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Hochschild Cohomology of Galois Coverings of Quantum Exterior Algebras

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Abstract In this paper, we consider the Hochschild cohomology of a class of quantum algebras Λ_{q}^n. We construct a minimal projective bimodule resolution of Λ_{q}^n, and calculate the $\Bbbk$-dimensions of all the Hochschild cohomology groups of Λ_{q}^n. Furthermore, we give the Hochschild cohomology ring structure of Λ_{q}^n for some special cases.

Keywords Hochschild cohomology; Galois covering; quantum exterior algebra.

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1. Introduction

Let Λ be a finite-dimensional algebra (associative with identity) over a field $\Bbbk$ and $\Lambda^e = \Lambda^{op} \otimes_{\Bbbk} \Lambda$ be the enveloping algebra of Λ. The i-th Hochschild cohomology group of Λ is defined by $HH^i(\Lambda) := \mathrm{Ext}^i_{\Lambda^e}(\Lambda, \Lambda)$, for each $i \geq 0$ (see [3]). It is well-known that the Hochschild cohomology groups are subtle invariance of associative algebras, such as Morita equivalent invariance, tilting invariance, derived equivalent invariance and so on, and have played a fundamental role in representation theory of artin algebras [3, 6, 10].

For the Hochschild homology and cohomology of finite-dimensional algebras, there are some problems and conjectures, such as Happel's question in [10], Han's conjecture [9], Snashall and Solberg's conjecture [15] and so on. In [10], Dieter Happel asked that if the Hochschild cohomology groups $HH^i(\Lambda)$ of a finite-dimensional algebra Λ over a field $\Bbbk$ vanish for all sufficiently large i, is the global dimension of Λ finite? In 2005, Buchweitz, Green, Madsen and Solberg first gave a negative answer to this question by exhibiting the Hochschild cohomology of the quantum exterior algebras $A_q = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ (see [2]). Moreover, these algebras were studied to exhibit some other pathologies [13, 14].

Recently, several results and tools from algebraic topology such as covering theory have been adapted to the representation theory of non-commutative finite dimensional associative algebras. It is well-known that there is a strong connection among skew group algebras, Galois coverings and smash products of graded algebras [5]. In [12], a comparison of the Hochschild cohomology of

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the algebras involved in a Galois covering of the Kronecker algebra with a cyclic group of order 2 was initiated. Moreover, it was proved in [18] that a finite-dimensional quiver $\Bbbk$-algebra is Koszul if and only if its finite Galois covering algebra with Galois group G satisfying $\mathrm{char}\Bbbk \nmid |G|$ is Koszul. The Hochschild cohomology behavior of $\mathbb{Z}_2$-Galois covering of the Grassmann algebra and $\mathbb{Z}_2 \times \mathbb{Z}_2$-Galois covering of the quantum exterior algebras were considered in [16] and [11].

In this paper, we consider the homological properties of the $\mathbb{Z}_n \times \mathbb{Z}_n$-Galois coverings of the quantum exterior algebras by calculating all the Hochschild cohomology groups. For a more general quantum algebra Λ_{q}^n, we construct a minimal projective bimodule resolution, and calculate the $\Bbbk$-dimension of all the Hochschild cohomology groups of Λ_{q}^n. In general, showing all the Hochschild cohomology groups for a given finite-dimensional algebra is very difficult. Our approach here is elementary and accurate. Furthermore, we give the Hochschild cohomology ring structure of Λ_{q}^n for some special case. Our results will be helpful to understand deeply the Hochschild cohomology of Galois covering algebras with finite Galois groups.

2. Minimal projective bimodule resolutions

From now on, we fix $\Bbbk$ a field, and denote by $\Lambda_{\mathrm{q}}^n = \Bbbk Q/I_{\mathrm{q}}$ the finite-dimensional $\Bbbk$-algebra given by the quiver Q with relations in ideal I_{q}, where Q is a torus-like finite quiver with n^2 vertices $\{(i,j) \mid i,j \in \mathbb{Z}_n\}$, and $2n^2$ arrows $\{a_{i,j} : (i,j) \to (i,j+1)\} \cup \{b_{i,j} : (i,j) \to (i+1,j)\}$ as follows:

