Journal of Mathematical Research with Applications
Sept., 2013, Vol. 33, No. 5, pp. 528–542
DOI:10.3770/j.issn:2095-2651.2013.05.002
Http://jmre.dlut.edu.cn

Product Zero Derivations on Strictly Upper Triangular Matrix Lie Algebras

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Abstract Let $\mathbb{F}$ be a field, $n \geq 3$, $\mathbf{N}(n,\mathbb{F})$ the strictly upper triangular matrix Lie algebra consisting of the $n \times n$ strictly upper triangular matrices and with the bracket operation $[x,y] = xy - yx$. A linear map φ on $\mathbf{N}(n,\mathbb{F})$ is said to be a product zero derivation if $[\varphi(x),y] + [x,\varphi(y)] = 0$ whenever $[x,y] = 0, x,y \in \mathbf{N}(n,\mathbb{F})$. In this paper, we prove that a linear map on $\mathbf{N}(n,\mathbb{F})$ is a product zero derivation if and only if φ is a sum of an inner derivation, a diagonal derivation, an extremal product zero derivation, a central product zero derivation and a scalar multiplication map on $\mathbf{N}(n,\mathbb{F})$.

Keywords product zero derivations; strictly upper triangular matrix Lie algebras; derivations of Lie algebras.

MR(2010) Subject Classification 15A04; 17A36

1. Introduction

A lot of attention has been paid to a linear preserver problem, which concerns the characterization of linear maps on matrix spaces or algebras that leave certain functions, subsets, relations, etc., invariant. Some of such linear maps generalize the usual automorphisms or derivations. The earliest paper on linear preserver problem dates back to 1897 (see [2]), and a great deal of effort has been devoted to the study of this type of question since then. One may consult the survey papers [3–6] for details. Let us mention particularly one type of classical example: linear maps preserving commutativity. The importance of this type example lies in the fact that the assumption of preserving commutativity of matrices can be considered as the assumption of preserving Lie products at the commuting elements on the related linear Lie algebra. Such type of linear preserver problem has been extensively studied on matrix algebras as well as on more general rings and operator algebras. For example, the commutativity preserving linear maps on triangular matrices were done in [7]; the commutativity preserving linear maps on strictly triangular matrices were studied in [8]; the nonlinear commutativity preserving maps on the algebra of full matrices were studied in [9] and [10]. For more references about commutativity preserving maps on associated algebras one may consult the survey paper [11].

Received May 30, 2012; Accepted August 15, 2012
Supported by the National Natural Science Foundation of China (Grant No. 11101084) and the Natural Science Foundation of Fujian Province (Grant No. 2013J01005).
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For linear preserver problems concerning Lie algebras we just find several papers [12–14]. The author in [12] characterized the invertible linear maps on simple Lie algebras of linear types preserving Lie products at the commuting elements. Recently, a few papers study nonlinear maps on Lie algebras [15–19]. Radjavi and Šemrl in [15] characterized the nonlinear maps on the general linear Lie algebra and the special linear Lie algebra which preserve solvability in both directions. The nonlinear bijective maps on triangular matrix Lie algebras (resp., simple Lie algebras) preserving Lie products were determined in [16] (resp., [18]). The nonlinear Lie derivations on upper triangular matrix Lie algebras (resp., parabolic subalgebras of simple Lie algebras) were determined in [17] (resp., [19]).

In [13], the authors introduced a new concept: product zero derivations of Lie algebras. Such maps behave Lie derivations only on pairs of commuting elements. In [13], it is shown that any product zero derivation on parabolic subalgebras of simple Lie algebras is a sum of an inner derivation and a scalar multiplication map. In this paper, we determine the product zero derivations on strictly upper triangular matrix Lie algebras $\mathbf{N}(n,\mathbb{F})$, where $\mathbb{F}$ is a field and $n \geq 3$. A linear map $\varphi : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F})$ is a product zero derivation if $[\varphi(A), B] + [A, \varphi(B)] = 0$ whenever $[A, B] = 0$, $A, B \in \mathbf{N}(n,\mathbb{F})$. Obviously, $\varphi(0) = 0$. Since a Lie derivation δ on $\mathbf{N}(n,\mathbb{F})$ is a linear map satisfying $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ whether $[A, B]$ is equal to zero or not (in other words, δ is a linear map derivable at any pair (A, B)), so any Lie derivation on $\mathbf{N}(n,\mathbb{F})$ is a product zero derivation. From [20, Theorem 3.2], any derivation on $\mathbf{N}(n,\mathbb{F})$ is a sum of an inner derivation, a diagonal derivation, an extremal derivation and a central derivation. In this paper, we will prove that any product zero derivation on $\mathbf{N}(n,\mathbb{F})$ is a sum of an inner derivation, a diagonal derivation, an extremal product zero derivation, a central product zero derivation and a scalar multiplication map (see Theorem 3.7), which generalizes the results about derivation [20, Theorem 3.2]. From our results we will see some difference between a product zero derivation and a derivation on $\mathbf{N}(n,\mathbb{F})$.

Here we specify some notations for later use in this paper. Let $\mathbb{F}$ be a field, $gl(n,\mathbb{F})$ the general linear Lie algebra consisting of all $n \times n$ matrices over $\mathbb{F}$ and with the bracket operation $[x, y] = xy - yx$. We denote by E_{ij} the matrix in $\mathbf{N}(n,\mathbb{F})$ whose sole nonzero entry 1 is in the (i, j)-position, where $1 \leq i < j \leq n$. Let $\mathcal{D}$ be the set of all diagonal matrices in $gl(n,\mathbb{F})$.

Clearly, the following sets $\mathbf{N}_k = \{X \in \mathbf{N}(n,\mathbb{F}) | X = \sum_{j-i\geq k} x_{ij}E_{ij}\}$, $k = 1, 2, \ldots, n-1$, are the ideals of the $\mathbb{F}$-algebra $\mathbf{N}(n,\mathbb{F})$, $\mathbf{N}_1 = \mathbf{N}(n,\mathbb{F})$, $\mathbf{N}_{n-1} = \mathbb{F}E_{1n}$, $\mathbf{N}_1 \supseteq \mathbf{N}_2 \supseteq \mathbf{N}_3 \supseteq \cdots \supseteq \mathbf{N}_{n-1}$, and $\mathbf{N}_{n-1}$ is the center of $\mathbf{N}(n,\mathbb{F})$. We assume that $\mathbf{N}_k = \{0\}$ for $k > n$. It is easy to check that $[\mathbf{N}_k, \mathbf{N}_l] \subseteq \mathbf{N}_{k+l}$.

Lemma 1.1 *If δ is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$, then $\delta(E_{1n}) = cE_{1n}$ for some $c \in \mathbb{F}$.*

Proof For any $X \in \mathbf{N}(n,\mathbb{F})$, we have $[X, E_{1n}] = 0$. By definition, $[\delta(X), E_{1n}] + [X, \delta(E_{1n})] = 0$, i.e.,

$$[X, \delta(E_{1n})] = 0.$$

Thus $\delta(E_{1n})$ is in the center of $\mathbf{N}(n,\mathbb{F})$, and so the lemma holds.

