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Congruences on Orthodox Semirings Whose Additive Idempotents Satisfy Permutation Identities

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Abstract The investigation of congruences on generalized inverse semigroups is initiated. Following some properties of such semigroups, the congruences on an orthodox semiring whose idempotents satisfy permutation identities are established. In addition, we give a structure theorem of homomorphic image of this kind of orthodox semirings.

Keywords inverse semiring; congruence; band semiring.

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1. Introduction and preliminaries

An algebraic structure $(S, +, \cdot)$ is called a semiring if $(S, +)$ and $(S, \cdot)$ are semigroups, and for each $a, b, c \in S$, $a \cdot (b + c) = a \cdot b + a \cdot c$, $(a + b) \cdot c = a \cdot c + b \cdot c$. Usually, we write $(S, +, \cdot)$ simply as S, and for any $a, b \in S$, we write $a \cdot b$ simply as ab.

Let S be a semiring. An element a of S is said to be an idempotent if it satisfies $a + a = a \cdot a = a$. If each element of S is idempotent, S is said to be an idempotent semiring. An idempotent semiring S is said to be a band semiring [3], if it satisfies the following conditions:

$$a + ab + a = a, a + ba + a = a$$

for any $a, b \in S$. In [7], the authors proved that band semirings are always regular band semirings.

Green's $\mathcal{L} - [\mathcal{R}-]$ relation on the additive reduct $(S, +)$ will be denoted by $\overset{+}{\mathcal{L}}$ $[\overset{+}{\mathcal{R}}]$. Also, we denote by $E^+(S)$ the set of all additive idempotents (if there exist) of a semiring S. Clearly, $E^+(S)$ is an ideal of the multiplicative reduct $(S, \cdot)$.

Let D be a distributive lattice. For each $\alpha \in D$, let S_α be a semiring and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in D$ such that $\alpha \leq \beta$, let $\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ be a semiring homomorphism such that

(1) $\varphi_{\alpha,\alpha} = 1_{S_\alpha}$;

(2) $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$, if $\alpha \leq \beta \leq \gamma$;

(3) $\varphi_{\alpha,\beta}$ is an injective, if $\alpha \leq \beta$;

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(4) $S_\alpha\varphi_{\alpha,\gamma}S_\beta\varphi_{\beta,\gamma}\subseteq S_{\alpha\beta}\varphi_{\alpha\beta,\gamma}$, if $\alpha+\beta\leq\gamma$.

On $S=\cup_{\alpha\in D}S_\alpha$, $+$ and $\cdot$ are defined as follows: For $a\in S_\alpha$ and $b\in S_\beta$,

(5) $a+b=a\varphi_{\alpha,\alpha+\beta}+b\varphi_{\beta,\alpha+\beta}$;

(6) $ab=(a\varphi_{\alpha,\alpha+\beta}b\varphi_{\beta,\alpha+\beta})\varphi^{-1}_{\alpha\beta,\alpha+\beta}$.

With the above operations, S is a semiring, and each S_α is a subsemiring of S. Write S as $[D;S_\alpha,\varphi_{\alpha,\beta}]$, and call it a strong distributive lattice D of semirings S_α (see [2]).

A semiring is said to be additively regular if for each $a\in S$, there exists $a^{-1}\in S$ such that $a=a+a^{-1}+a$. For each element $a\in S$, let

$$V^+(a)=\{x\in S:a+x+a=a,x+a+x=x\}$$

be the set of inverse of a in S. Let S be a semiring, A a subset of S. Then A is said to satisfy the permutation identity if

$$(\forall x_1,x_2,\ldots,x_n\in S)\ x_1+x_2+\cdots+x_n=x_{p_1}+x_{p_2}+\cdots+x_{p_n},$$

where $(p_1p_2\cdots p_n)$ is a nontrivial permutation of $(12\cdots n)$. [9] has given the orthodox semiring whose additive idempotents satisfy permutation identities and discussed the structure of such semirings.

After Yamada [8] has given a complete classification of inverse semigroups satisfying the same condition (called generalized inverse semigroups), Clifford has given the characterization of congruences on generalized semigroups [1]. In this paper, we will show that every congruence on an orthodox semiring whose additive idempotents satisfy permutation identities is uniquely determined by a congruence on its associated left normal band semiring, d-inverse semiring, and right normal band semiring. The converse also holds. Throughout this paper we shall use the terminology and notations of [4].

