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The Fixed Point and Mann Iteration of a Modified Isotonic Operator

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Abstract This paper consists of two parts. In the first part, we discuss the Hölder continuity of Cauchy-type integral operator T of isotonic functions and the relationship between $\|T[f]\|_\alpha$ and $\|f\|_\alpha$. In the second part, firstly, we introduce a modified Cauchy-type integral operator T' and demonstrate that the operator T' has a unique fixed point by the Contraction Mapping Principle. Then we give the Mann iterative sequence and prove that the Mann iterative sequence strongly converges to the fixed point of the modified Cauchy-type integral operator T'.

Keywords Clifford analysis; isotonic operator; the fixed point theorem; Mann iteration.

MR(2010) Subject Classification 47H05; 30G30

1. Introduction

The isotonic functions are the functions defined in the even dimensional Euclidean space R^n with values in the complex Clifford algebra $\mathcal{C}_{0,n}$ and satisfy the isotonic system $\partial_{\underline{x_1}} f + i\widetilde{f}\partial_{\underline{x_2}} = 0$, where $\partial_{\underline{x_1}} = \sum_{j=1}^m e_j \partial x_j$, $\partial_{\underline{x_2}} = \sum_{j=1}^m e_j \partial x_{m+j}$. Isotonic Clifford analysis is a new field in Clifford analysis. It is a generalization of complex Clifford algebra. Recently, Blaya, Sommen and some other experts [1–4] have studied isotonic functions, and obtained a series of results such as the integral representation of the isotonic functions and cauchy integral formulas, etc. In addition, they have found that the isotonic functions are closely related to hermitian monogenic functions in Clifford analysis and they have done much research on hermitian monogenic functions by applying isotonic functions. So to study the isotonic functions can generalize the further applications of Clifford analysis to the field in mathematics and other subjects, and thus it is valuable to study it both in theory and practice.

Fixed point problem of operators is an important branch in Clifford analysis. So it is necessary to study the existence of the fixed point and the iterative approximation for the isotonic operator. There are many iterative schemes to approximate the fixed point of an operator, such as Picard iterative scheme, Mann iterative scheme, Ishikawa iterative scheme, Projection

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iterative scheme, Hybrid iterative scheme, etc [5–9]. And different iterative schemes have different approximation degrees and different complexity degrees in the process of approximation.

On the basis of [10], [5] and [12], this article studies some properties of the Cauchy-type integral operator T of isotonic functions. Firstly, we discuss the Hölder continuity of the Cauchy-type integral operator T and the relationship between $\|T[f]\|_\alpha$ and $\|f\|_\alpha$. Secondly, we introduce a modified Cauchy-type integral operator $T^{'}$ and prove that the operator T' has a unique fixed point by the contraction mapping principle. Finally, we give the Mann iterative sequence and prove that this sequence strongly converges to the fixed point of the modified Cauchy-type integral operator T'. These results make the theory of Clifford analysis more perfect as well as lay a theoretical foundation for the study of the properties of singularity integral operator in Clifford analysis.

2. Preliminaries

Let $e_1,\ldots,e_n$ be an orthogonal basis of the Euclidean space R^n and $\mathcal{C}_{0,n}$ be the complex Clifford algebra with basis $\{e_A : e_A = e_{\alpha_1}\cdots e_{\alpha_h}\}$, where $A = \{\alpha_1,\ldots,\alpha_h\} \subseteq \{1,\ldots,n\}, 1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_h \leq n$ and $e_A = e_0 = 1$ for $A = \emptyset$.

The noncommutative multiplication of the basis in $\mathcal{C}_{0,n}$ is governed by the rules:

$$\begin{cases} e_i^2 = -1, \;\; i = 1,2,\ldots,n, \\ e_ie_j = -e_je_i, \;\; 1 \leq i,j \leq n, \;\; i \neq j. \end{cases}$$

Any Clifford number $a \in \mathcal{C}_{0,n}$ can be written as $a = \sum_A C_Ae_A$, $C_A \in \mathcal{C}$. For any $a \in \mathcal{C}_{0,n}$, $|a| = \left(\sum_A |C_A|^2\right)^{\frac{1}{2}}$, $\overline{a} = \sum_A \overline{C}_A\overline{e}_A$ and $\widetilde{a} = \sum_A C_A\widetilde{e}_A$, where $\widetilde{e}_A = (-1)^{|A|}e_A$, $\overline{e}_A = (-1)^{\frac{|A|(|A|+1)}{2}}e_A$. For $k = 0,1,\ldots,n$, ${\mathcal{C}_{0,n}}^{(k)} = \{a \in \mathcal{C}_{0,n} : a = \sum_{|A|=k} C_Ae_A\}$ is the subspace of k-vectors. Thus we can obtain that $\mathcal{C}_{0,n} = \bigoplus_{k=0}^n {\mathcal{C}_{0,n}}^{(k)}$ and for any $a \in \mathcal{C}_{0,n}$, $a = \sum_{k=0}^n [a]_k$, where $[\cdot]_k$ is the projection operator on $\mathcal{C}_{0,n}^{(k)}$.