2. Certain product zero derivations on $\mathbf{N}(n,\mathbb{F})$

In this section, we construct certain standard product zero derivations on $\mathbf{N}(n,\mathbb{F})$, $n \geq 3$. Such maps will be used to describe arbitrary product zero derivations on $\mathbf{N}(n,\mathbb{F})$.

(A) Inner derivations:

For any $A=(a_{ij})_{n\times n} \in \mathbf{N}(n,\mathbb{F})$, the map

$$\text{ad } A : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F}), X \mapsto [A,X]$$

is a derivation of $\mathbf{N}(n,\mathbb{F})$, called an inner derivation on $\mathbf{N}(n,\mathbb{F})$. By [1, Section 1.3], any inner derivation is a (usual) Lie derivation, and so a product zero derivation.

(B) Diagonal derivations:

Let $d \in \mathcal{D}$. Then the map

$$\eta_d : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F}), X \mapsto [d,X]$$

is a Lie derivation of $\mathbf{N}(n,\mathbb{F})$, called a diagonal derivation [20, Section 2(B)].

(C) Extremal product zero derivations:

For any $a \in \mathbb{F}$, $i=1,n-1$ and $j=1,2$, we define four linear maps $e_a^{(ij)}$:

(C-1) $e_a^{(11)} : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F})$ determined by

$$X=\sum_{1\leq i<j\leq n} x_{ij}E_{ij} \mapsto a x_{12}E_{2n};$$

(C-2) $e_a^{(n-1,1)} : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F})$ determined by

$$X=\sum_{1\leq i<j\leq n} x_{ij}E_{ij} \mapsto a x_{n-1,n}E_{1,n-1};$$

(C-3) $e_a^{(12)} : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F})$ determined by

$$X=\sum_{1\leq i<j\leq n} x_{ij}E_{ij} \mapsto a x_{12}E_{3n} + a x_{13}E_{2n};$$

(C-4) $e_a^{(n-1,2)} : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F})$ determined by

$$X=\sum_{1\leq i<j\leq n} x_{ij}E_{ij} \mapsto a x_{n-1,n}E_{1,n-2} + a x_{n-2,n}E_{1,n-1}.$$

It is easy to check that they are product zero derivations of $\mathbf{N}(n,\mathbb{F})$. In particular, for any $a\in\mathbb{F}$, $e_a^{(11)}$ and $e_a^{(n-1,1)}$ are Lie derivations of $\mathbf{N}(n,\mathbb{F})$, called extremal derivations [20, Section 2(D)]. We call the above maps $e_a^{(ij)}$ and their sums extremal product zero derivations.

(D) Central product zero derivations:

Let $f : \mathbf{N}(n,\mathbb{F}) \to \mathbb{F}$ be a linear function. Then there is a linear map $\varphi_f : \mathbf{N}(n,\mathbb{F}) \to \mathbf{N}(n,\mathbb{F})$ given by

$$\varphi_f(X) = f(X)E_{1n}$$

for any $X \in \mathbf{N}(n,\mathbb{F})$. It is easily verified that φ_f is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$, called a central product zero derivation.

Moreover, if $f(\mathbf{N}_2) = 0$, then φ_f is a derivation, which is called a central derivation [20, Section 2(C)].

(E) Scalar multiplication maps:

For $c \in \mathbb{F}$, define

$$\varphi_c : \mathbf{N}(n, \mathbb{F}) \to \mathbf{N}(n, \mathbb{F}), \quad X \mapsto cX,$$

for all $X \in \mathbf{N}(n, \mathbb{F})$. We call φ_c a scalar multiplication map on $\mathbf{N}(n, \mathbb{F})$.

It is easily verified that any scalar multiplication map is a product zero derivation on $\mathbf{N}(n, \mathbb{F})$. Note that if the characteristic of $\mathbb{F}$ is not equal to 2, then φ_c is not a Lie derivation unless c is equal to 0.

3. Decomposition of product zero derivations

For any product zero derivation δ on $\mathbf{N}(n, \mathbb{F})$, assume that

$$\delta(E_{i,i+1}) \equiv \sum_{j=1}^{n-1} a_{ji} E_{j,j+1} \text{ mod } \mathbf{N}_2 \text{ for } i = 1, 2, \ldots, n-1,$$

i.e., $\delta(E_{i,i+1}) - \sum_{j=1}^{n-1} a_{ji} E_{j,j+1} \in \mathbf{N}_2$ for $i = 1, 2, \ldots, n-1$, where $a_{ij} \in \mathbb{F}, i, j \in \{1, 2, \ldots, n-1\}$. Then δ determines a $(n-1) \times (n-1)$ matrix

$$A(\delta) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix},$$

where the entries a_{ij} are dependent on δ. We call $A(\delta)$ the matrix attached to the product zero derivation δ.

Lemma 3.1 *If δ is a product zero derivation on $\mathbf{N}(n, \mathbb{F})$, $n \geq 5$, then $A(\delta)$ is a diagonal matrix.*

Proof We prove that $a_{ij} = 0$ for any $i \neq j$, $i, j \in \{1, 2, \ldots, n-1\}$. We prove it in the following cases.

Case 1 $i \neq 1$ or $n-1$.

In this cases, $i-1, i+1 \in \{1, 2, \ldots, n-1\}$.

Case 1.1 $i < j$.

Since $[E_{i-1,i}, E_{j,j+1}] = 0$, $[\delta(E_{i-1,i}), E_{j,j+1}] + [E_{i-1,i}, \delta(E_{j,j+1})] = 0$. Comparing the coefficients of $E_{i-1,i+1}$ on the both sides of the above equality, we have $a_{ij} = 0$.

Case 1.2 $i > j$.

Since $[E_{j,j+1}, E_{i+1,i+2}] = 0$, $[\delta(E_{j,j+1}), E_{i+1,i+2}] + [E_{j,j+1}, \delta(E_{i+1,i+2})] = 0$. Comparing the coefficients of $E_{i,i+2}$ on the both sides of the above equality, we have $a_{ij} = 0$.

Case 2 $i = 1$.

Since $i \neq j$, $j \neq 1$, i.e., $j \geq 2$.

Case 2.1 $2 \leq j < n-1$.

Since $[E_{j,j+1}, E_{2n}] = 0$, $[\delta(E_{j,j+1}), E_{2n}] + [E_{j,j+1}, \delta(E_{2n})] = 0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $a_{1j} = 0$, i.e., $a_{ij} = 0$.

Case 2.2 $j = n-1$.

Since $[E_{23}, E_{n-1,n}] = 0$, $[\delta(E_{23}), E_{n-1,n}] + [E_{23}, \delta(E_{n-1,n})] = 0$. Comparing the coefficients of E_{13} on the both sides of the above equality, we have $a_{1,n-1} = 0$, i.e., $a_{ij} = 0$.

Case 3 $i = n-1$.

Since $i \neq j$, $j \neq n-1$, i.e., $j \leq n-2$.

Case 3.1 $1 < j \leq n-2$.

Since $[E_{1,n-1}, E_{j,j+1}] = 0$, $[\delta(E_{1,n-1}), E_{j,j+1}] + [E_{1,n-1}, \delta(E_{j,j+1})] = 0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $a_{n-1,j} = 0$, i.e., $a_{ij} = 0$.

Case 3.2 $j = 1$.