We first recall some results about band semirings.

Theorem 1.1 ([6]) *A semiring S is a rectangular band semiring if and only if S is isomorphic to the direct product of a left zero band semiring and a left zero band semiring.*

Theorem 1.2 ([6]) *A semiring S is a normal band semiring if and only if S is a strong distributive lattice of rectangular band semirings.*

Let T be a d-inverse semiring whose distributive lattice of additive idempotents is D, $L=[D;L_\alpha,\varphi_{\alpha,\beta}]$ a strong distributive lattice of left zero band semirings L_α and $R=[D;R_\alpha,\psi_{\alpha,\beta}]$ a strong distributive lattice of right zero band semirings R_α. Let

$$M=\{(e,a,f)\in L\times T\times R;e\in L_{a+a^{-1},a\in T,f\in R_{a^{-1}+a}}\}.$$

Define addition "+" and multiplication "·" as follows:

$$(e,a,f)+(u,b,v)=(e+g,a+b,h+v),$$
$$(e,a,f)\cdot(u,b,v)=(eu,ab,fv),$$

where $g\in L_{a+b+(a+b)^{-1}}$, $h\in R_{(a+b)^{-1}+a+b}$. We denote M by $QS(L,T,R;D)$. In [9], the author proved that $(M,+,\cdot)$ is an orthodox semiring.

Easily, we can prove the following lemma.

Lemma 1.3 *Let* $(e,a,f),(u,b,v)\in S=QS(L,T,R;D)$. *Then*

(1) $(e,a.f)\in E^+(S)\Longleftrightarrow a\in E^+(T)$,

(2) $(e,a,f)\overset{+}{\mathcal{L}}(u,b,v)\Longleftrightarrow a\overset{+}{\mathcal{L}}b, f=v$,

(3) $(e,a,f)\overset{+}{\mathcal{R}}(u,b,v)\Longleftrightarrow a\overset{+}{\mathcal{R}}b, e=u$.

2. The characterizations of congruence

Suppose that $S=QS(L,T,R;D)$, let ρ be a congruence on S. Define the relation as follows:

$$a\xi b\Longleftrightarrow(\exists(e,a,f),(u,b,v)\in S)(e,a,f)\rho(u,b,v).$$

Denote $\rho_T=\xi^\infty$, where $e\in L_\alpha, f\in R_\alpha, u\in L_\beta, v\in R_\beta$.

$$e\rho_L u\Longleftrightarrow(\exists s\in R_\alpha, t\in R_\beta)\ ((e,\alpha,f)+(u,\beta,t))\rho(u,\beta,t)\ ((u,\beta,t)+(e,\alpha,s))\rho(e,\alpha,s),$$

$$f\rho_R v\Longleftrightarrow(\exists p\in L_\alpha, q\in L_\beta)\ ((p,\alpha,f)+(q,\beta,v))\rho(p,\alpha,f)\ ((q,\beta,v)+(p,\alpha,f))\rho(q,\beta,v).$$

Proposition 2.1 *ρ_L is a congruence on L and ρ_R is a congruence on R.*

Proof It is easily seen that ρ_L is reflexive and symmetric. Suppose $e_1,e_2,e_3\in L$ such that $e_1\rho_L e_2, e_2\rho_L e_3$. Then there exist $\alpha,\beta,\gamma\in D$ such that $e_1\in L_\alpha, e_2\in L_\beta, e_3\in L_\gamma, s_1\in R_\alpha, s_2,s_4\in R_\beta, s_3\in R_\gamma$ satisfying:

$$((e_1,\alpha,s_1)+(e_2,\beta,s_2))\rho(e_2,\beta,s_2),\quad((e_2,\beta,s_2)+(e_1,\alpha,s_1))\rho(e_1,\alpha,s_1),$$

and

$$((e_2,\beta,s_2)+(e_3,\gamma,s_3))\rho(e_3,\gamma,s_3),\quad((e_3,\gamma,s_3)+(e_2,\beta,s_2))\rho(e_2,\beta,s_2).$$

