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Cycles Containing a Subset of a Given Set of Elements in Cubic Graphs

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Abstract The technique of contractions and the known results in the study of cycles in 3-connected cubic graphs are applied to obtain the following result. Let G be a 3-connected cubic graph, $X \subseteq V(G)$ with $|X| = 16$ and $e \in E(G)$. Then either for every 8-subset A of X, $A \cup \{e\}$ is cyclable or for some 14-subset A of X, $A \cup \{e\}$ is cyclable.

Keywords contraction; cubic graphs; cyclability.

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1. Contractions and reductions

The study of cycles containing sets of elements in cubic graphs arises from two important sources: the study of cycles in graphs [3, 5, 7] and classical graph colouring problems [6, 8]. The concept of contraction is the main tool in the study of cycles containing sets of elements in cubic graphs. We are interested in necessay and sufficient conditions for the existence of a cycle containing an arbitrary set of vertices with a specified cardinality. Such conditions have been obtained by means of contractions [3, 4]. The concept and techinque of contraction also play a key role in the main result of this note. Here we consider cycles containing comparatively large subsets of a given set of vertices and an edge in a cubic graph.

Let $G = (V, E)$ be a graph. A contraction of G is a partition $\{V_1, V_2, \ldots, V_s\}$ of the vertex set V such that for each $i = 1, 2, \ldots, s$, the induced subgraph $G|_{V_i}$ is connected. This partition gives rise to a natural mapping from G to a graph H, the contraction (graph) obtained from G under the contraction. The contraction (graph) H is the graph with

$$V(H) = \{V_1, V_2, \ldots, V_s\}, \quad E(H) = \{V_iV_j : i \neq j, [V_i, V_j] \neq \emptyset\}.$$

If $f : G \to H$ denotes a contraction (mapping) of G onto H, then by the definition above, $G|_{f^{-1}(u)}$ is a connected subgraph of G for each $u \in V(H)$. For a special and extreme example, the graph K_1 is a contraction of any connected graph G since $\{V\}$ is a partition of V and $G = G|_V$ is connected. Any automorphism of G is a contraction since it is a permutation of the trivial partition of V into single vertices. Hence a graph G is a contraction of itself. As we consider only cubic graphs, all contractions in this paper are cubic contractions.

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A contraction $f : G \to H$ is called edge-injective or faithful if $e_1, e_2 \in E(G)$ and $e_1 \neq e_2$ implies that $f(e_1) \neq f(e_2)$. As we are working with simple cubic graphs, H has no loops even if f is faithful.

Let $G = (V, E)$ be a graph and $A \subseteq V$. A subgraph of G is said to be cyclic if it contains a cycle. Let G be connected and $S \subseteq E$. If $G - S$ is not connected and each component of $G - S$ is cyclic, then S is said to be a cyclic separating set of edges. The size of a cyclic separating set of edges of G with the least number of edges is called the cyclic edge-connectivity of G. If the cyclic edge-connectivity of G is at least k, then G is said to be a cyclically k-edge-connected graph.

If G has a cycle containing A, then A is said to be cyclable in G. If every m-subset of V is cyclable in G, then G is said to be an m-cyclable graph. The largest integer m for which G is m-cyclable is called the cyclability of G.

Let G be a 3-connected cubic graph with a cyclic separating set $S = \{u_i v_i : 1 \leq i \leq 3\}$. Suppose that L and R are the two cyclic components of $G - S$ and $u_i \in L$, $v_i \in R$ (Note that $G - S$ has exactly two components). Also suppose $u, v \notin V(G)$ and $u \neq v$. Then the cubic graphs

$$H = G/R = L \cup \{u, uu_1, uu_2, uu_3\}, \quad J = G/L = R \cup \{v, vv_1, vv_2, vv_3\}$$

are called the reductions of G across S, and u and v are the new vertices of H and J, respectively. Since G is 3-connected, so are both H and J (see [6]). Clearly, $|H|, |J| \leq |G| - 2$ since $|L|, |R| \geq 3$.