Since $[E_{12}, E_{n-2,n-1}] = 0$, $[\delta(E_{12}), E_{n-2,n-1}] + [E_{12}, \delta(E_{n-2,n-1})] = 0$. Comparing the coefficients of $E_{n-2,n}$ on the both sides of the above equality, we have $a_{n-1,1} = 0$, i.e., $a_{ij} = 0$. $\square$

Lemma 3.2 *If δ is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$, $n \geq 5$, $A(\delta) = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_{n-1}\}$ is the unique diagonal matrix attached to δ, then we can construct a diagonal matrix $D = \operatorname{diag}\{0, \lambda_1, \lambda_1+\lambda_2, \dots, \lambda_1+\lambda_2+\dots+\lambda_{n-1}\}$ such that $A(\delta+\eta_D) = 0$, i.e.,*

$$(\delta+\eta_D)(E_{i,i+1}) \in \mathbf{N}_2, \quad \forall 1 \leq i \leq n-1.$$

Proof By easy computations. We omit the process of computations. $\square$

Lemma 3.3 *If δ is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$ and $A(\delta) = 0$, $n \geq 5$, then there exists a matrix $M'' \in \mathbf{N}(n,\mathbb{F})$ and elements $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that*

$$(e_{a_1}^{(11)} + e_{a_2}^{(n-1,1)} + e_{a_3}^{(12)} + e_{a_4}^{(n-1,2)} + \operatorname{ad} M'' + \delta)(E_{i,i+1}) \in \mathbf{N}_{n-1}, \quad \forall 1 \leq i \leq n-1.$$

Proof We prove this lemma by the following three steps.

Step 1. There exists a matrix $M \in \mathbf{N}(n,\mathbb{F})$ satisfying

$$(\operatorname{ad} M + \delta)(E_{i,i+1}) \in \mathbf{N}_{n-3}, \quad \forall 1 \leq i \leq n-1.$$

We use induction on k to prove that there exists $M_k \in \mathbf{N}(n,\mathbb{F})$, $k = 1, 2, 3, \dots, n-4$, satisfying that

$$(\operatorname{ad} M_k + \delta)(E_{i,i+1}) \in \mathbf{N}_{k+1}, \quad \forall 1 \leq i \leq n-1. \tag{3.1}$$

Let $M_1 = 0$. Then the equality (3.1) holds for $k = 1$ by conditions.

Assume that there exists a matrix $M_{k-1} \in \mathbf{N}(n,\mathbb{F})$ with $2 \leq k \leq n-4$ such that (3.1) is true for $k-1$, i.e., $(\operatorname{ad} M_{k-1}+\delta)(E_{i,i+1}) \in \mathbf{N}_k$, $\forall 1 \le i \le n-1$. Set

$$\delta' = \operatorname{ad} M_{k-1} + \delta.$$

It is clear that $A(\delta')$ is still a zero matrix. Assume that

$$\delta'(E_{i,i+1}) \equiv \sum_{j=1}^{n-k} a_{ji}^{(k)} E_{j,j+k} \bmod \mathbf{N}_{k+1}, \quad 1 \le i \le n-1. \tag{3.2}$$

For (3.2), we first prove the following claim:

Claim $a_{si}^{(k)} = 0$ for $s \ne i, i+1-k$.

(A-1) $s \le n-k-1$ and $s \ne i-k, i-k-1$. Applying δ' to $[E_{i,i+1}, E_{s+k,s+k+1}] = 0$, we have $[\delta'(E_{i,i+1}), E_{s+k,s+k+1}] + [E_{i,i+1}, \delta'(E_{s+k,s+k+1})] = 0$. Comparing the coefficients of $E_{s,s+k+1}$ on the both sides of the above equality, we have $a_{si}^{(k)} = 0$.

(A-2) $s \ge 2$ and $s \ne i+1, i+2$. Applying δ' to $[E_{s-1,s}, E_{i,i+1}] = 0$, we have $[\delta'(E_{s-1,s}), E_{i,i+1}] + [E_{s-1,s}, \delta'(E_{i,i+1})] = 0$. Comparing the coefficients of $E_{s-1,s+k}$ on the both sides of the above equality, we have $a_{si}^{(k)} = 0$.

Since $s = i-k$ or $i-k-1$ implies that $s \ne i+1, i+2$, it follows from (A-1) and (A-2) that for $2 \le s \le n-k-1$ the claim is true and it remains to consider the following cases (B-1), (B-2), (C-1) and (C-2).

(B-1) $s = 1$ and $s = i-k$.

In this case, $i = k+1 \le n-3$. Applying δ' to $[E_{i,i+1}, E_{i,i+2}] = 0$, we have $[\delta'(E_{i,i+1}), E_{i,i+2}] + [E_{i,i+1}, \delta'(E_{i,i+2})] = 0$. Comparing the coefficients of $E_{1,i+2}$ on the both sides of the above equality, we have $a_{1i}^{(i-1)} = 0$, i.e., $a_{si}^{(k)} = 0$.

(B-2) $s = 1$ and $s = i-k-1$.

In this case, $3 \le i = k+2 \le n-2$. Applying δ' to $[E_{i,i+1}, E_{i-1,i+2}] = 0$, we have $[\delta'(E_{i,i+1}), E_{i-1,i+2}] + [E_{i,i+1},\ \delta'(E_{i-1,i+2})] = 0$. Comparing the coefficients of $E_{1,i+2}$ on the both sides of the above equality, we have $a_{1i}^{(i-2)} = 0$, i.e., $a_{si}^{(k)} = 0$.

(C-1) $s = n-k$ and $s = i+1$.

In this case, since $k \le n-4$, we have $s \ge 4$. Applying δ' to $[E_{s-2,s}, E_{s-1,s}] = 0$ gives $[\delta'(E_{s-2,s}), E_{s-1,s}] + [E_{s-2,s}, \delta'(E_{s-1,s})] = 0$. Comparing the coefficients of $E_{s-2,s+k}$ on the both sides of the above equality, we have $a_{s,s-1}^{(k)} = 0$, i.e., $a_{si}^{(k)} = 0$.

(C-2) $s = n-k$ and $s = i+2$.

As in (C-1), $s \ge 4$. Applying δ' to $[E_{s-3,s}, E_{s-2,s-1}] = 0$, we have $[\delta'(E_{s-3,s}), E_{s-2,s-1}] + [E_{s-3,s}, \delta'(E_{s-2,s-1})] = 0$. Comparing the coefficients of $E_{s-3,s+k}$ on the both sides of the above equality, we have $a_{s,s-2}^{(k)} = 0$, i.e., $a_{si}^{(k)} = 0$.