Therefore,

$$\begin{aligned}&((e_1,\alpha,s_1)+(e_3,\gamma,s_3))\rho((e_1,\alpha,s_1)+(e_2,\beta,s_2)+(e_3,\gamma,s_3))\\&\quad=((e_1,\alpha,s_1)+(e_2,\beta,s_4)+(e_3,\beta,s_3))\rho((e_2,\beta,s_4)+(e_3,\gamma,s_3))\\&\quad=((e_2,\beta,s_2)+(e_3,\gamma,s_3))\rho(e_3,\gamma,s_3).\end{aligned}$$

Similarly, we can prove $((e_3,\gamma,s_3)+(e_1,\alpha,s_1))\rho(e_1,\alpha,s_1)$.

Now, we can prove ρ_L is compatible with addition and multiplication. Suppose that $e_1\rho_L e_2$. Then there exist $s\in L_\gamma, t\in R_\gamma, \gamma\in D$ such that

$$\begin{aligned}&(s+e_1,\gamma+\alpha,t+s_1)+(s+e_2,\gamma+\beta,t+s_2)\\&\quad=((s,\gamma,t)+(e_1,\alpha,s_1)+(e_2,\beta,s_2))\rho((s,\gamma,t)+(e_2,\beta,s_2))\\&\quad=(s+e_2,\gamma+\beta,t+s_2).\end{aligned}$$

Similarly, $(e_1+s,e_2+s)\in\rho_T$.

$$\begin{aligned}(se_1,\gamma\alpha,ts_1)+(se_2,\gamma\beta,ts_2)&=(s,\gamma,t)((e_1,\alpha,s_1)+(e_2,\beta,s_2))\rho(s,\gamma,t)(e_2,\beta,s_2)\\&=(se_2,\gamma\beta,ts_2).\end{aligned}$$

Also, $(e_1s, e_2s) \in \rho_L$. So ρ_L is a congruence on L. It is similar to prove ρ_R is a congruence on R. □

Proposition 2.2 *Let ρ be a congruence on S. Suppose $(e, a, f)\rho(u, b, v)$. Then*

(1) $(\forall s \in R_{a+a^{-1}}, t \in R_{b+b^{-1}})$ $((e, a+a^{-1}, s)+(u, b+b^{-1}, t))\rho(u, b+b^{-1}, t)$ $((u, b+b^{-1}, t)+(e, a+a^{-1}, s))\rho(e, a+a^{-1}, s)$,

(2) $(\forall g \in L_{a^{-1}+a}, h \in L_{b^{-1}+b})$ $((g, a^{-1}+a, f)+(h, b^{-1}+b, v))\rho(g, a^{-1}+a, f)$ $((h, b^{-1}+b, v)+(g, a^{-1}+a, f))\rho(h, b^{-1}+b, v)$.

Suppose S ia an orthodox semiring. Then we can define a relation σ on S as follows:

$$a\sigma b \text{ if and only if } V^+(a) = V^+(b).$$

In [9], the author proved that σ is the minimum d-inverse semiring congruence. Now, we give another characterization of the congruence σ.

Proposition 2.3 *Let S be an orthodox semiring, $E^+(S)$ be a normal band semiring. Then*

$$\sigma = \{(a, b) \in S \times S | a = a + a^{-1} + b + a^{-1} + a, b = b + b^{-1} + a + b^{-1} + b\}.$$

Proof If $x\sigma y$, then $V^+(x) = V^+(y)$. Suppose $x^{-1} \in V^+(x)(= V^+(y))$, then

$$x + x^{-1} + x = x, x^{-1} + x + x^{-1} = x^{-1}, y + x^{-1} + y = y, x^{-1} + y + x^{-1} = x^{-1}.$$

