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Majorization Properties for Certain Classes of Analytic Functions Involving a Generalized Differential Operator

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Abstract In this paper, we introduce new subclasses $S^{m,j,l}_{p,q,\lambda}[A,B;\gamma]$ and $H^{m,j,l}_{p,q,\lambda}(\alpha,\beta)$ of certain p-valent analytic functions defined by a generalized differential operator. Majorization properties for functions belonging to the classes $S^{m,j,l}_{p,q,\lambda}[A,B;\gamma]$ and $H^{m,j,l}_{p,q,\lambda}(\alpha,\beta)$ are investigated. Also, we point out some new or known consequences of our main results.

Keywords analytic functions; starlike functions; β-spiral functions; subordination; majorization property; differential operator.

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1. Introduction and definitions

Let f and g be two analytic functions in the open unit disk

$$\Delta=\{z\in C:|z|<1\}. \tag{1.1}$$

We say that f is majorized by g in Δ (see [1]) and write

$$f(z)\ll g(z)\ \ (z\in\Delta), \tag{1.2}$$

if there exists a function φ, analytic in Δ such that

$$|\varphi(z)|\le 1\ \text{ and }\ f(z)=\varphi(z)g(z)\ \ (z\in\Delta). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions.

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, if there exists a Schwarz function ω, which is analytic in Δ with

$$\omega(0)=0\ \text{ and }\ |\omega(z)|<1\ \ (z\in\Delta),$$

such that

$$f(z)=g(\omega(z))\ \ (z\in\Delta).$$

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We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function g is univalent in Δ, then

$$f(z) \prec g(z) \quad (z \in \Delta) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in N = \{1, 2, \dots\}), \tag{1.4}$$

that are analytic and p-valent in the open unit disk Δ. Also, let $A_1 = A$.

For a function $f \in A_p$, let $f^{(q)}$ denote qth-order ordinary differential operator by

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q}, \tag{1.5}$$

where $p > q$, $p \in N$, $q \in N_0 = N \cup \{0\}$ and $z \in \Delta$.

Next, we define the generalized differential operator $I_{p,\lambda}^{m,l} f^{(q)} : A_p \to A_p$ by

$$I_{p,\lambda}^{0,l} f^{(q)}(z) = f^{(q)}(z);$$

$$I_{p,\lambda}^{1,l} f^{(q)}(z) = (1-\lambda) f^{(q)}(z) + \lambda z^{1-l} (z^l f^{(q)}(z))';$$

and

$$I_{p,\lambda}^{m,l} f^{(q)}(z) = I_{p,\lambda}^{1,l} (I_{p,\lambda}^{m-1,l} f^{(q)}(z)). \tag{1.6}$$

If $f \in A_p$, then from (1.5) and (1.6), we can easily see that

$$I_{p,\lambda}^{m,l} f^{(q)}(z) = \frac{p![1+\lambda(p+l-q-1)]^m}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k![1+\lambda(k+l-q-1)]^m}{(k-q)!} a_k z^{k-q}, \tag{1.7}$$

where $m \in N_0;\ \lambda, l \geq 0;\ p > q;\ p \in N$ and $q \in N_0$.

We note that for suitable choices of $p,\ q,\ \lambda$ and l, we obtain the following operators studied by various authors.

(i) $I_{p,1}^{m,l} f^{(0)}(z) = I_p(m,l) f(z)$ (see Kumar et al. [2]);

(ii) $I_{1,1}^{m,l} f^{(0)}(z) = I_l^m f(z)$ (see Cho and Srivastava [3] and Cho and Kim [4]);

(iii) $I_{1,\lambda}^{m,0} f^{(0)}(z) = D_\lambda^m f(z)$ (see Al-Oboudi [5]);

(iv) $I_{p,1}^{m,0} f^{(q)}(z) = D^m f^{(q)}(z)$ (see Frasin [6] and Goswami and Aouf [19]);

(v) $I_{p,1}^{m,0} f^{(0)}(z) = D_p^m f(z)$ (see Kamali and Orhan [7] and Aouf and Mostafa [8]);

(vi) $I_{1,1}^{m,0} f^{(0)}(z) = D^m f(z)$ (see Salagean [9]).