Therefore the claim holds and the equality (3.2) may be written as

$$\delta'(E_{i,i+1}) \equiv a_{i+1-k,i}^{(k)} E_{i+1-k,i+1} + a_{ii}^{(k)} E_{i,i+k} \bmod \mathbf{N}_{k+1},\ 1 \le i \le n-1. \tag{3.3}$$

To complete the induction on k, we need again induction on l to prove that there exist $S_l \in \mathbf{N}(n, \mathbb{F})$, $l = 0, 1, \ldots, n-1$, such that for $1 \le i \le l$,

$$(\operatorname{ad} S_l + \delta')(E_{i,i+1}) \equiv 0 \bmod \mathbf{N}_{k+1} \tag{3.4}$$

and for $l+1 \le i \le n-1$,

$$(\operatorname{ad} S_l + \delta')(E_{i,i+1}) \equiv c_{i+1-k,i}^{(l)} E_{i+1-k,i+1} + c_{ii}^{(l)} E_{i,i+k} \bmod \mathbf{N}_{k+1}. \tag{3.5}$$

Let $S_0 = 0$. Then it follows from (3.3) that (3.5) with $c_{i+1-k,i}^{(0)} = a_{i+1-k,i}^{(k)}$ and $c_{ii}^{(0)} = a_{ii}^{(k)}$ is trivially true, and (3.4) does not occur. Assume that (3.4) and (3.5) hold for some $S_{l-1} \in \mathbf{N}(n,\mathbb{F})$ with $0 \le l-1 \le n-2$. In particular,

$$(\operatorname{ad} S_{l-1} + \delta')(E_{l,l+1}) \equiv c_{l+1-k,l}^{(l-1)} E_{l+1-k,l+1} + c_{ll}^{(l-1)} E_{l,l+k} \bmod \mathbf{N}_{k+1}.$$

Set $S_l = S_{l-1} - c_{l+1-k,l}^{(l-1)} E_{l+1-k,l} + c_{ll}^{(l-1)} E_{l+1,l+k}$. In fact, $c_{l+1-k,l}^{(l-1)} = 0$ if $l \ne k$. This is clear for $l < k$. And for $l > k$ applying $\operatorname{ad} S_{l-1} + \delta'$ to $[E_{l-k,l-k+1}, E_{l,l+1}] = 0$, we have $c_{l+1-k,l}^{(l-1)} E_{l-k,l+1} \equiv 0 \bmod \mathbf{N}_{k+2}$. Hence, $c_{l+1-k,l}^{(l-1)} = 0$. It is easy to check that (3.4) and (3.5) hold with

$$c_{i+1-k,i}^{(l)} = c_{i+1-k,i}^{(l-1)} + \delta_{l+k,i} c_{ll}^{(l-1)},$$

where δ_{ji} denotes the Kronecker delta, and $c_{ii}^{(l)} = c_{ii}^{(l-1)}$ for S_l hold. Thus the induction on l is completed. Set

$$M_k = S_{n-1} + M_{k-1}.$$

Then (3.1) is true for k, and the induction on k is completed. Thus Step 1 holds.

Step 2. There exist a matrix $M' \in \mathbf{N}(n,\mathbb{F})$ and two elements $a_3, a_4 \in \mathbb{F}$ satisfying

$$(e_{a_3}^{(12)} + e_{a_4}^{(n-1,2)} + \operatorname{ad} M' + \delta)(E_{i,i+1}) \in \mathbf{N}_{n-2}, \quad \forall 1 \le i \le n-1. \tag{3.6}$$

By Step 1, assume that

$$(\delta + \operatorname{ad} M)(E_{i,i+1}) \equiv \sum_{j=1}^{3} a_{ji}^{(n-3)} E_{j,j+n-3} \bmod \mathbf{N}_{n-2} \text{ for } 1 \le i \le n-1. \tag{3.7}$$

Repeating the arguments in (A-1), (A-2), (B-1) and (C-1) above with $k = n-3$, we obtain that in (3.7) $a_{si}^{(n-3)} = 0$ for $s \ne i, i+4-n$ except $a_{31}^{(n-3)}$ and $a_{1,n-1}^{(n-3)}$. Thus (3.7) may be rewritten as

$$(\operatorname{ad} M + \delta)(E_{12}) \equiv a_{11}^{(n-3)} E_{1,n-2} + a_{31}^{(n-3)} E_{3n} \bmod \mathbf{N}_{n-2},$$

$$(\operatorname{ad} M + \delta)(E_{i,i+1}) \equiv a_{i+4-n,i}^{(n-3)} E_{i+4-n,i+1} + a_{ii}^{(n-3)} E_{i,i+n-3} \bmod \mathbf{N}_{n-2} \text{ for } i = 2,3,\ldots,n-2,$$

$$(\operatorname{ad} M + \delta)(E_{n-1,n}) \equiv a_{1,n-1}^{(n-3)} E_{1,n-2} + a_{3,n-1}^{(n-3)} E_{3n} \bmod \mathbf{N}_{n-2}.$$

In the same argument as in Step 1, we can use induction to prove that there exists an $S \in \mathbf{N}(n,\mathbb{F})$ such that

$$(\operatorname{ad} S + \operatorname{ad} M + \delta)(E_{12}) \equiv a_{31}^{(n-3)} E_{3n} \bmod \mathbf{N}_{n-2},$$

$$(\operatorname{ad} S + \operatorname{ad} M + \delta)(E_{i,i+1}) \equiv 0 \bmod \mathbf{N}_{n-2} \text{ for } i = 2,3,\ldots,n-2,$$

$$(\operatorname{ad} S + \operatorname{ad} M + \delta)(E_{n-1,n}) \equiv a_{1,n-1}^{(n-3)} E_{1,n-2} \bmod \mathbf{N}_{n-2}.$$

Set

$$M' = S + M, a_3 = -a_{31}^{(n-3)}, a_4 = -a_{1,n-1}^{(n-3)}.$$

It is easy to check that $\operatorname{ad} M'$, $e_{a_3}^{(12)}$ and $e_{a_4}^{(n-1,2)}$ satisfy (3.6).

Step 3. There exist a matrix $M'' \in \mathbf{N}(n,\mathbb{F})$ and elements $a_1, a_2 \in \mathbb{F}$ such that

$$(e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M'' + \delta)(E_{i,i+1}) \in \mathbf{N}_{n-1}, \quad \forall 1 \le i \le n-1. \tag{3.8}$$

By (3.6), for any $1 \le i \le n-1$, we may assume that

$$(e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M' + \delta)(E_{i,i+1}) \equiv a^{(n-2)}_{1i} E_{1,n-1} + a^{(n-2)}_{2i} E_{2n} \bmod \mathbf{N}_{n-1}. \tag{3.9}$$

For convenience, we set

$$\delta'' = e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M' + \delta.$$

For (3.9), we first give the following observations (1)–(3).

(1) For any $2 \le i \le n-3$, $[E_{i,i+1}, E_{n-1,n}] = 0$, then $[\delta''(E_{i,i+1}), E_{n-1,n}] + [E_{i,i+1}, \delta''(E_{n-1,n})] = 0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $a^{(n-2)}_{1i} = 0$.

(2) For any $3 \le i \le n-2$, $[E_{12}, E_{i,i+1}] = 0$, then $[\delta''(E_{12}), E_{i,i+1}] + [E_{12}, \delta''(E_{i,i+1})] = 0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $a^{(n-2)}_{2i} = 0$.

(3) Since $[E_{12}, E_{n-1,n}] = 0$, we have $[\delta''(E_{12}), E_{n-1,n}] + [E_{12}, \delta''(E_{n-1,n})] = 0$. Comparing the coefficients of E_{1n} on the both sides of the above equality gives $a^{(n-2)}_{11} = -a^{(n-2)}_{2,n-1}$.