So

$$x = x + x^{-1} + y + (x^{-1} + x), y = y + x^{-1} + x + x^{-1} + y.$$

Conversely, if

$$x = x + x^{-1} + y + x^{-1} + x, y = y + y^{-1} + x + y^{-1} + y.$$

Then,

$$x^{-1} = x^{-1} + x + x^{-1} = x^{-1} + x + x^{-1} + y + x^{-1} + x + x^{-1} = x^{-1} + y + x^{-1}.$$

$$\begin{aligned} y + x^{-1} + y &= y + y^{-1} + x + y^{-1} + y + x^{-1} + y + y^{-1} + x + y^{-1} + y \\ &= y + y^{-1} + x + y^{-1} + y + x^{-1} + x + x^{-1} + x + y^{-1} + y \\ &= y + y^{-1} + x + x^{-1} + x + y^{-1} + y + x^{-1} + x + y^{-1} + y \\ &= y + y^{-1} + x + y^{-1} + y = y. \end{aligned}$$

Hence $x^{-1} \in V^+(y)$, and $V^+(y) = V^+(x)$ as required. □

Using Proposition 2.3, we can easily prove the following proposition.

Proposition 2.4 *Let ρ be a congruence on S. Then $(\sigma \vee \rho)/\rho = \sigma/\rho$.*

Proposition 2.5 *The mapping: $\phi : S \to T$, defined by $(e, a, f)\phi = a$ is a homomorphism and $\sigma = \ker \phi$.*

Proof It is clear that ϕ is a homomorphism. Since T is a d-inverse semiring, it follows from Proposition 2.3 that $\sigma \subseteq \ker \phi$. To show the reverse inclusion, suppose that $(e, a, f)\phi = (u, b, v)\phi$, then $a = b$. Therefore,

$$\begin{aligned}(e,a,f) &= (e, a+a^{-1}, s) + (u,b,v) + (t, a^{-1}+a, f)\\ &= (e,a,f) + (e,a^{-1},f) + (u,b,v) + (e,a^{-1},f) + (e,a,f)\\ &= (e,a,f) + (e,a,f)^{-1} + (u,b,v) + (e,a,f)^{-1} + (e,a,f),\\ (u,b,v) &= (u, b+b^{-1}, p) + (e,a,f) + (q, b^{-1}+b, v)\\ &= (u,b,v) + (u,b^{-1},v) + (e,a,f) + (u,b^{-1},v) + (u,b,v)\\ &= (u,b,v) + (u,b,v)^{-1} + (e,a,f) + (u,b,v)^{-1} + (u,b,v).\end{aligned}$$

Hence $(e,a,f)\sigma(u,b,v)$. So $\sigma = \ker\phi$. □

Proposition 2.6 *Let $(e,a,f),(u,b,v) \in S$, and ρ be a congruence on S. Then $(e,a,f)\sigma/\rho$ (u,b,v) if and only if $a\rho_T b$.*

Proof If $(e,a,f)\sigma/\rho(u,b,v)$, then it follows from Proposition 2.4 that

$$(e,a,f)\rho(e_1,a_1,f_1)\sigma(e_2,a_2,f_2)\rho\cdots\sigma(e_{2n},a_{2n},f_{2n})\rho(u,b,v)$$

for some $(e_1,a_1,f_1),\ldots,(e_{2n},a_{2n},f_{2n})$ in S. That is,

$$a\rho_T a_1, a_2\rho_T a_3, \ldots, a_{2n}\rho_T b,$$

and

$$a_1 = a_2, a_3 = a_4, \ldots, a_{2n-1} = a_{2n}.$$

Hence $a\rho_T b$.

Conversely, if $a\rho_T b$, there exist $a_0, a_1, \ldots, a_{2n} \in T$ such that $a = a_0\rho_T a_1\rho_T a_2\cdots\rho_T a_{2n}\rho_T a_{2n+1} = b$. So we have

$$(u_i,a_i,v_i)\rho(e_i,a_i,f_i)$$

for some $e_1,\ldots,e_{2n+1}; u_0,\ldots,u_{2n} \in L, f_1,\ldots,f_{2n+1}; v_0,\ldots,v_{2n} \in R$, where $i = 0,1,\ldots,2n$. From Lemma 2.5,

$$(e,a,f)\sigma(u_0,a_0,v_0)\rho(e_1,a_1,f_1)\sigma\cdots\rho(e_{2n+1},a_{2n+1},f_{2n+1})\sigma(u,b,v).$$

Thus, $(e,a,f)\rho\vee\sigma(u,b,v)$. So $(e,a,f)\sigma/\rho(u,b,v)$. □

Corollary 2.7 *ρ_T is a congruence on T.*

Theorem 2.8 *If ρ is a congruence on S, and $(e,a,f),(u,b,v) \in S$, then $(e,a,f)\rho(u,b,v)$ if and only if $e\rho_L u, a\rho_T b, f\rho_R v$.*

Proof If $(e,a,f)\rho(u,b,v)$, by Proposition 2.2 and definitions of ρ_L,ρ_T,ρ_R, it is easy to see that $e\rho_L u, a\rho_T b$ and $f\rho_R v$.