The Petersen graph, essential to the work of this note, is shown in Figure 1 where the labels of vertices will be used throughout the paper.

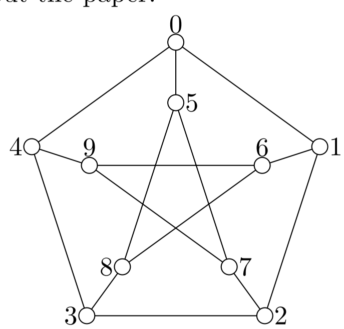

Figure 1 The Petersen graph P

In this notation, if G is a cubic graph and $f : G \to P$ is a faithful contraction, then

$$S = [f^{-1}(0), G - f^{-1}(0)] = \{f^{-1}(01), f^{-1}(04), f^{-1}(05)\}$$

(the preimage of the three edges incident with the top vertex) is a separating set of edges which separates $P|_{f^{-1}(0)}$ from $G - f^{-1}(0)$. If $G|_{f^{-1}(0)}$ contains a cycle (note that G is cubic graph $|f^{-1}(0)| > 1$ implies that $G|_{f^{-1}(0)}$ contains a cycle), then S is a cyclic separating set of G with $|S| = 3$.

2. Cycles containing elements

The following results will be used:

Theorem 2.1 ([7]) *Let G be a 3-connected cubic graph. If $A \subset V(G), |A| \leq 5$ and $e \in E(G)$, then $G - e$ has a cycle containing A.*

Theorem 2.2 ([7]) *Every 3-connected cubic graph is 9-cyclable.*

Let G be a 3-connected cubic graph and $A \subseteq V(G)$. If there is a faithful contraction $f : G \to P$ such that $f(A) = V(P)$, then A is not cyclable in G. Notice that this was the motivation of [4]. The converse of this is also true for $|A| = 12$:

Theorem 2.3 ([4]) *Let G be a 3-connected cubic graph, $A \subset V(G)$ and $|A| = 12$. Then either A is cyclable or there is a faithful contraction $f : G \to P$ such that $f(A) = V(P)$.*

If G has a cycle containing $A \subseteq V(G)$ that also contains $e \in E(G)$, then $A \cup \{e\}$ is said to be cyclable for brevity.

Theorem 2.4 ([3]) *Let G be a 3-connected cubic graph, $A \subset V(G)$, $|A| = 8$ and $e \in E(G)$. Then either $A \cup \{e\}$ is cyclable in G or there is a faithful contraction $f : G \to P$ such that $f(e) = 01$ and $f(A) = V(P) - \{0, 1\} = \{i : 2 \leq i \leq 9\}$.*

This result is the motivation for the work of the present paper. Note that since the Petersen graph is both vertex-transitive and edge-transitive, the labels of vertices are immaterial and aid the discussion only. A useful corollary to this theorem can now be easily stated which is a weaker form of the main result of [1]. An unavoidable edge given $A \subseteq V(G)$ is an edge e such that if C is any cycle containing A, then C contains e also.

Corollary 2.1 *Let G be a 3-connected cubic graph, $A \subset V(G)$, $|A| \leq 7$ and $e \in E(G)$. Then G has a cycle containing $A \cup \{e\}$.*

3. Cycles containing subsets

The main theorem of this paper is

Theorem 3.1 *Let G be a 3-connected cubic graph, $X \subset V(G)$ with $|X| = 16$ and $e \in E(G)$. Then, either for every 8-subset A of X, $A \cup \{e\}$ is cyclable or for some 14-subset A of X, $A \cup \{e\}$ is cyclable.*

Let G and H be cubic graphs and let

$$f : G \longrightarrow H$$

be a faithful contraction. If for $x \in V(H)$ both $L_x = G|_{f^{-1}(x)}$ and $R_x = G - L_x$ are cyclic subgraphs, then denote $H_x = G/R_x$ and $J_x = G/L_x$ with u_x the new vertex of H_x and v_x that of J_x. In particular, consider $H = P$. Then for $i \in V(P)$, $f_i : G \to H_i$ is the faithful contraction of $G - f^{-1}(i)$ to a new vertex u_i and $J_i = G/G|_{f^{-1}(i)}$ with the new vertex v_i. If $|f^{-1}(i)| > 1$, then since G is 3-connected and the three edges between $f^{-1}(i)$ and $G - f^{-1}(i)$ form a separating set, $G|_{f^{-1}(i)}$ is a cyclic subgraph of G. Since $P - i$ has many cycles, $J_i - v_i = G - f^{-1}(i)$ is a cyclic subgraph of G.