We construct an inner derivation

$$\sigma = \operatorname{ad}(a^{(n-2)}_{11} E_{2,n-1}) + \operatorname{ad}(a^{(n-2)}_{22} E_{3n}) + \operatorname{ad}(-a^{(n-2)}_{1,n-2} E_{1,n-2}).$$

Then

$$(\sigma + \delta'')(E_{12}) \equiv a^{(n-2)}_{21} E_{2n} \bmod \mathbf{N}_{n-1};$$

$$(\sigma + \delta'')(E_{i,i+1}) \equiv 0 \bmod \mathbf{N}_{n-1}, \quad 2 \le i \le n-2;$$

$$(\sigma + \delta'')(E_{n-1,n}) \equiv a^{(n-2)}_{1,n-1} E_{1,n-1} \bmod \mathbf{N}_{n-1}.$$

Let $a_1 = -a^{(n-2)}_{21}$, $a_2 = -a^{(n-2)}_{1,n-1}$. Then $(e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + \sigma + \delta'')(E_{i,i+1}) \equiv 0 \bmod \mathbf{N}_{n-1}$, $1 \le i \le n-1$, i.e.,

$$\begin{aligned}(e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M' + \operatorname{ad}(a^{(n-2)}_{11} E_{2,n-1}) + \\ \operatorname{ad}(a^{(n-2)}_{22} E_{3n}) + \operatorname{ad}(-a^{(n-2)}_{1,n-2} E_{1,n-2}) + \delta)(E_{i,i+1}) \in \mathbf{N}_{n-1}.\end{aligned}$$

Set

$$M'' = M' + a^{(n-2)}_{11} E_{2,n-1} + a^{(n-2)}_{22} E_{3n} - a^{(n-2)}_{1,n-2} E_{1,n-2}.$$

Thus the lemma holds. □

Lemma 3.4 *If δ is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$, $n \ge 4$, and $\delta(E_{i,i+1}) \in \mathbf{N}_{n-1}, 1 \le i \le n-1$, then there exist elements $b_s, b'_t \in \mathbb{F}$, $2 \le s \le n$, $1 \le t \le n-1$, satisfying the following conditions.*

(1) $\delta(E_{1s}) \equiv b_s E_{1s} \bmod \mathbf{N}_{n-1}$, $2 \le s \le n$;

(2) $\delta(E_{tn}) \equiv b'_t E_{tn} \bmod \mathbf{N}_{n-1}$, $1 \le t \le n-1$.

Proof (1) Set $b_2 = 0$ and $b_n = 0$. Then (1) holds for $s = 2$ by the condition, and for $s = n$ by Lemma 1.1. For $3 \le s \le n-1$, we set

$$\delta(E_{1s}) \equiv \sum_{1 \le l-k \le n-2} b^{(1s)}_{kl} E_{kl} \bmod \mathbf{N}_{n-1}, \quad 3 \le s \le n-1. \tag{3.10}$$

For (3.10), we first prove that $b_{kl}^{(1s)}=0$ for any $(k,l)\neq(1,s)$ or $(s+1,n)$, where $1\leq l-k\leq n-2$. We prove it in the following three cases.

Case 1 $l\neq s$ or n.

Since $[E_{l,l+1},E_{1s}]=0$, by conditions, $[E_{l,l+1},\delta(E_{1s})]=0$. Comparing the coefficients of $E_{k,l+1}$ on the both sides of the above equality, we have $b_{kl}^{(1s)}=0$.

Case 2 $l=s,k\neq 1$.

Since $k-1<l-1<l=s$, we have $[E_{k-1,k},E_{1s}]=0$, then by conditions, $[E_{k-1,k},\delta(E_{1s})]=0$. Comparing the coefficients of $E_{k-1,l}$ on the both sides of the above equality, we have $b_{kl}^{(1s)}=0$.

Case 3 $l=n,k\neq s+1$.

Since $l-k\leq n-2$ and $l=n$, we have $k\neq 1$. Applying δ to the equality $[E_{k-1,k},E_{1s}]=0$ yields $[E_{k-1,k},\delta(E_{1s})]=0$ by conditions. Comparing the coefficients of $E_{k-1,l}$ on the both sides of the above equality, we have $b_{kl}^{(1s)}=0$.

Thus

$$\delta(E_{1s})\equiv b_{1s}^{(1s)}E_{1s}+b_{s+1,n}^{(1s)}E_{s+1,n} \bmod \mathbf{N}_{n-1},\quad 3\leq s\leq n-2;$$

$$\delta(E_{1,n-1})\equiv b_{1,n-1}^{(1,n-1)}E_{1,n-1} \bmod \mathbf{N}_{n-1}.$$

For $3\leq s\leq n-2$, since $[E_{1s},E_{1,s+1}]=0$, we have

$$[\delta(E_{1s}),E_{1,s+1}]+[E_{1s},\delta(E_{1,s+1})]=0. \tag{3.11}$$

Comparing the coefficients of E_{1n} on the both sides of the above equality (3.11), we have $b_{s+1,n}^{(1s)}=0$ for any $3\leq s\leq n-2$. Set $b_s=b_{1s}^{(1s)}$ for $1\leq s\leq n-1$, and then the condition (1) holds.

The proof of (2) is dual to that of (1). $\square$

Lemma 3.5 *If δ is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$, $n\geq 4$, and $\delta(E_{i,i+1})\in\mathbf{N}_{n-1}, 1\leq i\leq n-1$, then for any $1\leq u\leq n-2$, $1\leq i\leq n-u$, we have $\delta(E_{i,i+u})\equiv\beta_u E_{i,i+u} \bmod \mathbf{N}_{n-1}$, where $\beta_1=0,\beta_u\in\mathbb{F}$, $u=2,3,\ldots,n-2$.*

Proof For $u=1$, the lemma holds by conditions with $\beta_1=0$. Let $2\leq u\leq n-2$. Set

$$\delta(E_{i,i+u})\equiv\sum_{1\leq l-k\leq n-2} b_{kl}^{(i,i+u)}E_{kl} \bmod \mathbf{N}_{n-1}.$$

First we prove that $b_{kl}^{(i,i+u)}=0$ for any $(k,l)\neq(i,i+u)$ and $1\leq l-k\leq n-2$. We prove it in the following cases.

Case 1 $k\neq 1$ or i.

Since $[E_{i,i+u},E_{1k}]=0$, we have $[\delta(E_{i,i+u}),E_{1k}]+[E_{i,i+u},\delta(E_{1k})]=0$. By Lemma 3.4, assume that $\delta(E_{1k})=\lambda E_{1k}+\mu E_{1n}$, where $\lambda,\mu\in\mathbb{F}$. Then $[E_{i,i+u},\delta(E_{1k})]=[E_{i,i+u},\lambda E_{1k}+\mu E_{1n}]=0$, and so $[\delta(E_{i,i+u}),E_{1k}]=0$. Comparing the coefficients of E_{1l} on the both sides of the above equality, we have $b_{kl}^{(i,i+u)}=0$.

Case 2 $k=1$.

In this case, since $l-k\le n-2$, we have $l\ne n$.

Case 2.1 $l\ne i+u$.

Since $[E_{i,i+u},E_{ln}]=0$, by Lemma 3.4, $[\delta(E_{i,i+u}),E_{ln}]=0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $b_{1l}^{(i,i+u)}=0$, i.e., $b_{kl}^{(i,i+u)}=0$.

Case 2.2 $l=i+u$.