Conversely, if $(e,a,f),(u,b,v) \in S$ satisfy $e\rho_L u, a\rho_T b$ and $f\rho_R v$, then from Propositions 2.2 and 2.6, there exist $s \in R_{a+a^{-1}}, t \in L_{a^{-1}+a}$ such that

$$(x,a,y)\rho((x,a+a^{-1},s)+(p,b,q)+(t,a^{-1}+a,y)),$$

where $(x,a,y),(p,b,q)\in S$.

On the other hand, by the definitions of ρ_L and ρ_R, for each $g\in R_{a+a^{-1}}, l\in L_{a^{-1}+a}, h\in R_{b+b^{-1}}, k\in L_{b^{-1}+b}$, then

$$((e,a+a^{-1},g)+(u,b+b^{-1},h))\rho(u,b+b^{-1},h),((k,b^{-1}+b,v)+(l,a^{-1}+a,f))\rho(k,b^{-1}+b,v).$$

So

$$\begin{aligned}(e,a,f)=&((e,a+a^{-1},g)+(x,a,y)+(l,a^{-1}+a,f))\rho((e,a+a^{-1},g)+(x,a+a^{-1},s)\\&(p,b,q)+(t,a^{-1}+a,y)+(l,a^{-1}+a,f))\\=&(e,a+a^{-1},g)+(p,b+b^{-1},h)+(u,b,v)+(k,b^{-1}+b,q)+(l,a^{-1}+a,f)\\=&((e,a+a^{-1},g)+(u,b+b^{-1},h)+(u,b,v)+(k,b^{-1}+b,v)+(l,a^{-1}+a,f))\rho\\&((u,b+b^{-1},h)+(u,b,v)+(k,b^{-1}+b,v))\\=&(u,b,v).\ \Box\end{aligned}$$

Now we can give the definition of congruence triple on orthodox semirings whose additive idempotents satisfy permutation identities.

Definition 2.9 *Let $S=QS(L,T,R;D)$, ρ_T a congruence on T, ρ_L and ρ_R the congruences on L and R respectively satisfying $\rho_T|_D=\rho_{L_D}=\rho_{R_D}$, where $\rho_{L_D}=\{(\alpha,\beta)\in D\times D|(\exists e\in L_\alpha, u,v\in L_{\alpha+\beta}, f\in L_\beta)\ e\rho_L u$ and $v\rho_L f\}, \rho_{R_D}=\{(\alpha,\beta)\in D\times D|(\exists e\in R_\alpha,u,v\in R_{\alpha+\beta},f\in R_\beta)\ e\rho_R u$ and $v\rho_R f\}$. Then (ρ_L,ρ_T,ρ_R) is said to be a congruence triple on S. Define a relation $\rho_{(\rho_L,\rho_T,\rho_R)}$ as follows:*

$$(e,a,f)\rho_{(\rho_L,\rho_T,\rho_R)}(u,b,v)\Longleftrightarrow e\rho_L u, a\rho_T b, f\rho_R v.$$

Proposition 2.10 *Let ρ_L be a congruence on a left normal band semiring $L=[D;L_\alpha,\varphi_{\alpha,\beta}]$. If $\alpha,\beta,\gamma\in D,\gamma\le\alpha,\beta$, and $\alpha\rho_L\beta$, then for each $e\in L_\gamma,(e+\gamma+\alpha+(\gamma+\alpha)^{-1})\rho_L(e+\gamma+\beta+(\gamma+\beta^{-1}))$.*

Theorem 2.11 *Let $S=QS(L,T,R;D)$. Then there exist congruences ρ_L,ρ_T and ρ_R on L,T and R respectively such that $\rho_{(\rho_L,\rho_T,\rho_R)}$ is the unique congruence which induces ρ_L,ρ_T and ρ_R.*

Conversely, for each congruence ρ, ρ_L,ρ_T and ρ_R can be defined as above, then (ρ_L,ρ_T,ρ_R) is the unique congruence triple on S satisfying $\rho_{(\rho_L,\rho_T,\rho_R)}=\rho$.