In this article, we assume that $n = 2m$ and denote the real Clifford vector $x = \sum_{j=1}^n e_jx_j$ by $x = \sum_{j=1}^m (e_jx_j + e_{m+j}x_{m+j})$. And for any real Clifford vector x, y, we have $|xy| \leq M_0|x||y|$. In addition, the corresponding Dirac operator can be written as $\partial_x = \sum_{j=1}^m (e_j\partial x_j + e_{m+j}\partial x_{m+j})$. Let $\vec{n} = \sum_{j=1}^m (e_jn_j(y) + e_{m+j}n_{m+j}(y))$ be the unit normal vector. We introduce the following Clifford vectors and their corresponding Dirac operators

$$\underline{x_1} = \sum_{j=1}^m e_jx_j, \;\; \underline{x_2} = \sum_{j=1}^m e_jx_{m+j}, \;\; \underline{n_1} = \sum_{j=1}^m e_jn_j(y),$$
$$\underline{n_2} = \sum_{j=1}^m e_jn_{m+j}(y), \;\; \partial_{\underline{x_1}} = \sum_{j=1}^m e_j\partial x_j, \;\; \partial_{\underline{x_2}} = \sum_{j=1}^m e_j\partial x_{m+j}.$$

In addition, let $I_j = \frac{1}{2}(1 + ie_je_{m+j}), j = 1,2,\ldots,m$. Then the primitive idempotent is given by $I = \prod_{j=1}^m$. So we have the following conversion relations $e_{m+j}I = ie_jI$, and therefore $e_{m+j}aI = i\tilde{a}e_jI$ with $a \in \mathcal{C}_{0,m}$. Using the front conversion relations, we have that

$$(y-x)\vec{n}fI = [(\underline{y_1} - \underline{x_1})(\underline{n_1}f + i\tilde{f}\underline{n_2}) + (f\underline{n_2} - i\underline{n_1}\tilde{f})(\underline{y_2} - \underline{x_2})]I. \tag{2.1}$$

Let $H^{\alpha}_{\partial\Omega}$ be the set of Hölder continuous functions defined on $\partial\Omega$ with the index α. For any $\varphi \in H^{\alpha}_{\partial\Omega}$, we define $\|\varphi\|_\alpha = C(\varphi,\partial\Omega) + H(\varphi,\partial\Omega,\alpha)$, where $C(\varphi,\partial\Omega) = \max_{t\in\partial\Omega}|\varphi(t)|$, $H(\varphi,\partial\Omega,\alpha) = \sup_{t_1\neq t_2, t_1,t_2\in\partial\Omega} \frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|^\alpha}$. Obviously, $H^{\alpha}_{\partial\Omega}$ is a Banach space. And for any $f, g \in H^{\alpha}_{\partial\Omega}$, we have

$$\|f+g\|_\alpha \leq \|f\|_\alpha + \|g\|_\alpha, \quad \|fg\|_\alpha \leq 2^n \|f\|_\alpha \|g\|_\alpha.$$

Throughout this article, we suppose that Ω is a nonempty connected open subset of R^{2m} and Ω^+, Ω^- are denoted as the interior and the exterior of Ω, respectively. We assume that its boundary $\partial\Omega$ is a differentiable, oriented and compact Liapunov surface.

Since $\partial\Omega$ is Liapunov surface, from the corresponding proof in [11], we have

$$|\mathrm{d}\sigma(x)| = |\mathrm{d}s(x)| = \Big|\frac{D(\xi_1,\ldots,\xi_{2m-1})}{D(\rho_0,\varphi_1,\ldots,\varphi_{2m-2})}\Big| |\mathrm{d}\rho_0 \mathrm{d}\varphi_1 \mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{2m-2}| \leq M_1 \rho_0^{2m-2} \mathrm{d}\rho_0,$$

where $M_1 > 0$ is a real constant number.