Since $[E_{i,i+u}+E_{i,i+1},E_{i+u,n}-E_{i+1,n}]=0$, by conditions and by Lemma 3.4, $[\delta(E_{i,i+u}),E_{i+u,n}-E_{i+1,n}]=0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $b_{1,i+u}^{(i,i+u)}-b_{1,i+1}^{(i,i+u)}=0$. By the above Case 2.1, $b_{1,i+1}^{(i,i+u)}=0$, and so $b_{1,i+u}^{(i,i+u)}=0$, i.e., $b_{kl}^{(i,i+u)}=0$.

Case 3 $k=i$.

In this case, since $(k,l)\ne(i,i+u)$ and $k=i$, we have $l\ne i+u$.

Case 3.1 $l<n$.

Since $[E_{i,i+u},E_{l,l+1}]=0$, by conditions, $[\delta(E_{i,i+u}),E_{l,l+1}]=0$. Comparing the coefficients of $E_{i,l+1}$ on the both sides of the above equality, we have $b_{il}^{(i,i+u)}=0$, i.e., $b_{kl}^{(i,i+u)}=0$.

Case 3.2 $l=n$.

Since $[E_{1i}+E_{1,i+u-1},E_{i,i+u}-E_{i+u-1,i+u}]=0$, by conditions and Lemma 3.4, $[E_{1i}+E_{1,i+u-1},\delta(E_{i,i+u})]=0$. Comparing the coefficients of E_{1n} on the both sides of the above equality, we have $b_{in}^{(i,i+u)}-b_{i+u-1,n}^{(i,i+u)}=0$. By the above Case 1, $b_{i+u-1,n}^{(i,i+u)}=0$, and so $b_{in}^{(i,i+u)}=0$, i.e., $b_{kl}^{(i,i+u)}=0$.

Thus

$$\delta(E_{i,i+u})\equiv b_{i,i+u}^{(i,i+u)}E_{i,i+u} \bmod \mathbf{N}_{n-1},\quad 1\le i\le n-u.$$

For $1\le i\le n-u$, applying δ to $[E_{i,i+u}+E_{i+1,i+1+u},E_{i,i+1}+E_{i+u,i+1+u}]=0$, we obtain that $[\delta(E_{i,i+u})+\delta(E_{i+1,i+1+u}),E_{i,i+1}+E_{i+u,i+1+u}]=0$ by conditions, so $(b_{i,i+u}^{(i,i+u)}-b_{i+1,i+u+1}^{(i+1,i+u+1)})E_{i,i+u+1}=0$, which implies that $b_{i,i+u}^{(i,i+u)}=b_{i+1,i+u+1}^{(i+1,i+u+1)}$. For any $2\le u\le n-2$, set

$$\beta_u=b_{1,1+u}^{(1,1+u)}.$$

Then the proof is completed. $\square$

Lemma 3.6 *If δ is a product zero derivation on $\mathbf{N}(n,\mathbb{F})$, $n\ge4$, and $\delta(E_{i,i+1})\in\mathbf{N}_{n-1},1\le i\le n-1$, then there exist a diagonal matrix $D'\in\mathcal{D}$ and a scalar $c\in\mathbb{F}$ such that for $1\le k<l\le n$,*

$$(\delta+\operatorname{ad}D'+\varphi_c)(E_{kl})\in\mathbf{N}_{n-1}. \tag{3.12}$$

Proof By Lemma 3.5, we may assume that $\delta(E_{i,i+u})\equiv\beta_uE_{i,i+u} \bmod \mathbf{N}_{n-1}$, where $\beta_1=0,\beta_u\in\mathbb{F},u=2,3,\dots,n-2$, $1\le i\le n-u$. Set

$$D'=\operatorname{diag}\{d_1,d_2,\dots,d_{n-1},d_n\}\in M_n(\mathbb{F})$$

with $d_u=\beta_u$, $1\le u\le n-2$, $d_{n-1}=d_2+d_{n-2},d_n=d_2+d_{n-1}$. Recall that $d_1=\beta_1=0$. We first show that

$$d_k+d_l=d_p+d_q\quad\text{for } 1\le k,l,p,q\le n \text{ and } k+l=p+q\le n+1. \tag{3.13}$$

If $k = p$, then (3.13) is clear. Assume that $k \neq p$. First consider the case of $k+l = p+q \leq n-1$. Applying δ to $[E_{1,1+k} + E_{1,1+p}, E_{1+k,1+k+l} - E_{1+p,1+p+q}] = 0$, then by Lemma 3.4, we have $\beta_k + \beta_l = \beta_q + \beta_q$, i.e., $d_k + d_l = d_p + d_q$. Next we consider the case of $k+l = p+q = n$. In this case, since $d_1 = 0$, it is enough to prove that

$$d_k + d_l = d_{n-1} \text{ for } 1 \leq k, l \leq n-1 \text{ and } k+l = n. \tag{3.14}$$

When $k = 1, 2$, (3.14) is clear. Assume that for k with $2 \leq k \leq \frac{n}{2} - 1$, (3.14) is true. Then $d_{k+1} + d_{n-k-1} = (d_{k+1} + d_1) + d_{n-k-1} = d_k + d_2 + d_{n-k-1} = d_k + (d_{n-k} + d_1) = d_k + d_{n-k} = d_{n-1}$. Thus (3.14) is true for $k+l = p+q = n$. Similarly, (3.14) holds for $k+l = p+q = n+1$. So (3.13) holds.

By (3.13), we have $d_2 + d_k + d_i = d_2 + (d_{k+i-1} + d_1) = d_2 + d_{k+i-1} = d_1 + d_{k+i} = d_{k+i}$, which implies that

$$d_k + d_i - d_{i+k} = -d_2 \ \text{ for } 1 \leq k \leq n \ \text{ and } \ 1 \leq i \leq n+1-k. \tag{3.15}$$

Then by (3.15), we have

$$(\mathrm{ad}\, D' + \delta)(E_{i,i+u}) \equiv (\beta_u + \beta_i - \beta_{i+u})(E_{i,i+u}) \bmod \mathbf{N}_{n-1} \equiv -d_2 E_{i,i+u} \bmod \mathbf{N}_{n-1},$$

i.e., $(\mathrm{ad}\, D' + \delta)(E_{kl}) \equiv -d_2 E_{kl} \bmod \mathbf{N}_{n-1}$ for any $1 \leq k < l \leq n$. Set $c = d_2$, then (3.12) holds. □

Theorem 3.7 *Let $n \geq 3$. A linear map δ on $\mathbf{N}(n, \mathbb{F})$ is a product zero derivation if and only if δ is a sum of an inner derivation, a diagonal derivation, an extremal product zero derivation, a central product zero derivation and a scalar multiplication map on $\mathbf{N}(n, \mathbb{F})$, i.e., there exist a matrix $N \in \mathbf{N}(n, \mathbb{F})$, a scalar $a \in \mathbb{F}$, a linear function f on $\mathbf{N}(n, \mathbb{F})$, a diagonal matrix $d \in \mathcal{D}$ and elements $b_1, b_2, b_3, b_4 \in \mathbb{F}$ such that*

$$\delta = \varphi_f + \varphi_a + ad\ N + \eta_d + e^{(11)}_{b_1} + e^{(n-1,1)}_{b_2} + e^{(12)}_{b_3} + e^{(n-1,2)}_{b_4}.$$

Proof It is easy to verify that a sum of several product zero derivations on $\mathbf{N}(n, \mathbb{F})$ is still a product zero derivation. Thus the sufficient direction of the theorem is obvious. Now we prove the essential direction of the theorem in the following cases $n \geq 5$, $n = 4$ and $n = 3$. Let δ be a product zero derivation on $\mathbf{N}(n, \mathbb{F})$, and $A(\delta)$ be the matrix attached to δ.