Using the operator $I_{p,\lambda}^{m,l} f^{(q)}(z)$, we now define the following classes of p-valent analytic functions.

Definition 1.1 *A function $f(z) \in A_p$ is said to be in the class $S_{p,q,\lambda}^{m,j,l}[A,B;\gamma]$ of p-valent functions of complex order $\gamma \neq 0$ in Δ if and only if*

$$\Big[1 + \frac{1}{\gamma}\Big(\frac{z(I_{p,\lambda}^{m,l} f^{(q)}(z))^{(j+1)}}{(I_{p,\lambda}^{m,l} f^{(q)}(z))^{(j)}} - p + j + m\Big)\Big] \prec \frac{1 + \frac{m}{\gamma} + Az}{1+Bz}, \tag{1.8}$$

where $z \in \Delta;\ -1 \le B < A \le 1;\ p > q;\ p \in N;\ m, j, q \in N_0;\ \lambda, l \ge 0$ *and* $\gamma \in C^* = C \setminus \{0\}$ *with*

$$|1 + \lambda(p + l - 1)| \ge |\lambda\gamma(A - B) + (1 + \lambda(p + l - m - 1))B|.$$

Clearly, we have the following relationships:

(i) $S^{m,j,l}_{p,q,\lambda}[1, -1; \gamma] = S^{m,j,l}_{p,q,\lambda}(\gamma)$;

(ii) $S^{m,j,l}_{p,0,1}[1, -1; \gamma] = S^{m,l}_{p,j}(\gamma)$;

(iii) $S^{m,j,0}_{p,0,1}[1, -1; \gamma] = S^{m}_{p,j}(\gamma)$;

(iv) $S^{m,0,0}_{p,0,1}[1, -1; \gamma] = S^{m}_{p}(\gamma)$;

(v) $S^{0,j,0}_{p,0,1}[1, -1; \gamma] = S_{p,j}(\gamma)$;

(vi) $S^{0,0,0}_{1,0,1}[1, -1; \gamma] = S(\gamma)\ \ (\gamma \in C^*)$;

(vii) $S^{0,1,0}_{1,0,1}[1, -1; \gamma] = K(\gamma)\ \ (\gamma \in C^*)$;

(viii) $S^{0,0,0}_{1,0,1}[1, -1; 1 - \alpha] = S^*(\alpha)\ \ (0 \le \alpha < 1)$.

The classes $S^{m,l}_{p,j}(\gamma)$ and $S_{p,j}(\gamma)$ were introduced by Goswami et al. [10] and Altintas and Srivastava [11], respectively. The classes $S(\gamma)$ and $K(\gamma)$ are said to be the classes of starlike and convex functions of complex order $\gamma \ne 0$ in Δ which were considered by Nasr and Aouf [12] and Wiatrowski [13], while $S^*(\alpha)$ denotes the class of starlike functions of order α in Δ.

Definition 1.2 *A function* $f(z) \in A_p$ *is said to be in the class* $H^{m,j,l}_{p,q,\lambda}(\alpha, \beta)$, *if and only if*

$$\mathrm{Re}\Big\{e^{i\beta}\frac{z(I^{m,l}_{p,\lambda}f^{(q)}(z))^{j+1}}{(I^{m,l}_{p,\lambda}f^{(q)}(z))^{j}}\Big\} > \alpha\cos\beta, \tag{1.9}$$

where $z \in \Delta;\ p > q;\ p \in N;\ m, j, q \in N_0;\ \lambda, l \ge 0;\ 0 \le \alpha < 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

It can be seen that, by specializing the parameters the class $H^{m,j,l}_{p,q,\lambda}(\alpha, \beta)$ reduces to many known subclasses of analytic functions.

(i) $H^{0,0,l}_{1,0\lambda}(\alpha, \beta) = S^*_\beta(\alpha)$; (ii) $H^{0,0,l}_{1,0\lambda}(\alpha, 0) = S^*(\alpha)$; (iii) $H^{0,0,l}_{1,0\lambda}(0, \beta) = S^*_\beta$.