Definition 2.1 *Let Ω, $\partial\Omega$ be the set as stated above. $f \in H^{\alpha}_{\partial\Omega}$ and $0 < \alpha < 1$. Then the integral*

$$(T[f])(x) = -\frac{1}{\omega_{2m}} \int_{\partial\Omega} \frac{(\underline{y_1} - \underline{x_1})(\underline{n_1} f + i\widetilde{f}\underline{n_2}) + (f\underline{n_2} - i\underline{n_1}\widetilde{f})(\underline{y_2} - \underline{x_2})}{|y-x|^{2m}} \mathrm{d}S_y$$

is called Cauchy-type integral operator of isotonic functions, where $\underline{y_1}$, $\underline{y_2}$, $\underline{x_1}, \underline{x_2}$, $\underline{n_1}$, $\underline{n_2}$ are defined as above, $\widetilde{f} = \sum_A f_A \widetilde{e}_A$ and $\mathrm{d}S_y$ is the area difference.

Remark (1) $(T[f])(x)$ is an isotonic function.

(2) If f is isotonic function, then we have

$$(T[f])(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in R^{2m} - \bar{\Omega}. \end{cases}$$

In addition, when $x \notin \partial\Omega$, it is clear that the integral is well defined. When $x \in \partial\Omega$, it is a singular integral. So, in the following, we give the definition of the Cauchy principal value.

Definition 2.2 *Let $\Omega, \partial\Omega$ be as stated above, $x_0 \in \partial\Omega$. Construct a sphere E with the center at x_0 and radiu $\delta > 0$, where $\partial\Omega$ is divided into two parts by E, and the part of $\partial\Omega$ lying in the interior of E is denoted by λ_δ. If $\lim_{\delta\to 0}(T[f])_\delta(x_0) = I(x_0)$, in which*

$$(T[f])_\delta(x_0) = -\frac{1}{\omega_{2m}} \int_{\partial\Omega-\lambda_\delta} \frac{(\underline{y_1} - \underline{x_1})(\underline{n_1} f + i\widetilde{f}\underline{n_2}) + (f\underline{n_2} - i\underline{n_1}\widetilde{f})(\underline{y_2} - \underline{x_2})}{|y-x|^{2m}} \mathrm{d}S_y,$$

then $I(x_0)$ is called the Cauchy principal value of singular integral $(T[f])(x_0)$ and denoted by $I(x_0) = (T[f])(x_0)$.

Lemma 2.3 ([10]) *For any x_1, y, $\underline{x_{1,1}}$, $\underline{x_{1,2}}$, $\underline{y_1}$, $\underline{y_2}$, x_2, $\underline{x_{2,1}}$, $\underline{x_{2,2}}$, we have*

$$\Big|\frac{\underline{y_1} - \underline{x_{2,1}}}{|y-x_2|^{2m}} - \frac{\underline{y_1} - \underline{x_{1,1}}}{|y-x_1|^{2m}}\Big| \leq \frac{C_0|x_2 - x_1|}{|y-x_2|^{2m}} \Big(1 + \sum_{k=1}^{2m-1} \frac{|y-x_2|^k}{|y-x_1|^k}\Big),$$

$$\Big|\frac{\underline{y_2}-\underline{x_{2,2}}}{|y-x_2|^{2m}}-\frac{\underline{y_2}-\underline{x_{1,2}}}{|y-x_1|^{2m}}\Big|\le\frac{C_0|x_2-x_1|}{|y-x_2|^{2m}}\Big(1+\sum_{k=1}^{2m-1}\frac{|y-x_2|^k}{|y-x_1|^k}\Big).$$

Lemma 2.4 ([1]) *For any $a,b\in\mathcal{C}_{0,m}$, $aI=bI$ if and only if $a=b$, where $I=\prod_{j=1}^m I_j, I_j=\frac{1}{2}(1+ie_je_{m+j})$.*

Lemma 2.5 ([10]) *Suppose $f\in H^\alpha_{\partial\Omega}$ and $0<\alpha<1$, $x_0\in\partial\Omega$. Then we have*

$$\begin{aligned}&-\frac{1}{\omega_{2m}}\int_{\partial\Omega}\frac{(\underline{y_1}-\underline{x_{0,1}})(\underline{n_1}f(x_0)+i\widetilde{f}(x_0)\underline{n_2})+(f(x_0)\underline{n_2}-i\underline{n_1}\widetilde{f}(x_0))(\underline{y_2}-\underline{x_{0,2}})}{|y-x_0|^{2m}}\mathrm{d}S_y\\&=\frac{1}{2}f(x_0).\end{aligned}$$