Proof Let G be a 3-connected cubic graph and $X \subseteq V(G)$ be any subset with $X = 16$. By Theorem 2.4, either for every $A \subseteq X$ with $|A| = 8$, $A \cup \{e\}$ is cyclable or there exists a faithful contraction

$$f : G \longrightarrow P$$

such that $f(e) = 01$ and $f(A) = V(P) - \{0, 1\}$. Suppose that for some $A \subseteq X$ with $|A| = 8$, G has no cycle containing $A \cup \{e\}$. We shall show that there exists $B \subseteq X$ with $|B| = 14$ such that $B \cup \{e\}$ is cyclable in G.

For each $i \in V(P)$ let $X_i = X \cap f^{-1}(i)$ and $n_i = |X_i|$ and call n_i the index of vertex i. Since $|X| = 16$ and $f(A) = V(P) - \{0, 1\}$, the integers n_i satisfy

$$\sum_{i=0}^{9} n_i = 16, \tag{1}$$

$$n_i \geq 1, \text{ for } 0 \leq i \leq 9. \tag{2}$$

Let $T_r = \{i \in V(P) : n_i = r\}$ and $t_r = |T_r|$. Then clearly

$$0 \leq t := \sum_{r \geq 5} t_r \leq 2. \tag{3}$$

Consider the following cases.

Case 1 $t = 2$. That is, there are precisely two indices that are at least 5. Let $n_i, n_j \geq 5$ for $i, j \in V(P) - \{0, 1\}$ (note that $n_0 = 0 = n_1$). Then for $k \in V(P) - \{0, 1, i, j\}$, $n_k = 1$ and $n_i + n_j \leq 10$. Hence $n_i = n_j = 5$. Also, $\{i, j\} \cap \{0, 1\} = \emptyset$, for otherwise, $|X| \geq 17$.

Let $\Gamma = \mathrm{Aut}(P)$ and denote by Γ_{01} the subgroup of Γ that fixes the edge 01. Then the orbit of this subgroup on $V(P)$ is

$$\mathrm{Orb}(\Gamma_{01}) = \{\{0, 1\}, \{2, 4, 5, 6\}, \{3, 7, 8, 9\}\}. \tag{4}$$

Hence the following pairwise inequivalent complete set of cases will be considered. In all these cases, denote $\gamma_i = f_i f^{-1}$ for simplicity. Note that γ_i is a mapping since f_i is a mapping. The relation f^{-1} is not necessarily a mapping but f^{-1} will provide a subset of vertices of G which f_i maps to a single vertex of P. Hence γ_i is a mapping from G to H_i.

Case 1.1 $i = 2$, $j = 3$. Consider the faithful contractions

$$f_2 : G \to H_2, \quad f_3 : G \to H_3.$$

By Corollary 2.1, H_2 has a cycle D_2 containing $X_2 \cup \{\gamma_2(23)\}$ and H_3 has a cycle D_3 containing $X_3 \cup \{\gamma_3(23)\}$.

If $\gamma_2(12) \notin D_2$ and $\gamma_3(34) \notin D_3$, then consider the cycle $C_P = 0168327940$.

If $|f^{-1}(k)| = 1$ for some $k \in \{4, 6, 7, 8, 9\}$, then the vertex $f^{-1}(k)$ and its two incident edges on C_P is a path of length 2 in G.