The classes $S^*_\beta(\alpha)$ and $S^*(\alpha)$ are said to be the classes of β-spiral-like and starlike functions of order α in Δ, which were studied by Libera [14] and Robertson [15], while S^*_β denotes the class of β-spiral-like functions in Δ considered by Spacek [16].

A majorization problem for the class $S^* = S^*(0)$ has been investigated by MacGregor [1]. Also, majorization problems for starlike functions of complex order $\gamma \ne 0$ and β-spiral-like of order α in Δ have recently been investigated by Altintas et al. [17], Goyal and Goswami [18], Goswami et al. [10, 19] and Abubaker et al. [20].

The main object of this paper is to investigate the problems of majorization of the classes $S^{m,j,l}_{p,q,\lambda}[A, B; \gamma]$ and $H^{m,j,l}_{p,q,\lambda}(\alpha, \beta)$ defined by a generalized differential operator.

In order to prove our main results, we need the following lemma.

Lemma 1.1 ([21]) *Let* $\varphi(z)$ *be analytic in* Δ *satisfying* $|\varphi(z)| \le 1$ *for* $z \in \Delta$. *Then,*

$$|\varphi'(z)| \le \frac{1 - |\varphi(z)|^2}{1 - |z|^2}. \tag{1.10}$$

2. Majorization problem for the class $S^{m,j,l}_{p,q,\lambda}[A, B; \gamma]$

We begin by proving the following result.

Theorem 2.1 *Let the function $f \in A_p$ and suppose that $g \in S^{m,j,l}_{p,q,\lambda}[A,B;\gamma]$. If $(I^{m,l}_{p,\lambda} f^{(q)}(z))^{(j)}$ is majorized by $(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}$ in Δ for $j \in N_0$, then*

$$|(I^{m+1,l}_{p,\lambda} f^{(q)}(z))^{(j)}| \le |(I^{m+1,l}_{p,\lambda} g^{(q)}(z))^{(j)}| \quad (|z| \le r_1), \tag{2.1}$$

where $r_1 = r_1(p,\gamma,\lambda,l,m,A,B)$ is the smallest positive root of the equation

$$\begin{aligned}&|\lambda\gamma(A-B)+(1+\lambda(p+l-m-1))B|r^3-[1+\lambda(p+l-1)+2\lambda|B|]r^2-\\&[|\lambda\gamma(A-B)+(1+\lambda(p+l-m-1))B|+2\lambda]r+|1+\lambda(p+l-1)|=0,\end{aligned} \tag{2.2}$$

$$(-1 \le B < A \le 1;\ p \in N;\ m \in N_0;\ \lambda, l \ge 0;\ \gamma \in C^*).$$

Proof Since $g \in S^{m,j,l}_{p,q,\lambda}[A,B;\gamma]$, we find from (1.8) that

$$\Big[1+\frac{1}{\gamma}\Big(\frac{z(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j+1)}}{(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}}-p+j+m\Big)\Big]=\frac{1+\frac{m}{\gamma}+A\omega(z)}{1+B\omega(z)}, \tag{2.3}$$

where $\omega(z)=c_1z+c_2z^2+\cdots$, $\omega \in P$, P denotes the well-known class of the bounded analytic functions in Δ and satisfies the conditions (see, for details, Goodman [22])

$$\omega(0)=0 \quad\text{and}\quad |\omega(z)| \le |z| \quad (z \in \Delta). \tag{2.4}$$

It follows from (2.3) that

$$\frac{z(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j+1)}}{(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}}=\frac{p-j+[\gamma(A-B)+B(p-j-m)]\omega(z)}{1+B\omega(z)}. \tag{2.5}$$