Lemma 2.6 ([10]) *Let $f\in H^\alpha_{\partial\Omega}, 0<\alpha<1, x_0\in\partial\Omega$, $(T[f])^+(x_0)=\lim_{x\to x_0,x\in\Omega^+}(T[f])(x)$ and $(T[f])^-(x_0)=\lim_{x\to x_0,x\in\Omega^-}(T[f])(x)$. Then we have*

$$\begin{cases}(T[f])^+(x_0)=\dfrac{1}{2}f(x_0)+(T[f])(x_0),\\(T[f])^-(x_0)=-\dfrac{1}{2}f(x_0)+(T[f])(x_0).\end{cases}$$

Lemma 2.7 ([13]) *(1) If $\varphi\in H^\alpha_{\partial\Omega}$, $0<\beta\le\alpha<1$, then $\varphi\in H^\beta_{\partial\Omega}$; (2) If $f_1(x),f_2(x)\in H^\alpha_{\partial\Omega}$, then $f_1(x)\pm f_2(x)\in H^\alpha_{\partial\Omega}$; (3) Let $f(x)=\sum_A f_A(x)e_A$. If $f_A(x)$ satisfies $f_A(x)\in H^\alpha_{\partial\Omega}$, then $f\in H^\alpha_{\partial\Omega}, 0<\alpha<1$.*

3. Some properties of the isotonic operator

Theorem 3.1 *Let Ω, $\partial\Omega$, Ω^-, $(T[f])^-$ be stated as above and $f\in H^\alpha_{\partial\Omega}$ $(0<\alpha<1)$. Then for any $x_1,x_2\in\partial\Omega$, we can obtain*

$$|(T[f])^-(x_1)-(T[f])^-(x_2)|\le JH(f,\partial\Omega,\alpha)|x_1-x_2|^\alpha,$$

where J is a positive constant independent of f.

Proof For any $x_1,x_2\in\partial\Omega$, let $|x_1-x_2|=\delta$. Then we construct a sphere E_1 centered at x_1 with radius 3δ. Then $\partial\Omega$ is divided into two parts by E_1. The part of $\partial\Omega$ lying in the interior of E_1 is denoted by $\lambda_{3\delta}$ and the other part $\partial\Omega$ lying in the outside of E_1 is denoted by $\partial\Omega\setminus\lambda_{3\delta}$. Assume that $6\delta<d$, where d is the same as the one in Liapunov surface definition. By Lemmas 2.5 and 2.6, we have

$$\begin{aligned}&(T[f])^-(x_1)-(T[f])^-(x_2)|\\&=|[(T[f])(x_1)-\frac{1}{2}f(x_1)]-[(T[f])(x_2)-\frac{1}{2}f(x_2)]|\\&=\Big|-\frac{1}{\omega_{2m}}\int_{\partial\Omega}\Big\{\frac{(\underline{y_1}-\underline{x_{1,1}})[\underline{n_1}(f(y)-f(x_1))+i(\widetilde{f}(y)-\widetilde{f}(x_1))\underline{n_2}]}{|y-x_1|^{2m}}+\\&\quad\frac{[(f(y)-f(x_1))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_1))](\underline{y_2}-\underline{x_{1,2}})}{|y-x_1|^{2m}}\Big\}\mathrm{d}S_y+\\&\quad\frac{1}{\omega_{2m}}\int_{\partial\Omega}\Big\{\frac{(\underline{y_1}-\underline{x_{2,1}})[\underline{n_1}(f(y)-f(x_2))+i(\widetilde{f}(y)-\widetilde{f}(x_2))\underline{n_2}]}{|y-x_2|^{2m}}+\end{aligned}$$

$$\frac{[(f(y)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_2))](\underline{y_2}-\underline{x_{2,2}})}{|y-x_2|^{2m}}\Big\}\mathrm{d}S_y\Big|$$
$$\leq C_1(L_1+L_2+L_3),$$