If $f^{-1}(k)$ has more than one vertex, say for $k = 4$, then consider H_4. Since $H_4 - \gamma_4(34)$ is 2-connected, it has a cycle D_4 containing $X_4 \cup \{u_4\}$ where $|X_4| = n_4 = 1$ and u_4 is the new vertex of H_4. $D_4 - u_4$ is a path in $H_4 - u_4$ containing X_4. Then the union of

$$(D_4 - u_4) \cup \{f^{-1}(04), f^{-1}(49)\}$$

and suitable paths in $G|_{f^{-1}(0)}$ and $G|_{f^{-1}(1)}$ gives a path in G containing X_4 and the edges required by $f^{-1}(C_P)$. Paths in H_k required by a cycle in G are obtained similarly for each

$k \in \{4,6,7,8,9\}$. Union of all these paths and $\{f^{-1}(e) : e \in E(C_P)\}$ is a cycle of G containing $(X - X_5) \cup \{e\}$ and $|X - X_5| = 15$. A cycle C in P lifts (via faithful contraction f) if there is a cycle $C_G \subseteq G$ such that $f(C_G) = C$. In all the following cases, we exhibit only the cycle needed in P and say that it lifts.

All possible pairwise inequivalent cases are shown in the following table with cycles that lift shown in the middle column.

Cases	Cycles of P that lift	$\|X - X_i\|$
$\gamma_2(12) \notin D_2$, $\gamma_3(34) \notin D_3$	0168327940	$\|X - X_5\| = 15$
$\gamma_2(12) \notin D_2$, $\gamma_3(38) \notin D_3$	0169432750	$\|X - X_8\| = 15$
$\gamma_2(27) \notin D_2$, $\gamma_3(34) \notin D_3$	0123869750	$\|X - X_4\| = 15$
$\gamma_2(27) \notin D_2$, $\gamma_3(38) \notin D_3$	0123496850	$\|X - X_7\| = 15$

Table 1 Case 1.1

Case 1.2 $i = 2$, $j = 4$. By Corollary 2.1, the faithful contraction H_2 has a cycle D_2 containing $X_2 \cup \{\gamma_2(27)\}$ and H_4 has a cycle D_4 containing $X_4 \cup \{\gamma_2(49)\}$. The complete cases are shown in the table below.

Cases	Cycles of P that lift	$\|X - X_i\|$
$\gamma_2(12) \notin D_2$, $\gamma_4(04) \notin D_4$	0169432750	$\|X - X_8\| = 15$
$\gamma_2(12) \notin D_2$, $\gamma_4(34) \notin D_4$	0168327940	$\|X - X_5\| = 15$
$\gamma_2(23) \notin D_2$, $\gamma_4(34) \notin D_4$	0127586940	$\|X - X_3\| = 15$

Table 2 Case 1.2

Case 1.3 $i = 2$, $j = 8$. The cases are similar to those considered in Cases 1.1–1.2. Here all possible cycles of H_2 that contain $X_2 \cup \{u_2\}$ and all possible cycles of H_8 that contain $X_8 \cup \{u_8\}$ are to be considered and the cycle in P is exhibited in each possible case.

Cases	Cycles of P that lift
$\gamma_2(12) \notin D_2$, $\gamma_8(38) \notin D_8$	0168572340
$\gamma_2(12) \notin D_2$, $\gamma_8(58) \notin D_8$	0168327940
$\gamma_2(12) \notin D_2$, $\gamma_8(68) \notin D_8$	0169723850
$\gamma_2(23) \notin D_2$, $\gamma_8(38) \notin D_8$	0127586940
$\gamma_2(23) \notin D_2$, $\gamma_8(58) \notin D_8$	0127968340
$\gamma_2(23) \notin D_2$, $\gamma_8(68) \notin D_8$	0127943850
$\gamma_2(27) \notin D_2$, $\gamma_8(38) \notin D_8$	0123496850
$\gamma_2(27) \notin D_2$, $\gamma_8(58) \notin D_8$	0123869750
$\gamma_2(27) \notin D_2$, $\gamma_8(68) \notin D_8$	0123857940

Table 3 Case 1.3

In all rows of this table, $|X - X_i| = 15$.