Now, using the following, easily verified from (1.7), identity

$$\lambda z(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j+1)}=(I^{m+1,l}_{p,\lambda} g^{(q)}(z))^{(j)}-[1+\lambda(j+l-1)](I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)} \tag{2.6}$$

in (2.5) and making simple calculations, we get

$$\frac{(I^{m+1,l}_{p,\lambda} g^{(q)}(z))^{(j)}}{(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}}=\frac{[1+\lambda(p+l-1)]+[\lambda\gamma(A-B)+(1+\lambda(p+l-m-1))B]\omega(z)}{1+B\omega(z)}, \tag{2.7}$$

which, in view of (2.4), immediately yields the inequality

$$\begin{aligned}&|(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}|\\&\le \frac{1+|B||z|}{|1+\lambda(p+l-1)|-|\lambda\gamma(A-B)+[1+\lambda(p+l-m-1)]B||z|}|(I^{m+1,l}_{p,\lambda} g^{(q)}(z))^{(j)}|.\end{aligned} \tag{2.8}$$

Next, since $(I^{m,l}_{p,\lambda} f^{(q)}(z))^{(j)}$ is majorized by $(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}$ in Δ, we have from (1.3)

$$(I^{m,l}_{p,\lambda} f^{(q)}(z))^{(j)}=\varphi(z)(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}. \tag{2.9}$$

Differentiating the equality (2.9) with respect to z and multiplying by z, we obtain

$$z(I^{m,l}_{p,\lambda} f^{(q)}(z))^{(j+1)}=z\varphi'(z)(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}+z\varphi(z)(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j+1)}. \tag{2.10}$$

Also, by using (2.6) in (2.10), we get

$$(I^{m+1,l}_{p,\lambda} f^{(q)}(z))^{(j)}=\lambda z\varphi'(z)(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}+\varphi(z)(I^{m+1,l}_{p,\lambda} g^{(q)}(z))^{(j)}. \tag{2.11}$$

Therefore, noting that $\varphi \in P$ satisfies the inequality (1.10) and using (2.8) in (2.11), we have

$$|(I_{p,\lambda}^{m+1,l} f^{(q)}(z))^{(j)}| \le \Big(|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2} \cdot \frac{\lambda|z|(1+|B||z|)}{|1+\lambda(p+l-1)| - |\lambda\gamma(A-B) + [1+\lambda(p+l-m-1)]B||z|}\Big) \cdot |(I_{p,\lambda}^{m+1,l} g^{(q)}(z))^{(j)}|,$$

which, upon setting

$$|z| = r \text{ and } |\varphi(z)| = \rho \quad (0 \le \rho \le 1)$$

leads us to the inequality

$$|(I_{p,\lambda}^{m+1,l} f^{(q)}(z))^{(j)}| \le \frac{\Phi(\rho)}{(1-r^2)[|1+\lambda(p+l-1)| - |\lambda\gamma(A-B) + [1+\lambda(p+l-m-1)]B|r]} \cdot |(I_{p,\lambda}^{m+1,l} g^{(q)}(z))^{(j)}|, \tag{2.12}$$

where

$$\Phi(\rho) = -\lambda r(1+|B|r)\rho^2 + (1-r^2)[|1+\lambda(p+l-1)| - |\lambda\gamma(A-B) + (1+\lambda(p+l-m-1))B|r]\rho + \lambda r(1+|B|r) \tag{2.13}$$

takes its maximum value at $\rho = 1$ with $r_1 = r_1(p,\gamma,\lambda,l,m,A,B)$, where $r_1 = r_1(p,\gamma,\lambda,l,m,A,B)$ is the smallest positive root of the equation (2.2). Furthermore, if $0 \le \delta \le r_1(p,\gamma,\lambda,l,m,A,B)$, then the function $\Psi(\rho)$ defined by

$$\Psi(\rho) = -\lambda\delta(1+|B|\delta)\rho^2 + (1-\delta^2)[|1+\lambda(p+l-1)| - |\lambda\gamma(A-B) + (1+\lambda(p+l-m-1))B|\delta]\rho + \lambda\delta(1+|B|\delta) \tag{2.14}$$

is an increasing function on the interval $0 \le \rho \le 1$, so that

$$\Psi(\rho) \le \Psi(1) = (1-\delta^2)[|1+\lambda(p+l-1)| - |\lambda\gamma(A-B) + (1+\lambda(p+l-m-1))B|\delta]$$

$$(0 \le \rho \le 1;\ 0 \le \delta \le r_1(p,\gamma,\lambda,l,m,A,B)).$$

Hence upon setting $\rho = 1$ in (2.14), we conclude that (2.1) of Theorem 2.1 holds true for $|z| \le r_1(p,\gamma,\lambda,l,m,A,B)$, which completes the proof of Theorem 2.1. □