where

$$L_1=\Big|\int_{\lambda_{3\delta}}\Big\{\frac{(\underline{y_1}-\underline{x_{1,1}})[\underline{n_1}(f(y)-f(x_1))+i(\widetilde{f}(y)-\widetilde{f}(x_1))\underline{n_2}]}{|y-x_1|^{2m}}+$$
$$\frac{[(f(y)-f(x_1))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_1))](\underline{y_2}-\underline{x_{1,2}})}{|y-x_1|^{2m}}\Big\}\mathrm{d}S_y\Big|$$
$$L_2=\Big|\int_{\lambda_{3\delta}}\Big\{\frac{(\underline{y_1}-\underline{x_{2,1}})[\underline{n_1}(f(y)-f(x_2))+i(\widetilde{f}(y)-\widetilde{f}(x_2))\underline{n_2}]}{|y-x_2|^{2m}}+$$
$$\frac{[(f(y)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_2))](\underline{y_2}-\underline{x_{2,2}})}{|y-x_2|^{2m}}\Big\}\mathrm{d}S_y\Big|$$
$$L_3=\Big|\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\frac{(\underline{y_1}-\underline{x_{2,1}})[\underline{n_1}(f(y)-f(x_2))+i(\widetilde{f}(y)-\widetilde{f}(x_2))\underline{n_2}]}{|y-x_2|^{2m}}+$$
$$\frac{[(f(y)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_2))](\underline{y_2}-\underline{x_{2,2}})}{|y-x_2|^{2m}}\Big\}\mathrm{d}S_y-$$
$$\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\frac{(\underline{y_1}-\underline{x_{1,1}})[\underline{n_1}(f(y)-f(x_1))+i(\widetilde{f}(y)-\widetilde{f}(x_1))\underline{n_2}]}{|y-x_1|^{2m}}+$$
$$\frac{[(f(y)-f(x_1))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_1))](\underline{y_2}-\underline{x_{1,2}})}{|y-x_1|^{2m}}\Big\}\mathrm{d}S_y\Big|.$$

From $|\underline{y_1}-\underline{x_{1,1}}|\leq|y-x_1|$, $|\underline{y_2}-\underline{x_{1,2}}|\leq|y-x_1|$ and the local generalized sphere coordinates transformation, we obtain

$$L_1\leq\int_{\lambda_{3\delta}}\frac{|y-x_1|C_2|f(y)-f(x_1)|+C_2|f(y)-f(x_1)||y-x_1|}{|y-x_1|^{2m}}|\mathrm{d}S_y|$$
$$=\int_{\lambda_{3\delta}}\frac{2C_2|f(y)-f(x_1)|}{|y-x_1|^{2m-1}}|\mathrm{d}S_y|$$
$$=2C_2\int_{\lambda_{3\delta}}\frac{|f(y)-f(x_1)|}{|y-x_1|^{\alpha}}\frac{|y-x_1|^{\alpha}}{|y-x_1|^{2m-1}}|\mathrm{d}S_y|$$
$$\leq 2C_2\int_{\lambda_{3\delta}}\frac{H(f,\partial\Omega,\alpha)}{|y-x_1|^{2m-1-\alpha}}|\mathrm{d}S_y|$$
$$\leq 2C_2\int_0^{3\delta}\frac{H(f,\partial\Omega,\alpha)}{\rho_0^{2m-1-\alpha}}M_1\rho_0^{2m-2}\mathrm{d}\rho_0$$
$$\leq C_3\int_0^{3\delta}H(f,\partial\Omega,\alpha)\rho_0^{\alpha-1}\mathrm{d}\rho_0=J_1H(f,\partial\Omega,\alpha)|x_1-x_2|^{\alpha}.$$

Similarly, we have $L_2\leq J_2H(f,\partial\Omega,\alpha)|x_1-x_2|^{\alpha}$. For L_3, we have

$$L_3\leq\Big|\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\Big(\frac{\underline{y_1}-\underline{x_{2,1}}}{|y-x_2|^{2m}}-\frac{\underline{y_1}-\underline{x_{1,1}}}{|y-x_1|^{2m}}\Big)[\underline{n_1}(f(y)-f(x_2))+i(\widetilde{f}(y)-\widetilde{f}(x_2))\underline{n_2}]+$$