Case 1.4 $i = 3$, $j = 7$. Here all possible cycles of H_3 that contain $X_3 \cup \{u_3\}$ and all possible cycles of H_7 that contain $X_7 \cup \{u_7\}$ are to be considered and the cycle in P is exhibited in each case.

Cases	Cycles of P that lift	$\lvert X - X_i \rvert$
$\gamma_3(23) \notin D_3$, $\gamma_7(27) \notin D_7$	0168349750	15
$\gamma_3(23) \notin D_3$, $\gamma_7(57) \notin D_7$	0127968340	15
$\gamma_3(23) \notin D_3$, $\gamma_7(79) \notin D_7$	012758340	14
$\gamma_3(34) \notin D_3$, $\gamma_7(27) \notin D_7$	0123857940	15
$\gamma_3(34) \notin D_3$, $\gamma_7(57) \notin D_7$	0168327940	15
$\gamma_3(34) \notin D_3$, $\gamma_7(79) \notin D_7$	016832750	14
$\gamma_3(38) \notin D_3$, $\gamma_7(27) \notin D_7$	0123496850	15
$\gamma_3(38) \notin D_3$, $\gamma_7(57) \notin D_7$	016972340	14
$\gamma_3(38) \notin D_3$, $\gamma_7(79) \notin D_7$	0168572340	15

Table 4 Case 1.4

Case 1.5 $i = 3$, $j = 8$. By Corollary 2.1, graph H_3 has a cycle through $X_3 \cup \{\gamma_3(38)\}$ and H_8 has a cycle containing $X_8 \cup \{\gamma_8(38)\}$. By the symmetry of the graph under Γ_{01}, two cases cover all possibilities: the cycles 0168349750 and 0169758340 both lift.

Case 2 $t = 1$. That is, there is precisely one index ≥ 5. Let $n_i \geq 5$, and for $j \neq i$, $n_j \leq 4$. Then

$$5 \leq n_i \leq 9.$$

Consider the following subcases.

Case 2.1 $n_i \in \{5, 6, 7\}$.

Case 2.1.1 $i = 0$. By Corollary 2.1, the faithful contraction H_0 has a cycle D_0 that contains $X_0 \cup \{\gamma_0(01)\}$. Since $n_0 \geq 5$, $t_1 \geq 3$. Let $n_j = 1$. If $j \in \{2, 4, 5, 6\}$, then let $j = 2$. Then consider the cycle 0168349750 and apply Theorem 2.1 to H_k for $k \neq 0, 2$. This gives a cycle in G through a 15-subset of X and e. If $j \in \{3, 7, 8, 9\}$, then consider the cycle 0127586940 that lifts.

Case 2.1.2 $i = 2$. Again there is an index $n_j = 1$. If $j = 3$, then 0127586940 lifts. If $j = 4$, then 0169723850 lifts. If $j = 6$, then 0127943850 lifts. If $j = 8$, then 0169432750 lifts.

Case 2.1.3 $i = 3$. Let $n_j = 1$ and consider j. If $j = 2$, then 0168349750 lifts. If $j = 5$, then 0168327940 lifts. If $j = 7$, then 0123496850 lifts. If $j = 8$, then 0169432750 lifts.

Case 2.2 $n_i = 8$.

Case 2.2.1 $i = 0$. Then $t_1 = 8$. If H_0 has a cycle through $X_0 \cup \{\gamma_0(01)\}$, then the proof is the same as that of Case 2.1.1. Hence assume that H_0 does not have a cycle through $X_0 \cup \{\gamma_0(01)\}$. By Theorem 2.4, there is a faithful contraction $f' : H_0 \to P'$ such that $f'(\gamma_0(01)) = 0'1' \in E(P')$ and $f'(X_0) = V(P') - \{0', 1'\} = \{i' : 2 \leq i \leq 9\}$ where P' is another copy of P with vertices labelled with i' for $i = 0, 1, \ldots, 9$. Let