As a special case of Theorem 2.1, when $A = 1$ and $B = -1$, we have

Corollary 2.1 *Let the function $f \in A_p$ and suppose that $g \in S_{p,q,\lambda}^{m,j,l}(\gamma)$. If $(I_{p,\lambda}^{m,l} f^{(q)}(z))^{(j)}$ is majorized by $(I_{p,\lambda}^{m,l} g^{(q)}(z))^{(j)}$ in Δ for $j \in N_0$, then*

$$|(I_{p,\lambda}^{m+1,l} f^{(q)}(z))^{(j)}| \le |(I_{p,\lambda}^{m+1,l} g^{(q)}(z))^{(j)}| \quad (|z| \le r_2), \tag{2.15}$$

where

$$r_2 = r_2(p,\gamma,\lambda,l,m) = \frac{\eta - \sqrt{\eta^2 - 4|1+\lambda(p+l-1)||1-\lambda(2\gamma+m-p-l+1)|}}{2|1-\lambda(2\gamma+m-p-l+1)|} \tag{2.16}$$

$$(\eta = 2\lambda + |1+\lambda(p+l-1)| + |1-\lambda(2\gamma+m-p-l+1)|;\ \lambda, l \ge 0;\ p \in N;\ m \in N_0;\ \gamma \in C^*).$$

Setting $\lambda = 1$ and $l = 0$ in Corollary 2.1, we get

Corollary 2.2 *Let the function $f \in A_p$ and suppose that $g \in S^{m,j,0}_{p,q,1}(\gamma)$. If $(D^m f^{(q)}(z))^{(j)}$ is majorized by $(D^m g^{(q)}(z))^{(j)}$ in Δ for $j \in N_0$, then*

$$|(D^{m+1} f^{(q)}(z))^{(j)}| \leq |(D^{m+1} g^{(q)}(z))^{(j)}| \quad (|z| \leq r_3),$$

where

$$r_3 = r_3(p, \gamma, m) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4p|2\gamma - p + m|}}{2|2\gamma - p + m|}$$

$$(\eta_1 = 2 + p + |2\gamma - p + m|;\ p \in N;\ m \in N_0;\ \gamma \in C^*).$$

Further putting $m = q = j = 0$ and $p = 1$ in Corollary 2.2, we obtain the result of Altintas et al. [17].

Corollary 2.3 *Let the function $f \in A$ be analytic and univalent in the open unit disk Δ and suppose that $g \in S(\gamma)$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq r_4),$$

where

$$r_4 = r_4(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|} \quad (\gamma \in C^*).$$

Also, for $\gamma = 1$, Corollary 2.3 reduces to the result of MacGregor [1].

Corollary 2.4 *Let the function $f \in A$ be analytic and univalent in the open unit disk Δ and suppose that $g \in S^*(0) = S^*$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \leq |g'(z)| \quad (|z| \leq 2 - \sqrt{3}).$$

Remark 2.1 (i) Taking $\lambda = 1$ and $q = 0$ in Theorem 2.1 and Corollary 2.1, we obtain the results of Goswami et al.[10, Theorem 2.1 and Corollary 2.1, respectively];

(ii) Taking $q = 0$ in Corollary 2.2, we get the result of Goswami et al. [10, Corollary 2.2].