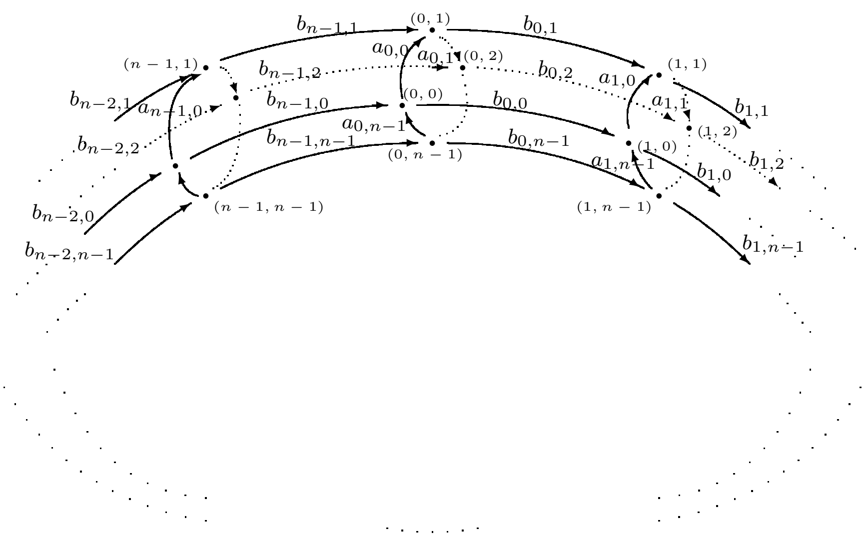

Figure 1 Quiver of Λ_{q}^n

and $I_q = \langle a_{i,j}a_{i,j+1},\ b_{i,j}b_{i+1,j},\ a_{i,j}b_{i,j+1} + q_{i,j}b_{i,j}a_{i+1,j} \mid i,j \in \mathbb{Z}_n, q_{i,j} \in \Bbbk \setminus \{0\}\rangle$. In particular, if we take $q_{ij} = q$ for all $i,j \in \mathbb{Z}_n$, then Λ_{q}^n are just the $\mathbb{Z}_n \times \mathbb{Z}_n$-Galois coverings of the quantum exterior algebras A_q (see [18]). For the details of path algebra and its quotient algebra we refer to [1].

Denote by $e_{i,j}$ the idempotent of the algebra Λ_{q}^n corresponding to the vertex (i,j). Then

$\mathcal{B} := \{e_{i,j},\ a_{i,j},\ b_{i,j},\ a_{i,j}b_{i,j+1} \mid i,j \in \mathbb{Z}_n\}$ is a $\Bbbk$-basis of $\Lambda^n_{\mathfrak{q}}$, where we identify the elements in $\mathcal{B}$ with their images in $\Lambda^n_{\mathfrak{q}}$ and identify $b_{i,n}$ with $b_{i,0}$. Next, we define an order "$\prec$" on the paths in $\Bbbk Q$ with length less than 2 by setting:

(i) $e_{i_1,j_1} \prec a_{i_2,j_2} \prec b_{i_3,j_3}$ for any $i_1,i_2,i_3,j_1,j_2,j_3 \in \mathbb{Z}_n$;

(ii) For any $x \in \{e,a,b\}$, $x_{i,j} \prec x_{i',j'}$ if and only if $i < i'$ or $i = i'$ but $j < j'$.

Then the length-left-lexicographic order induced by "$\prec$" provides an admissible order for all paths in $\Bbbk Q$, and $R = \{a_{i,j}a_{i,j+1}, b_{i,j}b_{i+1,j}, a_{i,j}b_{i,j+1} + q_{i,j}b_{i,j}a_{i+1,j} \mid i,j \in \mathbb{Z}_n\}$ forms a non-commutative quadratic reduced Gröber basis of I_q. Hence $\Lambda^n_{\mathfrak{q}}$ are Koszul algebras [7].

In this section, we will construct a minimal projective bimodule resolution for $\Lambda^n_{\mathfrak{q}}$. We first need some notations. Let

$F^0 :=\{f^0_{0,i,j} = e_{i,j} \mid i,j \in \mathbb{Z}_n\}$;

$F^1 :=\{f^1_{0,i,j} = a_{i,j},\ f^1_{1,i,j} = b_{i,j} \mid i,j \in \mathbb{Z}_n\}$;

$F^2 :=\{f^2_{0,i,j} = a_{i,j}a_{i,j+1},\ f^2_{1,i,j} = a_{i,j}b_{i,j+1} + q_{i,j}b_{i,j}a_{i+1,j},\ f^2_{2,i,j} = b_{i,j}b_{i+1,j} \mid i,j \in \mathbb{Z}_n\}$.

For $l \geq 3$, we define the set $F^l := \{f^l_{k,i,j} \mid 0 \leq k \leq l, i,j \in \mathbb{Z}_n\}$ by induction on l as follows:

$$f^l_{k,i,j} = a_{i,j}f^{l-1}_{p,i,j+1} + q_{i,j}q_{i,j+1}\cdots q_{i,j+l-k-1}b_{i,j}f^{l-1}_{p-1,i+1,j},$$

where $f^l_{k,i,j} = 0$ for all $k < 0$ or $k > l$.