$$
\begin{aligned}
&[(f(y)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_2))]\Big(\frac{\underline{y_2}-\underline{x_{2,2}}}{|y-x_2|^{2m}}-\frac{\underline{y_2}-\underline{x_{1,2}}}{|y-x_1|^{2m}}\Big)\Big\}\mathrm{d}S_y+\\
&\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\frac{\underline{y_1}-\underline{x_{1,1}}}{|y-x_1|^{2m}}[\underline{n_1}(f(y)-f(x_2))+i(\widetilde{f}(y)-\widetilde{f}(x_2))\underline{n_2}]+\\
&[(f(y)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_2))]\frac{\underline{y_2}-\underline{x_{1,2}}}{|y-x_1|^{2m}}\Big\}\mathrm{d}S_y-\\
&\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\frac{(\underline{y_1}-\underline{x_{1,1}})[\underline{n_1}(f(y)-f(x_1))+i(\widetilde{f}(y)-\widetilde{f}(x_1))\underline{n_2}]}{|y-x_1|^{2m}}+\\
&\frac{[(f(y)-f(x_1))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_1))](\underline{y_2}-\underline{x_{1,2}})}{|y-x_1|^{2m}}\Big\}\mathrm{d}S_y\Big|\\
\le&\Big|\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\Big(\frac{\underline{y_1}-\underline{x_{2,1}}}{|y-x_2|^{2m}}-\frac{\underline{y_1}-\underline{x_{1,1}}}{|y-x_1|^{2m}}\Big)[\underline{n_1}(f(y)-f(x_2))+i(\widetilde{f}(y)-\widetilde{f}(x_2))\underline{n_2}]+\\
&[(f(y)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(y)-\widetilde{f}(x_2))]\Big(\frac{\underline{y_2}-\underline{x_{2,2}}}{|y-x_2|^{2m}}-\frac{\underline{y_2}-\underline{x_{1,2}}}{|y-x_1|^{2m}}\Big)\Big\}\mathrm{d}S_y\Big|+\\
&\Big|\int_{\partial\Omega\setminus\lambda_{3\delta}}\Big\{\frac{(\underline{y_1}-\underline{x_{1,1}})[\underline{n_1}(f(x_1)-f(x_2))+i(\widetilde{f}(x_1)-\widetilde{f}(x_2))\underline{n_2}]}{|y-x_1|^{2m}}+\\
&\frac{[(f(x_1)-f(x_2))\underline{n_2}-i\underline{n_1}(\widetilde{f}(x_1)-\widetilde{f}(x_2))](\underline{y_2}-\underline{x_{1,2}})}{|y-x_1|^{2m}}\Big\}\mathrm{d}S_y\Big|\\
=&O_1+O_2.
\end{aligned}
$$

When $y\in\partial\Omega\setminus\lambda_{3\delta}$, $|y-x_2|\ge|y-x_1|-|x_1-x_2|\ge 3\delta-\delta=2\delta$. Thus

$$\frac{1}{2}\le\Big|\frac{y-x_2}{y-x_1}\Big|\le 2.$$

Again by Lemma 2.3, we get

$$
\begin{aligned}
O_1\le&\int_{\partial\Omega\setminus\lambda_{3\delta}}C_0C_4\frac{|x_2-x_1|}{|y-x_2|^{2m}}\Big(1+\sum_{k=1}^{2m-1}\frac{|y-x_2|^k}{|y-x_1|^k}\Big)|f(y)-f(x_2)||\mathrm{d}S_y|\\
\le&C_5\int_{\partial\Omega\setminus\lambda_{3\delta}}\frac{|x_2-x_1|}{|y-x_2|^{2m}}|f(y)-f(x_2)||\mathrm{d}S_y|\\
=&C_5\int_{\partial\Omega\setminus\lambda_{3\delta}}\frac{|x_2-x_1|}{|y-x_2|^{2m}}\frac{|f(y)-f(x_2)|}{|y-x_2|^{\alpha}}|y-x_2|^{\alpha}|\mathrm{d}S_y|\\
\le&C_5H(f,\partial\Omega,\alpha)\int_{\partial\Omega\setminus\lambda_{3\delta}}\frac{1}{|y-x_2|^{2m-\alpha}}|\mathrm{d}S_y||x_2-x_1|\\
\le&C_6H(f,\partial\Omega,\alpha)\int_{3\delta}^{\infty}\frac{1}{\rho_0^{2m-\alpha}}\rho_0^{2m-2}\mathrm{d}\rho_0|x_2-x_1|\\
=&J_3H(f,\partial\Omega,\alpha)|x_1-x_2|^{\alpha}.
\end{aligned}
$$

By Lemma 2.4 and the formula (2.1), we have

$$
\begin{aligned}
O_2=&\Big|\int_{\partial\Omega\setminus\lambda_{3\delta}}\frac{(y-x_1)n[f(x_1)-f(x_2)]}{|y-x_1|^{2m}}\mathrm{d}S_y\Big|\\
=&\Big|\int_{\partial\Omega\setminus\lambda_{3\delta}}\frac{(y-x_1)}{|y-x_1|^{2m}}\mathrm{d}\sigma y[f(x_1)-f(x_2)]\Big|
\end{aligned}
$$

$$\leq C_7\Big|\int_{\partial\Omega\backslash\lambda_{3\delta}}\frac{y-x_1}{|y-x_1|^{2m}}\mathrm{d}\sigma(y)\Big||f(x_1)-f(x_2)|.$$

Using the conclusion of regular functions, we get

$$\lim_{\delta\to 0}\Big[-\frac{1}{\omega_{2m}}\int_{\partial\Omega\backslash\lambda_{3\delta}}\frac{y-x_1}{|y-x_1|^{2m}}\mathrm{d}\sigma(y)\Big]=-\frac{1}{\omega_{2m}}\int_{\partial\Omega}\frac{y-x_1}{|y-x_1|^{2m}}\mathrm{d}\sigma(y)=\frac{1}{2}.$$