3. Majorization problem for the class $H^{m,j,l}_{p,q,\lambda}(\alpha, \beta)$

Next, we state and prove

Theorem 3.1 *Let the function $f \in A_p$ and suppose that $g \in H^{m,j,l}_{p,q,\lambda}(\alpha, \beta)$. If $(I^{m,l}_{p,\lambda} f^{(q)}(z))^{(j)}$ is majorized by $(I^{m,l}_{p,\lambda} g^{(q)}(z))^{(j)}$ in Δ for $j \in N_0$, then*

$$|(I^{m+1,l}_{p,\lambda} f^{(q)}(z))^{(j)}| \leq |(I^{m+1,l}_{p,\lambda} g^{(q)}(z))^{(j)}| \quad (|z| \leq r_1), \tag{3.1}$$

where

$$r_1 = r_1(\lambda, l, j, \alpha, \beta) = \frac{\eta - \sqrt{\eta^2 - 4|1 + \lambda(j + l)||2\lambda(1 - \alpha)\cos\beta - [1 + \lambda(j + l)]e^{i\beta}|}}{2|2\lambda(1 - \alpha)\cos\beta - [1 + \lambda(j + l)]e^{i\beta}|} \tag{3.2}$$

with $\eta = 2\lambda + |1+\lambda(j+l)| + |2\lambda(1-\alpha)\cos\beta - [1+\lambda(j+l)]e^{i\beta}|$ *and* $|1+\lambda(j+l)| \ge |2\lambda(1-\alpha)\cos\beta - [1+\lambda(j+l)]e^{i\beta}|$,

$$(j \in N_0;\ \lambda, l \ge 0;\ 0 \le \alpha < 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}).$$

Proof Since $g \in H^{m,j,l}_{p,q,\lambda}(\alpha,\beta)$, we find from (1.9) that

$$e^{i\beta}\frac{z(I^{m,l}_{p,\lambda}g^{(q)}(z))^{(j+1)}}{(I^{m,l}_{p,\lambda}g^{(q)}(z))^{(j)}} = \frac{1+(1-2\alpha)\omega(z)}{1-\omega(z)}\cos\beta + i\sin\beta, \tag{3.3}$$

where $\omega(z)$ is defined as (2.4).

From (3.3), we get

$$\frac{z(I^{m,l}_{p,\lambda}g^{(q)}(z))^{(j+1)}}{(I^{m,l}_{p,\lambda}g^{(q)}(z))^{(j)}} = \frac{e^{i\beta} + [2(1-\alpha)\cos\beta - e^{i\beta}]\omega(z)}{e^{i\beta}[1-\omega(z)]}. \tag{3.4}$$

Now, using the identity (2.6) in (3.4) and making simple calculations, we obtain

$$\frac{(I^{m+1,l}_{p,\lambda}g^{(q)}(z))^{(j)}}{(I^{m,l}_{p,\lambda}g^{(q)}(z))^{(j)}} = \frac{[1+\lambda(j+l)]e^{i\beta} + [2\lambda(1-\alpha)\cos\beta - (1+\lambda(j+l))e^{i\beta}]\omega(z)}{e^{i\beta}[1-\omega(z)]}, \tag{3.5}$$

which, in view of (2.4), immediately yields the following inequality

$$|(I^{m,l}_{p,\lambda}g^{(q)}(z))^{(j)}| \le \frac{1+|z|}{|1+\lambda(j+l)| - |2\lambda(1-\alpha)\cos\beta - [1+\lambda(j+l)]e^{i\beta}||z|}|(I^{m+1,l}_{p,\lambda}g^{(q)}(z))^{(j)}|. \tag{3.6}$$

Next, making use of (1.10) and (3.6) in (2.11), and just as the proof of Theorem 2.1, we have

$$\begin{aligned}&|(I^{m+1,l}_{p,\lambda}f^{(q)}(z))^{(j)}|\\ &\le \Big(\frac{\lambda|z|(1-|\varphi(z)|^2)}{(1-|z|)[|1+\lambda(j+l)| - |2\lambda(1-\alpha)\cos\beta - (1+\lambda(j+l))e^{i\beta}||z|]} + |\varphi(z)|\Big)\cdot\\ &\quad |(I^{m+1,l}_{p,\lambda}g^{(q)}(z))^{(j)}|,\end{aligned} \tag{3.7}$$

which upon setting $|z| = r$ and $|\varphi(z)| = \rho$ $(0 \le \rho \le 1)$ leads us to the inequality