For any path p in $\Bbbk Q$, we denote by $\mathfrak{o}(p)$ and $\mathfrak{t}(p)$ the originals and terminus of p, respectively. Recall that a non-zero element $x = \sum_{i=1}^{s} k_i p_i \in \Bbbk Q$ with $k_i \in \Bbbk$ is said to be uniform if there exist vertices u and v in Q such that $\mathfrak{o}(p_i) = u$ and $\mathfrak{t}(p_i) = v$ for all paths p_i. It is easy to see that the elements $f^l_{k,i,j}$ are uniform. For all $f \in F^l$, we denote by $\mathfrak{o}(f)$ and $\mathfrak{t}(f)$ the common originals and terminus of all paths occurring in f.

For convenience, we denote the tensor product over $\Bbbk$ by $\otimes$. Define

$$P_l := \bigoplus_{f \in F^l} \Lambda^n_{\mathfrak{q}}\mathfrak{o}(f) \otimes \mathfrak{t}(f)\Lambda^n_{\mathfrak{q}}$$

for all $l \geq 0$, and differentials

$$\begin{aligned} d_l\big(\mathfrak{o}(f^l_{k,i,j}) \otimes \mathfrak{t}(f^l_{k,i,j})\big) =& a_{i,j} \otimes \mathfrak{t}(f^{l-1}_{k,i,j+1}) + (-1)^l \mathfrak{o}(f^{l-1}_{k-1,i,j}) \otimes b_{i+k-1,j+l-k} + \\ & q_{i,j}q_{i,j+1}\cdots q_{i,j+l-k-1}b_{i,j} \otimes \mathfrak{t}(f^{l-1}_{k-1,i+1,j}) \\ & (-1)^l q_{i,j+l-k-1}\cdots q_{i+k-1,j+l-k-1}\mathfrak{o}(f^{l-1}_{k,i,j}) \otimes a_{i+k,j+l-k-1} \end{aligned}$$

for all $l \geq 1$. Then, one can check that $\mathbb{P} := (P_l, d_l)$ is a complex. Moreover, following Theorem 2.1 in [8], we have the following result:

Proposition 1 *The complex*

$$\mathbb{P}: \quad \cdots \longrightarrow P_l \xrightarrow{d_l} P_{l-1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda^n_{\mathfrak{q}} \longrightarrow 0$$

is a minimal projective bimodule resolution of $\Lambda^n_{\mathfrak{q}}$, where π is the multiplication map.

3. Hochschild cohomology groups

In this section, we explicitly calculate the $\Bbbk$-dimensions of all the Hochschild cohomology groups of algebra $\Lambda_{\mathfrak{q}}^n$. Not that if $n=1$, $\Lambda_{\mathfrak{q}}^1$ is just the quantum exterior algebra; if $n=2$, the Hochschild homology and cohomology of $\Lambda_{\mathfrak{q}}^2$ have been considered in [11]. Henceforth, we assume $n \geq 3$.

We first give a description of the Hochschild cochain complex obtained from minimal projective bimodule resolution in Proposition 2.1 by the parallel paths and use this resolution to calculate the Hochschild cohomology groups. Using the parallel path to calculate the Hochschild cohomology group for a given finite-dimensional algebra can be traced back to Cibils's work [4]. Recall that a pair of uniform elements (x,y) in $\Bbbk Q$ is called parallel if $\mathfrak{o}(x)=\mathfrak{o}(y)$ and $\mathfrak{t}(x)=\mathfrak{t}(y)$, and denoted by $x//y$. Let X and Y be two sets of uniform elements in $\Bbbk Q$. We define

$$X//Y := \{(x,y) \in X \times Y \mid x//y\},$$

and denote by $\Bbbk(X//Y)$ the $\Bbbk$-vector space spanned by $X//Y$. Consider the $\Bbbk$-vector spaces $V^l := \Bbbk(\mathcal{B}//F^l)$ for all $l \geq 0$. For any $f^l_{k,i',j'}$, we have

- $e_{i,j}//f^l_{k,i',j'}$ if and only if $(i,j)=(i',j')$ and $l=l'n$, $k=k'n$ for some integers l',k' with $0 \leq k' \leq l'$;
- $a_{i,j}//f^l_{k,i',j'}$ if and only if $(i,j)=(i',j')$ and $l=l'n+1$, $k=k'n$ for some integers l',k' with $0 \leq k' \leq l'$;
- $b_{i,j}//f^l_{k,i',j'}$ if and only if $(i,j)=(i',j')$ and $l=l'n+1$, $k=k'n+1$ for some integers l',k' with $0 \leq k' \leq l'$;
- $a_{i,j}b_{i,j+1}//f^l_{k,i',j'}$ if and only if $(i,j)=(i',j')$ and $l=l'n+2$, $k=k'n+1$ for some integers l',k' with $0 \leq k' \leq l'$.