Case 1 $n \geq 5$.

By Lemma 3.1, $A(\delta)$ is a diagonal matrix. Assume that

$$A(\delta) = \mathrm{diag}\,\{a_{11}, a_{22}, \ldots, a_{n-1,n-1}\}.$$

By Lemma 3.2, there is a diagonal matrix

$$D = \mathrm{diag}\,\{0, a_{11}, a_{11} + a_{22}, \ldots, a_{11} + a_{22} + \cdots + a_{n-1,n-1}\} \in \mathcal{D},$$

such that $A(\delta + \eta_D) = 0$. Since $\delta + \eta_D$ is still a product zero derivation, by Lemma 3.3, there exist a matrix $M \in \mathbf{N}(n, \mathbb{F})$ and elements $a_1, a_2, a_3, a_4 \in \mathbb{F}$ such that $(e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \mathrm{ad}\, M + \eta_D + \delta)(E_{i,i+1}) \in \mathbf{N}_{n-1}, 1 \leq i \leq n-1$. By Lemma 3.6, there exist a matrix $D' \in \mathcal{D}$ and a scalar $c \in \mathbb{F}$ such that $(\eta_{D'} + \varphi_c + e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M + \eta_D + \delta)(E_{kl}) \in \mathbf{N}_{n-1}$ for any $1 \le k < l \le n$. We may assume that

$$(\eta_{D'} + \varphi_c + e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M + \eta_D + \delta)(E_{kl}) = r_{kl}E_{1n},$$

where $r_{kl} \in \mathbb{F}$. Let $f : \mathbf{N}(n, \mathbb{F}) \to \mathbb{F}$ is a linear function on $\mathbf{N}(n, \mathbb{F})$ determined by $f(E_{kl}) = r_{kl}$, $1 \le k < l \le n$. Then $\eta_{D'} + \varphi_c + e^{(11)}_{a_1} + e^{(n-1,1)}_{a_2} + e^{(12)}_{a_3} + e^{(n-1,2)}_{a_4} + \operatorname{ad} M + \eta_D + \delta$ is a linear functional φ_f on $\mathbf{N}(n, \mathbb{F})$. So

$$\begin{aligned}\delta =& - e^{(11)}_{a_1} - e^{(n-1,1)}_{a_2} - e^{(12)}_{a_3} - e^{(n-1,2)}_{a_4} - \operatorname{ad} M - \eta_D - \eta_{D'} - \varphi_c + \varphi_f\\ =& e^{(11)}_{-a_1} + e^{(n-1,1)}_{-a_2} + e^{(12)}_{-a_3} + e^{(n-1,2)}_{-a_4} + \operatorname{ad}(-M) + \eta_{-D-D'} + \varphi_{-c} + \varphi_f.\end{aligned}$$

We set

$$N = -M,\ d = -D - D',\ a = -c,\ b_1 = -a_1,\ b_2 = -a_2,\ b_3 = -a_3,\ b_4 = -a_4.$$

Then the lemma holds for $n \ge 5$.

Case 2 $n = 4$.

Since $[E_{12}, E_{34}] = 0$, we have

$$[\delta(E_{12}), E_{34}] + [E_{12}, \delta(E_{34})] = 0. \tag{3.16}$$

Comparing the coefficients of E_{24} (resp., E_{13}) on the both sides of the above equality (3.16), we have $a_{21} = 0$ (resp. $a_{23} = 0$). Since $[E_{23}, E_{24}] = 0$, we have

$$[\delta(E_{23}), E_{24}] + [E_{23}, \delta(E_{24})] = 0. \tag{3.17}$$

Comparing the coefficients of E_{14} on the above equality (3.17), we have $a_{12} = 0$. Since $[E_{13}, E_{23}] = 0$, we have

$$[\delta(E_{13}), E_{23}] + [E_{13}, \delta(E_{23})] = 0. \tag{3.18}$$

Comparing the coefficients of E_{14} on the above equality (3.18), we have $a_{32} = 0$.

Then

$$\delta(E_{12}) \equiv a_{11}E_{12} + a_{31}E_{34} \bmod \mathbf{N}_2;$$

$$\delta(E_{23}) \equiv a_{22}E_{23} \bmod \mathbf{N}_2;$$

$$\delta(E_{34}) \equiv a_{13}E_{12} + a_{33}E_{34} \bmod \mathbf{N}_2.$$

Thus

$$(e^{(12)}_{-a_{31}} + e^{(32)}_{-a_{13}} + \delta)(E_{i,i+1}) \equiv a_{ii}E_{i,i+1} \bmod \mathbf{N}_2, \quad 1 \le i \le 3.$$

Let

$$D = \operatorname{diag}\{0, a_{11}, a_{11} + a_{22}, a_{11} + a_{22} + a_{33}\}.$$

Then $(e^{(12)}_{-a_{31}} + e^{(32)}_{-a_{13}} + \eta_D + \delta)(E_{i,i+1}) \in \mathbf{N}_2$. Assume that

$$(e^{(12)}_{-a_{31}} + e^{(32)}_{-a_{13}} + \eta_D + \delta)(E_{i,i+1}) \equiv c^{(2)}_{1i}E_{13} + c^{(2)}_{2i}E_{24} \bmod \mathbf{N}_3, \quad 1 \le i \le 3.$$

Applying $e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta$ to the equality $[E_{12},E_{34}]=0$, we have

$$[(e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta)(E_{12}),E_{34}]+[E_{12},(e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta)(E_{34})]=0.$$

Comparing the coefficients of E_{14} on the both sides of the above equality, we have $c^{(2)}_{11}=-c^{(2)}_{23}$. Let

$$\sigma=\mathrm{ad}(c^{(2)}_{11}E_{23}+c^{(2)}_{22}E_{34}-c^{(2)}_{12}E_{12}).$$

Then

$$(\sigma+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta)(E_{12})\equiv c^{(2)}_{21}E_{24} \bmod \mathbf{N}_3;$$

$$(\sigma+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta)(E_{23})\equiv 0 \bmod \mathbf{N}_3;$$

$$(\sigma+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta)(E_{34})\equiv c^{(2)}_{13}E_{13} \bmod \mathbf{N}_3.$$

So

$$(\sigma+e^{(11)}_{-c^{(2)}_{21}}+e^{(31)}_{-c^{(2)}_{13}}+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta)(E_{i,i+1})\equiv 0 \bmod \mathbf{N}_3.$$