$$\begin{aligned}&|(I^{m+1,l}_{p,\lambda}f^{(q)}(z))^{(j)}|\\ &\le \frac{\Phi_1(\rho)}{(1-r)[|1+\lambda(j+l)| - |2\lambda(1-\alpha)\cos\beta - (1+\lambda(j+l))e^{i\beta}|r]}|(I^{m+1,l}_{p,\lambda}g^{(q)}(z))^{(j)}|,\end{aligned} \tag{3.8}$$

where the function $\Phi_1(\rho)$ defined by

$$\Phi_1(\rho) = -\lambda r\rho^2 + (1-r)[|1+\lambda(j+l)| - |2\lambda(1-\alpha)\cos\beta - (1+\lambda(j+l))e^{i\beta}|r]\rho + \lambda r \tag{3.9}$$

takes its maximum value at $\rho = 1$ with $r_1 = r_1(\lambda, l, j, \alpha, \beta)$ given by (3.2). Moreover, if $0 \le \sigma \le r_1(\lambda, l, j, \alpha, \beta)$, then the function

$$\Psi_1(\rho) = -\lambda\sigma\rho^2 + (1-\sigma)[|1+\lambda(j+l)| - |2\lambda(1-\alpha)\cos\beta - (1+\lambda(j+l))e^{i\beta}|\sigma]\rho + \lambda\sigma \tag{3.10}$$

increases on the interval $0 \le \rho \le 1$, so that $\Psi_1(\rho)$ does not exceed

$$\Psi_1(1) = (1-\sigma)[|1+\lambda(j+l)| - |2\lambda(1-\alpha)\cos\beta - (1+\lambda(j+l))e^{i\beta}|\sigma] \quad (0 \le \sigma \le r_1(\lambda, l, j, \alpha, \beta)).$$

Therefore, from this fact, (3.8) gives the inequality (3.1). This completes the proof of Theorem 3.1. □

Taking $\lambda = 1$ and $l = 0$ in Theorem 3.1, we immediately obtain the following result.

Corollary 3.1 *Let the function $f \in A_p$ and suppose that $g \in H_{p,q,1}^{m,j,0}(\alpha,\beta)$. If $(D^m f^{(q)}(z))^{(j)}$ is majorized by $(D^m g^{(q)}(z))^{(j)}$ in Δ for $j \in N_0$, then*

$$|(D^{m+1} f^{(q)}(z))^{(j)}| \le |(D^{m+1} g^{(q)}(z))^{(j)}| \quad (|z| \le r_2), \tag{3.11}$$

where

$$r_2 = r_2(j,\alpha,\beta) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4|1+j||2(1-\alpha)\cos\beta - (1+j)e^{i\beta}|}}{2|2(1-\alpha)\cos\beta - (1+j)e^{i\beta}|} \tag{3.12}$$

with $\eta_1 = 2 + |1+j| + |2(1-\alpha)\cos\beta - (1+j)e^{i\beta}|$ and $|1+j| \ge |2(1-\alpha)\cos\beta - (1+j)e^{i\beta}|$,

$$(j \in N_0;\ 0 \le \alpha < 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}).$$

Further, putting $m = q = j = 0$ and $p = 1$ in Corollary 3.1, we also obtain the result of Altintas et al. [17].

Corollary 3.2 *Let the function $f \in A$ and suppose that $g \in S^*((\alpha-1)e^{i\beta}) = S^*_\beta(\alpha)$, where $0 \le \alpha < 1$ and $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$. If $f(z)$ is majorized by $g(z)$ in Δ, then*

$$|f'(z)| \le |g'(z)| \quad (|z| \le r_3),$$

where

$$r_3 = r_3(\alpha,\beta) = \frac{\eta_2 - \sqrt{\eta_2^2 - 4|2(1-\alpha)\cos\beta - e^{i\theta}|}}{2|2(1-\alpha)\cos\beta - e^{i\beta}|}$$

$$(\eta_2 = 3 + |2(1-\alpha)\cos\beta - e^{i\beta}|;\ 0 \le \alpha < 1;\ -\frac{\pi}{2} < \beta < \frac{\pi}{2}),$$

which contains the well-known result of MacGregor [1] for $\alpha = \beta = 0$.

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