That is,

$$\dim_{\Bbbk} V^l = \begin{cases} (l'+1)n^2, & \text{if } l=l'n \text{ for some integer } l'; \\ 2(l'+1)n^2, & \text{if } l=l'n+1 \text{ for some integer } l'; \\ (l'+1)n^2, & \text{if } l=l'n+2 \text{ for some integer } l'; \\ 0, & \text{otherwise.} \end{cases} \tag{*}$$

Denote by $\Gamma := (\Lambda_{\mathfrak{q}}^n)^{op} \otimes \Lambda_{\mathfrak{q}}^n$ the enveloping algebra of $\Lambda_{\mathfrak{q}}^n$. Applying the functor $\mathrm{Hom}_\Gamma(-,\Lambda_{\mathfrak{q}}^n)$ to the deleted complex of the minimal projective bimodule resolution $\mathbb{P}$, we obtain the Hochschild cochain complex $\mathbf{C}^*(\mathbb{P})$:

$$0 \to \mathrm{Hom}_\Gamma(P_0,\Lambda_{\mathfrak{q}}^n) \xrightarrow{\delta^1} \mathrm{Hom}_\Gamma(P_1,\Lambda_{\mathfrak{q}}^n) \to \cdots \to \mathrm{Hom}_\Gamma(P_{l-1},\Lambda_{\mathfrak{q}}^n) \xrightarrow{\delta^l} \mathrm{Hom}_\Gamma(P_l,\Lambda_{\mathfrak{q}}^n) \to \cdots$$

where $\delta^l := \mathrm{Hom}_\Gamma(d_l,\Lambda_{\mathfrak{q}}^n)$, for all $l \geq 1$. Note that there are isomorphisms

$$\mathrm{Hom}_\Gamma(P_l,\Lambda_{\mathfrak{q}}^n) \cong \bigoplus_{f\in F^l} \mathrm{Hom}_\Gamma\big(\Lambda_{\mathfrak{q}}^n\mathfrak{o}(f) \otimes \mathfrak{t}(f)\Lambda_{\mathfrak{q}}^n\big) \cong \bigoplus_{f\in F^l} \mathfrak{o}(f)\Lambda_{\mathfrak{q}}^n\mathfrak{t}(f) \cong V^l.$$

for all $l \geq 0$. The corresponding isomorphism $\varphi_l : V^l \to \mathrm{Hom}_\Gamma(P_l, \Lambda^n_{\mathrm{q}})$ is given by $(x, f) \mapsto \xi_{(x,f)}$, where $\xi_{(x,f)}(\mathfrak{o}(f') \otimes \mathfrak{t}(f'))$ is x if $f = f'$ and is 0 otherwise. For each $l \geq 1$, we define a map $\eta^l : V^{l-1} \to V^l$ by setting

$$\begin{aligned}\eta^l(x,\ f^{l-1}_{k,i,j}) =&(a_{i,j-1}x,\ f^l_{k,i,j-1}) + (-1)^l q_{i,j+l-k-1} \cdots q_{i+k-1,j+l-k-1}(xa_{i+k,j+l-k-1},\ f^l_{k,i,j})+\\ &q_{i-1,j} \cdots q_{i-1,j+l-k-2}(b_{i-1,j}x,\ f^l_{k+1,i-1,j}) + (-1)^l(xb_{i+k,j+l-k-1},\ f^l_{k+1,i,j}).\end{aligned}$$

Then we get a cochain complex $\mathbb{V} := (V^l, \eta^l)$:

$$0 \longrightarrow V^0 \xrightarrow{\eta^1} V^1 \xrightarrow{\eta^2} V^2 \longrightarrow \cdots \longrightarrow V^{l-1} \xrightarrow{\eta^l} V^l \longrightarrow \cdots.$$

More importantly, we have the following result.

Lemma 2 *The Hochschild cochain complex* $\mathbf{C}^*(\mathbb{P})$ *is isomorphism to the complex* $\mathbb{V}$.

Therefore, we can calculate the Hochschild cohomology group of the algebra Λ^n_{q} using the cochain complex $\mathbb{V} := (V^l, \eta^l)$. By the definition, $HH^l(\Lambda^n_{\mathrm{q}}) = \mathrm{Ker}\,\eta^{l+1}/\mathrm{Im}\,\eta^l$, and so that

$$\begin{aligned}\dim_{\Bbbk} HH^l(\Lambda^n_{\mathrm{q}}) &= \dim_{\Bbbk}\mathrm{Ker}\,\eta^{l+1} - \dim_{\Bbbk}\mathrm{Im}\,\eta^l\\ &= \dim_{\Bbbk}V^l - \dim_{\Bbbk}\mathrm{Im}\,\eta^{l+1} - \dim_{\Bbbk}\mathrm{Im}\,\eta^l.\end{aligned}$$

Thus, to calculate the $\Bbbk$-dimensions of Hochschild cohomology groups, we only need to determine $\mathrm{Im}\,\eta^l$ for all $l \geq 0$.