Proof Denote $\rho=\rho_{(\rho_L,\rho_T,\rho_R)}$. We immediately see ρ is an equivalence relation on S. If $(e,a,f)\rho(u,b,v)$, then $e\rho_L u, a\rho_T b, f\rho_R v$. Since ρ_T is a congruence on T, then for each $(i,x,j)\in S$, we have $(x+a)\rho_T(x+b)$ and

$$(x+a+(x+a)^{-1})\rho_T(x+b+(x+b)^{-1}),((x+a)^{-1}+x+a)\rho_T((x+b)^{-1}+x+b).$$

Following the definition of congruence triple and Proposition 2.10, we have

$$\begin{aligned}&(i+x+a+(x+a)^{-1})\rho_L(i+x+b+(x+b)^{-1}),\\&(i+(x+a)^{-1}+x+a)\rho_L(i+(x+b)^{-1}+x+b).\end{aligned}$$

Hence,

$$(i,x,j)+(e,a,f)=(i+x+a+(x+a)^{-1},x+a,(x+a)^{-1}+x+a+f)\rho$$

$$(i+x+b+(x+b)^{-1},x+b,(x+b)^{-1}+x+b+v)$$
$$=(i,x,j)+(u,b,v).$$

Similarly,

$$((e,a,f)+(i,x,j))\rho((u,b,v)+(i,x,j)).$$

At the same time, we have

$$(i,x,j)(e,a,f)=(ie,xa,jf)\rho(iu,xb,jv)=(i,x,j)(u,b,v),$$
$$(e,a,f)(i,x,j)=(ei,ax,fj)\rho(ui,bx,vj)=(u,b,v)(i,x,j).$$

So ρ is a congruence on S.

Conversely, following Proposition 2.1 and Corollary 2.7, we just need to prove (ρ_L,ρ_T,ρ_R) satisfies the condition of Definition 2.9. If $\alpha,\beta\in D, a\rho_T b$, from the Propositions 2.3 and 2.6, there exist $(e,\alpha,f),(t,\alpha,s),(u,\beta,v),(p,\beta,q)\in S$ such that

$$(e,\alpha,f)\rho((e,\alpha,s)+(u,\beta,v)+(t,\alpha,f)),(u,\beta,v)\rho((u,\beta,q)+(e,\alpha,f)+(p,\beta,v)).$$

So

$$(e,\alpha,f)\rho(e+\alpha+\beta+(\alpha+\beta)^{-1},\alpha+\beta,(\alpha+\beta)^{-1}+\alpha+\beta+f),$$
$$(u,\beta,v)\rho(u+\beta+\alpha+(\beta+\alpha)^{-1},\beta+\alpha,(\beta+\alpha)^{-1}+\beta+\alpha+v).$$

Thus, $e\rho_L(e+\alpha+\beta+(\alpha+\beta)^{-1})$, $u\rho_L(u+\beta+\alpha+(\beta+\alpha)^{-1})$, $f\rho_R((\alpha+\beta)^{-1}+\alpha+\beta+f)$, $v\rho_R((\beta+\alpha)^{-1}+\beta+\alpha+v)$. Therefore, $\alpha\rho_{L_D}\beta$ and $\alpha\rho_{R_D}\beta$.