By Lemma 3.6, there exist a diagonal matrix $D'\in\mathcal{D}$ and a scalar $c\in\mathbb{F}$ such that $(\sigma+e^{(11)}_{-c^{(2)}_{21}}+e^{(31)}_{-c^{(2)}_{13}}+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta+\eta_{D'}+\varphi_c)(E_{kl})\in\mathbf{N}_3$, where $1\le k<l\le 4$. Assume that $(\sigma+e^{(11)}_{-c^{(2)}_{21}}+e^{(31)}_{-c^{(2)}_{13}}+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta+\eta_{D'}+\varphi_c)(E_{kl})=r_{kl}E_{14}$, where $r_{kl}\in\mathbb{F}, 1\le k<l\le 4$. Let $f:\mathbf{N}(4,\mathbb{F})\to\mathbb{F}$ be a linear function determined by $f(E_{kl})=r_{kl}$, $1\le k<l\le 4$. Then $\sigma+e^{(11)}_{-c^{(2)}_{21}}+e^{(31)}_{-c^{(2)}_{13}}+e^{(12)}_{-a_{31}}+e^{(32)}_{-a_{13}}+\eta_D+\delta+\eta_{D'}+\varphi_c$ is the central product zero derivation φ_f attached to f. Thus $\delta=\mathrm{ad}(-c^{(2)}_{11}E_{23}-c^{(2)}_{22}E_{34}+c^{(2)}_{12}E_{12})+e^{(11)}_{c^{(2)}_{21}}+e^{(31)}_{c^{(2)}_{13}}+e^{(12)}_{a_{31}}+e^{(32)}_{a_{13}}+\eta_{-D-D'}+\varphi_{-c}+\varphi_f$. Set

$$N=-c^{(2)}_{11}E_{23}-c^{(2)}_{22}E_{34}+c^{(2)}_{12}E_{12},\ b_1=c^{(2)}_{21},\ b_2=c^{(2)}_{13},$$

$$b_3=a_{31},\ b_4=a_{13},\ d=-D-D',\ a=-c.$$

Then the lemma holds for $n=4$.

Case 3 $n=3$.

Let $e^{(11)}_{-a_{21}}$ and $e^{(21)}_{-a_{12}}$ be the extremal product zero derivations defined in Section 2(C). Then

$$(e^{(11)}_{-a_{21}}+e^{(21)}_{-a_{12}}+\delta)(E_{12})\equiv a_{11}E_{12} \bmod \mathbf{N}_2;$$

$$(e^{(11)}_{-a_{21}}+e^{(21)}_{-a_{12}}+\delta)(E_{23})\equiv a_{22}E_{23} \bmod \mathbf{N}_2.$$

Let η_D be the diagonal derivation defined in Section 2(B), where

$$D=\mathrm{diag}\{0,a_{11},a_{11}+a_{22}\}.$$

Then $(\eta_D+e^{(11)}_{-a_{21}}+e^{(21)}_{-a_{12}}+\delta)(E_{i,i+1})\equiv 0 \bmod \mathbf{N}_2$, $i=1,2$.

Assume that $(\eta_D+e^{(11)}_{-a_{21}}+e^{(21)}_{-a_{12}}+\delta)(E_{kl})=r_{kl}E_{13}$, where $1\le k<l\le 3$. Let $f:\mathbf{N}(3,\mathbb{F})\to\mathbb{F}$ be a linear function determined by $f(E_{kl})=r_{kl}$. Thus $\eta_D+e^{(11)}_{-a_{21}}+e^{(21)}_{-a_{12}}+\delta$ is the central product zero derivation φ_f attached to f. Thus $\delta=e^{(11)}_{a_{21}}+e^{(21)}_{a_{12}}+\eta_{-D}+\varphi_f$. Then the lemma holds for $n=3$. $\Box$

Remark If $n=2$, $\mathbf{N}(n,\mathbb{F})=\mathbb{F}E_{12}$ is a one-dimensional linear space, and any linear map on $\mathbf{N}(n,\mathbb{F})$ is a product zero derivation. We leave this case in the above theorem.

By Theorem 3.7, we can obtain the following results about derivations of $\mathbf{N}(n,\mathbb{F})$, which appeared in [20, Theorem 3.2].

Corollary 3.8 *Let $n\geq 3$, Char $\mathbb{F}\neq 2$. A linear map δ on $\mathbf{N}(n,\mathbb{F})$ is a Lie derivation if and only if δ is a sum of an inner derivation, a diagonal derivation, an extremal derivation and a central derivation, i.e., there exist a matrix $N\in\mathbf{N}(n,\mathbb{F})$, a diagonal matrix $d\in\mathcal{D}$, elements $b_1,b_2\in\mathbb{F}$ and a linear function f with $f(\mathbf{N}_2)=0$ on $\mathbf{N}(n,\mathbb{F})$ such that*

$$\delta=\varphi_f+\operatorname{ad}N+\eta_d+e_{b_1}^{(11)}+e_{b_2}^{(n-1,1)}.$$

Proof It is clear that a map on $\mathbf{N}(n,\mathbb{F})$ of the form above is a Lie derivation. Conversely, assume that δ is any Lie derivation on $\mathbf{N}(n,\mathbb{F})$. Then δ is a product zero derivation. By Theorem 3.7, we may assume that $\delta=\varphi_f+\varphi_a+\operatorname{ad}N+\eta_d+e_{b_1}^{(11)}+e_{b_2}^{(n-1,1)}+e_{b_3}^{(12)}+e_{b_4}^{(n-1,2)}$. Since $\operatorname{ad}N$, η_d, $e_{b_1}^{(11)}$, $e_{b_2}^{(n-1,1)}$ are derivations on $\mathbf{N}(n,\mathbb{F})$, $\varphi_f+\varphi_a+e_{b_3}^{(12)}+e_{b_4}^{(n-1,2)}$ is also a Lie derivation. Since $E_{n-2,n}=[E_{n-2,n-1},E_{n-1,n}]$, we have

$$\begin{aligned}(\varphi_f+\varphi_a+e_{b_3}^{(12)}+e_{b_4}^{(n-1,2)})(E_{n-2,n})=&[(\varphi_f+\varphi_a+e_{b_3}^{(12)}+e_{b_4}^{(n-1,2)})(E_{n-2,n-1}),E_{n-1,n}]+\\&[E_{n-2,n-1},(\varphi_f+\varphi_a+e_{b_3}^{(12)}+e_{b_4}^{(n-1,2)})(E_{n-1,n})].\end{aligned}\tag{3.19}$$

By computation, the coefficients of $E_{n-2,n}$ (resp., $E_{1,n-1}$) on the left-hand side is a (resp., b_4), and the coefficients of $E_{n-2,n}$ on the right-hand side is $2a$ (resp., $-b_4$), then $a=2a$ (resp., $b_4=-b_4$), and so $a=0$ (resp., $b_4=0$). Similarly, applying $\varphi_f+\varphi_a+e_{b_3}^{(12)}+e_{b_4}^{(n-1,2)}$ to the equality $[E_{12},E_{23}]=E_{13}$, we have $b_3=0$. Thus $\varphi_a=0$, and $e_{b_3}^{(12)}=e_{b_4}^{(n-1,2)}=0$. For any $X,Y\in\mathbf{N}(n,\mathbb{F})$, $\varphi_f([X,Y])=[\varphi_f(X),Y]+[X,\varphi_f(Y)]$, i.e., $f([X,Y])E_{1n}=[f(X)E_{1n},Y]+[X,f(Y)E_{1n}]=0$. Thus $f([X,Y])=0$ for any $X,Y\in\mathbf{N}(n,\mathbb{F})$, i.e., $f(\mathbf{N}_2)=0$. Thus φ_f is a central derivation, and the corollary holds. □

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