Note that $\mathcal{B} := \{e_{i,j},\ a_{i,j},\ b_{i,j},\ a_{i,j}b_{i,j+1} \mid i,j \in \mathbb{Z}_n\}$ is an ordered $\Bbbk$-basis of Λ^n_{q} under the length-left-lexicographic order "$\prec$". Now we define an order on the set $\mathcal{B}//F^l$ for each l as follows:

$$(x, f^l_{k,i,j}) \prec (x', f^l_{k',i',j'}) \quad \text{if and only if } k < k',\ \text{or } k = k' \text{ but } x \prec x'.$$

Then the $\Bbbk$-vector space V^l has an ordered $\Bbbk$-basis. We denote by η^l the matrix of the differential η^l under this ordered basis. Let $\mu_i := \Pi^{n-1}_{j=0} q_{ij}$, $\nu_j := \Pi^{n-1}_{i=0} q_{ij}$, I_n be the $n \times n$ identity matrix. By the description of η^l, it is not difficult to see that $\eta^l = 0$ if $l \neq l'n + 1$ or $l'n + 2$ for any integer $l' \geq 0$, and

- if $l = l'n + 1$ for some integer l', then $\eta^l = \mathrm{diag}\{(A_0|B_0), (A_1|B_1), \ldots, (A_{l'}|B_{l'})\}$, where $A_i := \mathrm{diag}\{A_{i,0}, A_{i,1}, \ldots, A_{i,n-1}\}$ with $n \times n$ matrices

$$A_{ij} = \begin{pmatrix} (-1)^l\nu^i_0 & & & 1\\ 1 & (-1)^l\nu^i_1 & & \\ & \ddots & \ddots & \\ & & 1 & (-1)^l\nu^i_{n-1} \end{pmatrix},$$

and

$$B_i = \begin{pmatrix} (-1)^l I_n & & & \mu^{l'-i}_0 I_n\\ \mu^{l'-i}_1 I_n & (-1)^l I_n & & \\ & \ddots & \ddots & \\ & & \mu^{l'-i}_{n-1} I_n & (-1)^l I_n \end{pmatrix}_{n^2\times n^2};$$

- if $l = l'n + 2$ for some integer l', then $\eta^l = \mathrm{diag}\left\{\begin{pmatrix} C_0\\ \hline D_0\end{pmatrix}, \begin{pmatrix} C_1\\ \hline D_1\end{pmatrix}, \ldots, \begin{pmatrix} C_{n-1}\\ \hline D_{n-1}\end{pmatrix}\right\}$, where

$D_i := \mathrm{diag}\{D_{i,0}, D_{i,1}, \ldots, D_{i,n-1}\}$ with $n\times n$ matrices

$$D_{ij} = \begin{pmatrix} (-1)^{l+1}\nu_0^i & & & 1 \\ 1 & (-1)^{l+1}\nu_1^i & & \\ & \ddots & \ddots & \\ & & 1 & (-1)^{l+1}\nu_{n-1}^i \end{pmatrix},$$

and

$$C_i = \begin{pmatrix} (-1)^l I_n & & & -\mu_0^{l'-i} I_n \\ -\mu_1^{l'-i} I_n & (-1)^l I_n & & \\ & \ddots & \ddots & \\ & & -\mu_{n-1}^{l'-i} I_n & (-1)^l I_n \end{pmatrix}_{n^2\times n^2}.$$

Let $\mathfrak{q} := \Pi_{i,j=0}^{n-1} q_{ij}$. Then we can get the $\mathbb{k}$-dimension of the image of map η^l for all $l \geq 0$, by the following two lemmas.