Secondly, if $\alpha\rho_{L_D}\beta$, then there exist $e\in L_\alpha$, $u,v\in L_{\alpha+\beta}$, $f\in L_\beta$ such that $e\rho_L u, v\rho_L f$. Hence, for some $s\in R_\alpha$, $p\in R_\beta$, $t,q\in R_{\alpha+\beta}$,

$$((e,\alpha,s)+(u,\alpha+\beta,t))\rho(u,\alpha+\beta,t),((u,\alpha+\beta,t)+(e,\alpha,s))\rho(e,\alpha,s),$$
$$((f,\beta,p)+(v,\alpha+\beta,q))\rho(v,\alpha+\beta,q),((v,\alpha+\beta,q)+(f,\beta,p))\rho(f,\beta,p).$$

Then,

$$(f,\beta,p)\rho[(f,\beta,p)+(v,\alpha+\beta,q)+(f,\beta,p)]$$
$$=((f,\beta,p)+(u,\alpha+\beta,t)+(f,\beta,p))\rho((f,\beta,p)+(e,\alpha,s)+(u,\alpha+\beta,t)+(f,\beta,p))$$
$$=((f,\beta,p)+(e,\alpha,s)+(v,\alpha+\beta,q)+(f,\beta,p))\rho((f,\beta,p)+(e,\alpha,s)+(f,\beta,p)).$$

Similarly, $(e,\alpha,s)\rho((e,\alpha,s)+(f,\beta,p)+(e,\alpha,s))$. That is, $\alpha\rho_T\beta, \rho_T|_D=\rho_{L_D}$. It is similar to prove $\rho_T|_D=\rho_{R_D}$. $\square$

3. The characterization of homomorphism

Based on Theorem 3.3 ([9]), it is easy to prove that additive idempotents of the homomorphism image of an orthodox semiring still satisfy permutation identities. Now we give the characterization of homomorphism image of an orthodox semiring whose additive idempotents still satisfy permutation identities.

Theorem 3.1 *Let $S = QS(L, T, R; D)$, $\rho_{(\rho_L,\rho_T,\rho_R)}$ be a congruence on S and $\rho = \rho_{(\rho_L,\rho_T,\rho_R)}$. Then*

$$S/\rho = QS(L/\rho_L, T/\rho_T, R/\rho_R; D/\rho_T|_D),$$

where for each $A \in D/\rho_T|_D$, $L_A = \cup_{\alpha\in A} L_\alpha$, $\rho_{L_A} = \rho_L|_A$, $R_A = \cup_{\alpha\in A} R_\alpha$, $\rho_{R_A} = \rho_R|_A$ and each $A, B \in D/\rho_T|_D$, $A \le B$, $\alpha \in A$, $\beta \in B$, $\alpha \le \beta$, $i \in L_\alpha$, $j \in L_\beta$

$$\begin{aligned}\overline{\varphi}_{A,B} :& L_A/\rho_{L_A} \longrightarrow L_B/\rho_{L_B}\\ & i\rho_L \longmapsto (i\varphi_{\alpha,\beta})\rho_L,\\ \overline{\psi}_{A,B} :& R_A/\rho_{R_A} \longrightarrow R_B/\rho_{R_B}\\ & j\rho_R \longmapsto (j\psi_{\alpha,\beta})\rho_R.\end{aligned}$$

Proof Let $A, B \in D/\rho_T|_D$, $A \le B$, $\alpha \in A$, $\beta \in B$. Then $\alpha \le \alpha + \beta \in B$. Suppose $\alpha \le \beta$, if $\gamma \in A$, $\delta \in B$, $\gamma \le \delta$, $i \in L_\alpha$, $j \in L_\gamma$ such that $i\rho_L j$. From Proposition 2.10, we have

$$(i + \alpha + \beta + \delta + (\alpha + \beta + \delta)^{-1})\rho_L(j + \gamma + \beta + \delta + (\gamma + \beta + \delta)^{-1}),$$
$$(i + \alpha + \beta + (\alpha + \beta)^{-1})\rho_L(i + \alpha + \beta + \delta + (\alpha + \beta + \delta)^{-1}),$$

and

$$(j + \gamma + \delta + (\gamma + \delta)^{-1})\rho_L(j + \gamma + \beta + \delta + (\gamma + \beta + \delta)^{-1}).$$

So $(i + \alpha + \beta + (\alpha + \beta)^{-1})\rho_L(j + \gamma + \delta + (\gamma + \delta)^{-1})$. Therefore, $\overline{\varphi}_{A,B}$ is well-defined. Similarly, $\overline{\psi}_{A,B}$ is well-defined.

