) $\Rightarrow$ (18). Since $\max_{1\le k\le n}|a_{nk}X_k|\le 2\max_{1\le k\le n}|\sum_{i=1}^{k}a_{ni}X_i|$, then from (19) we have

$$\sum_{n=1}^{\infty}n^{r-2}P\big(\max_{1\le k\le n}|a_{nk}X_k|>\epsilon n^{1/p}\big)<\infty,\quad\forall\epsilon>0. \tag{35}$$

When $r\ge 2$, it is obvious that $P(\max_{1\le k\le n}|a_{nk}X_k|>\epsilon n^{1/p})\to 0$ as $n\to\infty$. Combining with the hypotheses of Theorem, for $r>1$, we have $P(\max_{1\le k\le n}|a_{nk}X_k|>\epsilon n^{1/p})\to 0$ as $n\to\infty$. Therefore, by Lemma 2, we have

$$\sum_{i=1}^{n}P(|a_{ni}X_i|>\epsilon n^{1/p})\ll P\big(\max_{1\le k\le n}|a_{nk}X_k|>\epsilon n^{1/p}\big). \tag{36}$$

Substituting (36) into (35), we get

$$\sum_{n=1}^{\infty}n^{r-2}\sum_{i=1}^{n}P(|a_{ni}X_i|>\epsilon n^{1/p})<\infty. \tag{37}$$

By (27), we have

$$E|X|^{p(r-1)}\approx\sum_{n=1}^{\infty}n^{r-2}\sum_{i=1}^{n}P(|a_{ni}X_i|>\epsilon n^{1/p}). \tag{38}$$

Therefore (18) holds. □

Proof of Theorem 2 Let $p=2$, $q=2$. Applying the same notations and method as in Theorem 1, we need only to give the different parts. Similarly to the proof of (26) and (27), noting that $E|X|^{2(r-1)}\log(1+|X|)<\infty$, we have

$$\sum_{n=1}^{\infty}n^{r-2}\sum_{i=1}^{n}P(|a_{ni}X_i|>\epsilon n^{1/2})\approx\sum_{n=1}^{\infty}n^{r-2}\sum_{k=n}^{\infty}(k/n)^{r-1}P(k<|Y|^2\le(k+1))$$

$$= \sum_{k=1}^{\infty} k^{r-1} P(k < |Y|^2 \le (k+1)) \sum_{n=1}^{k} n^{-1} \approx \sum_{k=1}^{\infty} k^{r-1} \log(1+k) P(k < |Y|^2 \le (k+1))$$
$$\approx E|Y|^{2(r-1)} \log(1+|Y|) \approx E|X|^{2(r-1)} \log(1+|X|) < \infty. \tag{39}$$

Choose $M > \max\{2, 2(r-1)\}$. Since $E|X|^{2(r-1)}\log(1+|X|) < \infty$ implies $E|X|^{2(r-1)} < \infty$, for $p = 2$, by (28), (29) and (30), we have

$$\sum_{n=1}^{\infty} n^{r-2-M/2} \sum_{i=1}^{n} E|a_{ni}X_i|^M I(|a_{ni}X_i| \le n^{1/2}) < \infty. \tag{40}$$

For $r-1 \le 1$, noting that $r-2-M(r-1)/2 < -1$, by (22) and Markov's inequality, we obtain

$$\sum_{n=1}^{\infty} n^{r-2-M/2} (\log_2 n)^M \big(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le n^{1/2})\big)^{M/2}$$
$$\le \sum_{n=1}^{\infty} n^{r-2-M/2} n^{M/2-M(r-1)/2} (\log_2 n)^M \big(\sum_{i=1}^{n} E|a_{ni}X_i|^{2(r-1)} I(|a_{ni}X_i| \le n^{1/2})\big)^{M/2}$$
$$\ll \sum_{n=1}^{\infty} n^{r-2-M(r-1)/2} (\log_2 n)^M < \infty. \tag{41}$$

For $r-1 > 1$, choosing sufficiently large M such that $r-2-\frac{M}{2(r-1)} < -1$, by Hölder's inequality and (22), we have

$$\sum_{n=1}^{\infty} n^{r-2-M/2} (\log_2 n)^M \big(\sum_{i=1}^{n} E|a_{ni}X_i|^2 I(|a_{ni}X_i| \le n^{1/2})\big)^{M/2}$$
$$\le \sum_{n=1}^{\infty} n^{r-2-M/2} (\log_2 n)^M \big(\sum_{i=1}^{n} |a_{ni}|^2\big)^{M/2}$$
$$\le \sum_{n=1}^{\infty} n^{r-2-M/2} (\log_2 n)^M \Big(\big(\sum_{i=1}^{n} a_{ni}^{2(r-1)}\big)^{\frac{1}{r-1}} \big(\sum_{i=1}^{n} 1\big)^{\frac{r-2}{r-1}}\Big)^{M/2}$$
$$\ll \sum_{n=1}^{\infty} n^{r-2-\frac{M}{2(r-1)}} (\log_2 n)^M < \infty. \tag{42}$$

Let (22) take the place of (25). Similarly to the proof of (34), we have

$$\frac{1}{n^{1/2}} \max_{1\le k\le n} \big|\sum_{i=1}^{k} Ea_{ni}X_i I(|a_{ni}X_i| \le n^{1/2})\big| \to 0 \quad \text{as} \quad n \to \infty. \tag{43}$$

Thus, we have proved (23) $\Rightarrow$ (24). Now we proceed to prove (24) $\Rightarrow$ (23). Using the same arguments as those in the necessary part of Theorem 1, by (39), we can easily prove

$$E|X|^{2(r-1)} \log(1+|X|) \approx \sum_{n=1}^{\infty} n^{r-2} \sum_{i=1}^{n} P(|a_{ni}X_i| > \epsilon n^{1/2}). \tag{44}$$

Therefore (23) holds. □

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