$$Q = (P - 0)\bigcup\{11', 44', 55'\}\bigcup(P' - 0').$$

(Note that the other alternative gives an isomorphic copy of Q). Then there is a faithful contraction

$$\phi : G \to Q$$

such that $\phi(X) = \{i : 2 \le i \le 9\} \cup \{i' : 2 \le i \le 9\}$ and $\phi(e) = 11'$. Now the cycle

$$D_Q = 169723855'8'3'2'7'9'6'1'1$$

lifts via ϕ^{-1} to give rise to a cycle in G that contains a 14-subset of X as well as e by applying Theorem 2.1 to the faithful contractions corresponding to $\phi^{-1}(x)$, $x \in V(Q)$.

Case 2.2.2 $i = 2$. Then $t_1 \ge 6$ and if $j \ne i$, then $n_j \le 2$. By Theorem 2.2, H_2 has a cycle D_2 containing $X_2 \cup \{u_2\}$. If $\gamma_2(12) \notin D_2$, then 0169723850 lifts. If $\gamma_2(23) \notin D_2$, then 0127943850 lifts. If $\gamma_2(27) \notin D_2$, then 0123496850 lifts.

Case 2.2.3 $i = 3$. Then again $t_1 \ge 6$ and if $j \ne i$, then $n_j \le 2$. By Theorem 2.2, H_3 has a cycle D_3 containing $X_3 \cup \{u_3\}$. This is the same as Case 2.1.3.

Case 2.3 $n_i = 9$. Then $i \notin \{0, 1\}$, $t_1 = 7$ and for $j \ne 0, 1, i$, we have $n_j = 1$. If H_i has a cycle D_i through $X_i \cup \{u_i\}$, then a cycle in P lifts. If there is no such cycle D_i in H_i, then by Theorem 2.3 there is a faithful contraction $f_i : H_i \to P'$ such that $f_i(X_i \cup \{u_i\}) = V(P')$.

Case 2.3.1 $i = 2$. Let

$$Q = (P - 2)\bigcup\{11', 33', 77'\}\bigcup(P' - 2').$$

Then there is a faithful contraction $\phi_2 : G \to Q$ such that $\phi_2(X) = V(Q) - \{0, 1\}$ and $\phi_2(e) = 01$. Now the cycle

$$D_Q = 011'6'8'5'0'4'9'7'7586940$$

lifts via ϕ_2^{-1} to give a desired cycle through $X - \phi_2^{-1}(3) - \phi_2^{-1}(3')$ and e in G.

Case 2.3.2 $i = 3$. Let

$$Q = (P - 3)\bigcup\{22', 44', 88'\}\bigcup(P' - 3').$$

Then there is a faithful contraction $\phi_3 : G \to Q$ such that $\phi_3(X) = V(Q) - \{0, 1\}$ and $\phi_3(e) = 01$. Now the cycle

$$D_Q = 0169722'7'9'6'1'0'5'8'8'5'0$$

lifts.

Case 3 $t = 0$. That is, for $0 \le i \le 9$, $n_i \le 4$. There is $i \in \{2, \ldots, 9\}$ such that $n_i \le 2$. Consider the two distinct subcases for such i.

1) If $i = 2$, then the cycle 0168349750 lifts.
2) If $i = 3$, then the cycle 0127586940 lifts.

This completes the proof. □

Let G be a 3-connected cubic graph, $X \subseteq V(G)$ with $|X| = p$ and $e \in E(G)$. If either for each $A \subset X$ with $|A| = r$, $A \cup \{e\}$ is cyclable or for some $A \subset X$ with $|A| = q$, $A \cup \{e\}$ is cyclable, then denote $G \in \mathcal{C}(r; 1)_{(p,q)}$.

There ought to be a result on $\mathcal{C}(8; 1)_{(18,16)}$. But sets of 6 vertices and two edges that are cyclable in these graphs must first be determined.

It should not be too difficult to determine $\mathcal{C}(9; 1)_{(16,14)}$ and $\mathcal{C}(9; 1)_{(18,16)}$, but the proofs will be tedious under the present method.

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