Lemma 3 *Let $n \geq 3$ and $\mathfrak{q}$ an r-th primitive root of unity. If $l = l'n+1$ or $l'n+2$ for some integer l', then*

$$\mathrm{rank}\,\eta^l = \begin{cases} n^2-1, & \text{if } l' = 0; \\ (l'+1)n^2 - k' - 1, & \text{if } n \text{ is even or } l' \text{ is even, and } l' = k'r; \\ (l'+1)n^2, & \text{otherwise.} \end{cases}$$

Proof We first let $l = l'n+1$ for some integer l' and consider the matrix η^l. If $l' = 0$, i.e., $l = 1$, it is easy to see that $\mathrm{rank}\,\eta^l = n-1$. For the case $l' > 0$, by elementary transformations, we can change the matrix $(A_i|B_i)$ into

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & (-1)^l I_n & & 0 & 0 \\ \vdots & & & \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & & (-1)^l I_n & 0 \\ 0 & \cdots & 0 & A_{i,n-1} & 0 & \cdots & 0 & F_i \end{pmatrix}$$

for all $0 \leq i \leq l'$, where $F_i = (-1)^l I_n + (-1)^{(l-1)(n-1)}\mathfrak{q}^{l'-i} I_n$. Thus $\mathrm{rank}(A_i|B_i) = (n-1)n + \mathrm{rank}(A_{i,n-1}|F_i)$. Note that $\mathrm{rank}(A_{i,n-1}|F_i) = n$ or $n-1$, and $\mathrm{rank}(A_{i,n-1}|F_i) = n-1$ if and only if

$$\mathrm{rank}\,A_{i,n-1} = n-1 \text{ and } F_i = 0 \iff \begin{cases} (-1)^{l'n^2}\mathfrak{q}^{l'-i} = 1, \\ (-1)^{l'n^2}\mathfrak{q}^{i} = 1. \end{cases}$$

That is, $\mathrm{rank}(A_i|B_i) = n^2-1$ if and only if n is even or l' is even and $r|i$, $r|l'$. Therefore, if n is even or l' is even and $l' = k'r$, then the number i such that $\mathrm{rank}(A_i|B_i) = n^2-1$ is k', and so that $\mathrm{rank}\,\eta^l = (l'+1)n^2 = k'$ in this case. Thus we get the first part.

Next, we let $l = l'n+2$ for some integer l' and consider the matrix η^l. It is easy to see that $\mathrm{rank}\,\eta^2 = n-1$. For the case $l' > 0$, by elementary transformations, we can change the matrix

$(\frac{C_i}{D_i})$ into

$$\begin{pmatrix} (-1)^l I_n & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & (-1)^l I_n & 0 \\ 0 & \cdots & 0 & E_i \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & 0 \\ 0 & \cdots & 0 & D_{i,n-1} \end{pmatrix}$$

for all $0 \le i \le l'$, where $E_i = (-1)^l I_n - (-1)^{l(n-1)} \mathbb{q}^{l'-i} I_n$. Thus $\operatorname{rank}(\frac{C_0}{D_0}) = (n-1)n + \operatorname{rank}(\frac{D_{i,n-1}}{E_i})$. Similarly to the $l = l'n+1$ case, we have $\operatorname{rank}(\frac{C_i}{D_i}) = n-1$ if and only if $\operatorname{rank} D_{i,n-1} = n-1$ and $E_i = 0$, if and only if n is even or l' is even and $r|i$, $r|l'$. Hence we get the second part and complete the proof of this lemma. □

Lemma 4 *Assume $n \ge 3$, $\mathbb{q}$ is not a root of unity, and $l = l'n+1$ or $l'n+2$ for some integer l'. Then*

$$\operatorname{rank} \eta^l = \begin{cases} n^2 - 1, & \text{if } l' = 0; \\ (l'+1)n^2, & \text{otherwise.} \end{cases}$$

Proof We can get $\operatorname{rank} \eta^1$ and $\operatorname{rank} \eta^2$ by direct calculation. If $l' > 0$, similarly to the proof of Lemma 3, we can change the matrix η^l into some simple matrices and obtain this lemma. □

Now, we can obtain all the $\mathbb{k}$-dimensions of the Hochschild cohomology groups of $\Lambda_{\mathbb{q}}^n$.

Theorem 5 *Let $n \ge 3$. We have*

$$\dim_{\mathbb{k}} HH^0(\Lambda_{\mathbb{q}}^n) = \dim_{\mathbb{k}} HH^2(\Lambda_{\mathbb{q}}^n) = 1 \ \text{ and } \ \dim_{\mathbb{k}} HH^1(\Lambda_{\mathbb{q}}^n) = 2.$$

For the higher degree Hochschild cohomology groups, i.e., $l \ge 3$, we have

(1) if $\mathbb{q}$ is an r-th primitive root of unity, then

$$\dim_{\mathbb{k}} HH^l(\Lambda_{\mathbb{q}}^n) = \begin{cases} 2k'+2, & \text{if } l = l'n+1, n \text{ is even or } l' \text{ is even, and } l' = k'r; \\ k'+1, & \text{if } l = l'n \text{ or } l = l'n+2, n \text{ is even or } l' \text{ is even, and } l' = k'r; \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $\mathbb{q}$ is not a root of unity, then $\dim_{\mathbb{k}} HH^l(\Lambda_{\mathbb{q}}^n) = 0$ for all $l \ge 3$, and so that the Hochschild cohomology ring $HH^*(\Lambda_{\mathbb{q}}^n)$ is a 4-dimension $\mathbb{k}$-algebra.