Hence for any $\varepsilon>0$, there exists a number $\delta_1>0$, such that when $0<\delta<\delta_1$, the following formula is right.

$$\Big|-\frac{1}{\omega_{2m}}\int_{\partial\Omega\backslash\lambda_{3\delta}}\frac{y-x_1}{|y-x_1|^{2m}}\mathrm{d}\sigma(y)-\frac{1}{2}\Big|<\varepsilon.$$

Thus taking $\varepsilon=\frac{1}{2}$, we have

$$0<\Big|-\frac{1}{\omega_{2m}}\int_{\partial\Omega\backslash\lambda_{3\delta}}\frac{y-x_1}{|y-x_1|^{2m}}\mathrm{d}\sigma(y)\Big|<1.$$

By $f\in H^{\alpha}_{\partial\Omega}$, we get

$$O_2\leq J_4H(f,\partial\Omega,\alpha)|x_1-x_2|^{\alpha}.$$

Hence

$$|(T[f])^{-}(x_1)-(T[f])^{-}(x_2)|\leq JH(f,\partial\Omega,\alpha)|x_1-x_2|^{\alpha}. \tag{3.1}$$

In addition, when $6|x_1-x_2|\geq d$, the above inequality can also be obtained. This completes the proof. □

Remark (1) From Theorem 3.1, we obviously obtain that $(T[f])^{-}$ is Hölder continuous operator with the index α.

(2) It is easily obtained by applying Lemmas 2.6 and 2.7 that $(T[f])^{+}$, $T[f]$ are both Hölder continuous operators with the index α.

Theorem 3.2 *Let $\Omega,\partial\Omega,\Omega^{-}$ and $(T[f])^{-}$ be stated as above. Suppose $f\in H^{\alpha}_{\partial\Omega}(0<\alpha<1)$ and $\|f\|_{\alpha}\leq M$, $x_0\in\partial\Omega$. Then $\|(T[f])^{-}\|_{\alpha}\leq K\|f\|_{\alpha}$, where K is a positive constant independent of f.*

Proof By Theorem 3.1, we know

$$H((T[f])^{-},\partial\Omega,\alpha)\leq JH(f,\partial\Omega,\alpha). \tag{3.2}$$

Again from Lemma 2.4 and $-\frac{1}{\omega_{2m}}\int_{\partial\Omega}\frac{y-x_0}{|y-x_0|^{2m}}\mathrm{d}\sigma(y)=\frac{1}{2}$, we get

$$\begin{aligned}|(T[f])^{-}(x_0)|&=\Big|-\frac{1}{2}f(x_0)+(T[f])(x_0)\Big|\\&\leq J_5\Big|\frac{1}{\omega_{2m}}\int_{\partial\Omega}\frac{y-x_0}{|y-x_0|^{2m}}\mathrm{d}\sigma(y)\Big||f(x_0)-f(y)|\\&\leq J_5\cdot\frac{1}{2}\cdot 2\max_{x_0\in\partial\Omega}|f(x_0)|=J_5\max_{x_0\in\partial\Omega}|f(x_0)|.\end{aligned}$$

Thus

$$\max_{x_0\in\partial\Omega}|(T[f])^{-}(x_0)|\leq J_5\max_{x_0\in\partial\Omega}|f(x_0)|.$$

Therefore

$$C((T[f])^-, \partial\Omega) \le J_5 C(f, \partial\Omega). \tag{3.3}$$

So, by (3.2) and (3.3), we obtain

$$\|(T[f])^-\|_\alpha = C((T[f])^-, \partial\Omega) + H((T[f])^-, \partial\Omega, \alpha) \le K\|f\|_\alpha, \tag{3.4}$$

where K is a positive constant independent of f. □

Theorem 3.3 *Let $\Omega, \partial\Omega, \Omega^+, (T[f])^+$ be stated as above, $f \in H^\alpha_{\partial\Omega}(0 < \alpha < 1)$ and $\|f\|_\alpha \le M, x_0 \in \partial\Omega$. Then $\|(T[f])^+\|_\alpha \le L\|f\|_\alpha$, where L is a positive constant independent of f.*

Proof By Lemma 2.6 we get

$$|(T[f])^+(x_0)| \le |(T[f])^-(x_0)| + |f(x_0)|.$$

Hence,

$$\max_{x_0\in\partial\Omega} |(T[f])^+(x_0)| \le \max_{x_0\in\partial\Omega} |(T[f])^-(x_0)| + \max_{x_0\in\partial\Omega} |f(x_0)|.$$