Clearly, $L/\rho_L = [D/\rho_T|_D, L_A/\rho_{L_A}, \overline{\varphi}_{A,B}]$ and $R/\rho_R = [D/\rho_T|_D,\ R_A/\rho_{R_A},\ \overline{\psi}_{A,B}]$ are left normal band semiring and right normal band semiring, respectively. Since ρ_T is a congruence, T/ρ_T is a d-inverse semiring. Let $\overline{S} = QS(L/\rho_L, T/\rho_T, R/\rho_R; D/\rho_T|_D)$. We define a mapping θ as follows:

$$\begin{aligned}\theta : S &\longrightarrow \overline{S},\\ (e, a, f) &\longmapsto (e\rho_L, a\rho_T, f\rho_R).\end{aligned}$$

For any $(e, a, f), (u, b, v) \in S$, let $A = (a + a^{-1})\rho_T|_D$, $B = (a + b + (a + b)^{-1})\rho_T|_D$, $C = (b^{-1} + b)\rho_T|_D$, $D = ((a + b)^{-1} + a + b)\rho_T|_D$. Then

$$\begin{aligned}((e, a, f) + (u, b, v))\theta =&(e + a + b + (a + b)^{-1}, a + b, (a + b)^{-1} + a + b + v)\theta\\ =&((e + a + b + (a + b)^{-1})\theta, (a + b)\theta, (a + b)^{-1} + a + b + v)\theta)\\ =&((e + a + b + (a + b)^{-1})\rho_L, (a + b)\rho_T, (a + b)^{-1} + a + b + v)\rho_R)\\ =&(e\rho_L + (a + b)\rho_L + (a + b)^{-1}\rho_L, a\rho_T + b\rho_T, (a + b)^{-1}\rho_R+\\ &(a + b)\rho_R + v\rho_R)\\ =&(e\rho_L, a\rho_T, f\rho_R) + (u\rho_L, b\rho_T, v\rho_R)\\ =&(e, a, f)\theta + (u, b, v)\theta,\end{aligned}$$

and

$$[(e, a, f)(u, b, v)]\theta = (eu, ab, fv)\theta = ((eu)\theta, (ab)\theta, (fv)\theta)$$

$$= ((eu)\rho_L, (ab)\rho_T, (fv)\rho_R) = (e\rho_L, a\rho_T, f\rho_R)(u\rho_L, b\rho_T, v\rho_R)$$
$$= (e, a, f)\theta(u, b, v)\theta.$$

Therefore, θ is a homomorphism.

If $(i\rho_L, a\rho_T, j\rho_R) \in \overline{S}$, where $i \in L_\alpha, j \in R_\beta$, then $\alpha\rho_T(a + a^{-1}), \beta\rho_T(a^{-1} + a)$. So $(\alpha + a)\rho_T a, (a+\beta)\rho_T a$ and $(\alpha+a+\beta+(\alpha+a+\beta)^{-1})\rho_T(a+a^{-1})$, $((\alpha+a+\beta)^{-1}+\alpha+a+\beta)\rho_T(a^{-1}+a)$, $(\alpha + a + \beta)\rho_T a$. Hence,

$$(i, a, j)\rho(i + \alpha + a + \beta + (\alpha + a + \beta)^{-1}, \alpha + a + \beta, (\alpha + a + \beta)^{-1} + \alpha + a + \beta + j).$$

Then,

$$i\rho_L(i + \alpha + a + \beta + (\alpha + a + \beta)^{-1}), j\rho_R((\alpha + a + \beta)^{-1} + \alpha + a + \beta + j).$$

So

$$(i + \alpha + a + \beta + (\alpha + a + \beta)^{-1}, \alpha + a + \beta, (\alpha + a + \beta)^{-1} + \alpha + a + \beta + j) \in S,$$

and

$$(i + \alpha + a + \beta + (\alpha + a + \beta)^{-1}, \alpha + a + \beta, (\alpha + a + \beta)^{-1} + \alpha + a + \beta + j)\theta = (i\rho_L, a\rho_T, j\rho_R).$$

Thus, θ is a homomorphism.

It is easily seen $\ker\theta = \rho$. Therefore, $S/\rho \cong \overline{S} = QS(L/\rho_L, T/\rho_T, R/\rho_R; D/\rho_T|_D)$. □

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