Proof By Lemmas 3 and 4, formula $(\star)$ and formula:

$$\dim_k HH^l(\Lambda_{\mathbb{q}}^n) = \dim_k V^l - \dim_k \operatorname{Im} \eta^{l+1} - \dim_k \operatorname{Im} \delta^l,$$

we obtain this theorem immediately. □

As a consequence of Theorem 5, we get the Hilbert series of the algebra $\Lambda_{\mathbb{q}}^n$ for some special cases by the simple calculation.

Corollary 6 *Let $n \geq 3$. The Hilbert series of the algebra $\Lambda_{\mathbb{q}}^n$ is*

$$\sum_{l=0}^{\infty} \dim_{\mathbb{k}} HH^l(\Lambda_{\mathbb{q}}^n)t^l = \begin{cases} \frac{1+2t+t^2}{(1-t^n)^2}, & \text{if } \mathbb{q}=1 \text{ and } n \text{ is even;} \\ \frac{2+4t+2t^2}{(1-t^n)^2} - \frac{1+2t+t^2}{1-t^n}, & \text{if } \mathbb{q}=1 \text{ and } n \text{ is odd;} \\ 1+2t+t^2, & \text{if } \mathbb{q} \text{ is not a root of unity.} \end{cases}$$

4. Hochschild cohomology ring

In this section, we determine the structure of Hochschild cohomology ring of $\Lambda_{\mathbb{q}}^n$ when $\mathbb{q} = \prod_{i,j=0}^{n-1} q_{i,j}$ is not a root of unity, and thus provide another family of counterexamples to Happel's conjecture.

For any finite-dimensional $\mathbb{k}$-algebra Λ, it is well-known that the Hochschild cohomology ring

$$HH^*(\Lambda) := \bigoplus_{l \geq 0} HH^l(\Lambda) = \bigoplus_{l \geq 0} \mathrm{Ext}_{\Lambda^e}^l(\Lambda, \Lambda)$$

is a graded commutative algebra under the Yoneda product, and that it is closely related to the homological property and the representation theory of Λ (see [1, 3, 6]). But in general, $HH^*(\Lambda)$ is not finitely generated and its Yoneda product $\sqcup : HH^l(\Lambda) \times HH^k(\Lambda) \to HH^{l+k}(\Lambda)$ is difficult to be given explicitly.

It follows from Theorem 5 that the Hochschild cohomology ring $HH^*(\Lambda_{\mathbb{q}}^n)$ is a 4-dimension $\mathbb{k}$-algebra. Describing the elements in $HH^l(\Lambda_{\mathbb{q}}^n)$ by the parallel paths, we can give the multiplication structure of $HH^*(\Lambda_{\mathbb{q}}^n)$ and show that $HH^*(\Lambda_{\mathbb{q}}^n)$ is just the exterior algebra in two variables.

Proposition 7 *If $\mathbb{q}$ is not a root of unity, then $HH^*(\Lambda_{\mathbb{q}}^n) \cong \wedge(u, v)$, the exterior algebra in two variables.*

Proof First for convenience, we identify the parallel path in V^l with its image in $HH^l(\Lambda_{\mathbb{q}}^n)$ for all $l \geq 0$. By the description of the parallel path in V^l, it is not difficult to see:

$$\begin{aligned} HH^0(\Lambda_{\mathbb{q}}^n) =& \mathbb{k}\{\sum_{i,j=0}^{n-1}(e_{ij}, f_{0,i,j}^0)\} \cong \mathbb{k}; \\ HH^1(\Lambda_{\mathbb{q}}^n) =& \mathbb{k}\{\sum_{i,j=0}^{n-1}(a_{i,j}, f_{0,i,j}^1), \sum_{i,j=0}^{n-1}(b_{i,j}, f_{1,i,j}^1)\}; \\ HH^2(\Lambda_{\mathbb{q}}^n) =& \mathbb{k}\{\sum_{i,j=0}^{n-1}(a_{i,j}b_{i,j+1}, f_{1,i,j}^2)\} \cong \mathbb{k}. \end{aligned}$$

Under the isomorphism $\varphi_l : V^l \to \mathrm{Hom}_\Gamma(P_l, \Lambda_{\mathbb{q}}^n)$ defined by $(x, f) \mapsto \xi_{(x,f)}$, where $\xi_{(x,f)}(\mathfrak{o}(f') \otimes \mathfrak{t}(f'))$ is x if $f = f'$ and is 0 otherwise, we have $\xi_a := \sum_{i,j=0}^{n-1} \xi_{(a_{i,j}, f_{0,i,j}^1)}$ and $\xi_b := \sum_{i,j=0}^{n-1} \xi_{(b_{i,j}, f_{1,i,j}^1)}$ also form a $\mathbb{k}$-basis of $HH^1(\Lambda_{\mathbb{q}}^n)$. Denote $\xi_{ab}^2 = \sum_{i,j=0}^{n-1} \xi_{(a_{i,j}b_{i,j+1}, f_{1,i,j}^2)}$, then $HH^2(\Lambda_{\mathbb{q}}^n) = \mathbb{k}\{\xi_{ab}^2\}$.

Next, we consider the Yoneda product $\sqcup : HH^l(\Lambda_{\mathbb{q}}^n) \times HH^k(\Lambda_{\mathbb{q}}^n) \to HH^{l+k}(\Lambda_{\mathbb{q}}^n)$ by the projective resolution $\mathbb{P} = (P_l, d_l)$ constructed in Section 2. For all $i \geq 0$, we define maps

$\phi_i : P_{i+1} \to P_i$ by

$$\phi_0 : \quad P_1 \to P_0, \quad \begin{cases} \mathfrak{o}(f^1_{0,i,j}) \otimes \mathfrak{t}(f^1_{0,i,j}) \mapsto & 0, \\ \mathfrak{o}(f^1_{1,i,j}) \otimes \mathfrak{t}(f^1_{1,i,j}) \mapsto & b_{i,j} \otimes \mathfrak{t}(f^1_{1,i,j}); \end{cases}$$

$$\phi_1 : \quad P_2 \to P_1, \quad \begin{cases} \mathfrak{o}(f^2_{0,i,j}) \otimes \mathfrak{t}(f^2_{0,i,j}) \mapsto & 0, \\ \mathfrak{o}(f^2_{1,i,j}) \otimes \mathfrak{t}(f^2_{1,i,j}) \mapsto & -q_{i,j} b_{i,j} \otimes \mathfrak{t}(f^1_{0,i+1,j}), \\ \mathfrak{o}(f^2_{2,i,j}) \otimes \mathfrak{t}(f^2_{2,i,j}) \mapsto & -b_{i,j} \otimes \mathfrak{t}(f^1_{1,i+1,j}); \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

It is easy to check that the following diagram with $\Lambda^n_{\mathbb{q}}$-bimodule homomorphisms is commutative:

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\phi_1"] & P_1 \arrow[r] \arrow[d, dashed, "\phi_0"] \arrow[dr, "\xi_b"] & \cdots & \\ \cdots \arrow[r] & P_1 \arrow[r, "d_1"] \arrow[d, "\xi_a"] & P_0 \arrow[r, "\pi"] & \Lambda^n_{\mathbb{q}} \arrow[r] & 0. \\ & \Lambda^n_{\mathbb{q}} & & & \end{tikzcd}$$

Thus the composition $\xi_a \circ \phi_1 : P_2 \to \Lambda^n_q$ is just the Yoneda product $\xi_a \sqcup \xi_b$ in $HH^2(\Lambda^n_{\mathbb{q}})$, and one can check that $\xi_a \sqcup \xi_b = \xi_{ab}$. On the other hand, by the graded commutativity of Ext algebra $HH^*(\Lambda^n_{\mathbb{q}})$, we have $\xi_a \sqcup \xi_b = -\xi_b \sqcup \xi_a$ and $\xi_a \sqcup \xi_a = 0 = \xi_b \sqcup \xi_b$ when char$\mathbb{k} \neq 2$, and these still hold by a direct calculation when char$\mathbb{k} = 2$.

Finally, define map $\Phi : HH^*(\Lambda^n_{\mathbb{q}}) \to \wedge(u, v)$ by $\xi_a \mapsto u$ and $\xi_b \mapsto v$. It is easy to see that Φ is an algebra isomorphism. $\Box$

Remark (1) It follows from this proposition that the algebras $\Lambda^n_{\mathbb{q}}$ provide a family of counterexamples to Happel's question when $\mathbb{q}$ is not a root of unity.

(2) In fact, $\Lambda^n_{\mathbb{q}}$ is a class of algebras with some strange homological properties. For example, in [17], the authors considered the one-point coextension algebra $\widetilde{\Lambda}^n_{\mathbb{q}}$ of $\Lambda^n_{\mathbb{q}}$ and provided a class of counterexamples to Snashall-Solberg's conjecture, i.e., the Hochschild cohomology ring of $\widetilde{\Lambda}^n_{\mathbb{q}}$ modulo nilpotence is infinitely generated.

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