Namely

$$C((T[f])^+, \partial\Omega) \le C((T[f])^-, \partial\Omega) + C(f, \partial\Omega).$$

Again from

$$|(T[f])^+(x_1) - (T[f])^+(x_2)| \le |(T[f])^-(x_1) - (T[f])^-(x_2)| + |f(x_1) - f(x_2)|,$$

we have

$$\begin{aligned} &\sup_{x_1,x_2\in\partial\Omega} \frac{|(T[f])^+(x_1) - (T[f])^+(x_2)|}{|x_1 - x_2|^\alpha} \\ &\le \sup_{x_1,x_2\in\partial\Omega} \frac{|(T[f])^-(x_1) - (T[f])^-(x_2)|}{|x_1 - x_2|^\alpha} + \sup_{x_1,x_2\in\partial\Omega} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|^\alpha}. \end{aligned}$$

Namely

$$H((T[f])^+, \partial\Omega, \alpha) \le H((T[f])^-, \partial\Omega, \alpha) + H(f, \partial\Omega, \alpha). \tag{3.6}$$

Thus by (3.4), (3.5) and (3.6), we have

$$\|(T[f])^+\|_\alpha \le \|(T[f])^-\|_\alpha + \|f\|_\alpha \le L\|f\|_\alpha,$$

where L is a positive constant independent of f. □

Theorem 3.4 *Let $\Omega, \partial\Omega, \Omega^+, (T[f])^+, (T[f])^-$ be stated as above, $f \in H^\alpha_{\partial\Omega}(0 < \alpha < 1)$ and $\|f\|_\alpha \le M$. Then $\|T[f]\|_\alpha \le A\|f\|_\alpha$, where A is a positive constant independent of f.*

Proof By Lemma 2.6, we have

$$(T[f])(x) = \frac{(T[f])^+(x) + (T[f])^-(x)}{2}, \quad x \in \partial\Omega.$$

Hence

$$\sup_{x_1,x_2\in\partial\Omega} \frac{|(T[f])(x_1) - (T[f])(x_2)|}{|x_1 - x_2|^\alpha}$$

$$\leq \frac{1}{2}\Big[\sup_{x_1,x_2\in\partial\Omega}\frac{|(T[f])^+(x_1)-(T[f])^+(x_2)|}{|x_1-x_2|^\alpha}+$$
$$\sup_{x_1,x_2\in\partial\Omega}\frac{|(T[f])^-(x_1)-(T[f])^-(x_2)|}{|x_1-x_2|^\alpha}\Big].$$

Namely

$$H(T[f],\partial\Omega,\alpha)\leq\frac{1}{2}[H((T[f])^+,\partial\Omega,\alpha)+H((T[f])^-,\partial\Omega,\alpha)].$$

In addition

$$\max_{x\in\partial\Omega}|(T[f])(x)|\leq\frac{1}{2}[\max_{x\in\partial\Omega}|(T[f])^+(x)|+\max_{x\in\partial\Omega}|(T[f])^-(x)|].$$

Namely

$$C(T[f],\partial\Omega)\leq\frac{1}{2}[C((T[f])^+,\partial\Omega)+C((T[f])^-,\partial\Omega)].$$

So by Theorems 3.2 and 3.3, we can obtain that

$$\|T[f]\|_\alpha\leq\frac{1}{2}[\|(T[f])^+\|_\alpha+\|(T[f])^-\|_\alpha]\leq\frac{1}{2}(K+L)\|f\|_\alpha=A\|f\|_\alpha, \tag{3.7}$$

where A is a positive constant independent of f. □

4. The fixed point and Mann iteration of the operator T'

Definition 4.1 *Let Ω be defined as above. Then the integral operator T'*

$$(T'[f])(x)=\lambda(T[f])(x)=\frac{-\lambda}{\omega_{2m}}\int_{\partial\Omega}\frac{(\underline{y_1}-\underline{x_1})(\underline{n_1}f+i\widetilde{f}\underline{n_2})+(f\underline{n_2}-i\underline{n_1}\widetilde{f})(\underline{y_2}-\underline{x_2})}{|y-x|^{2m}}\mathrm{d}S_y$$

is called a modified Cauchy-type integral operator of isotonic functions, where $\lambda\in R$, $0<|\lambda|<\frac{\beta}{A}$ $(0<\beta<1)$, A is the same in Theorem 3.4 and others are defined as in Definition 2.1.

Definition 4.2 ([5]) *Let X be a linear space, $B\subset X$, $T:B\to B$, and $\{\beta_n\}$ be a sequence contained in $[0,1]$. Then for a given $x_1\in B$, the sequence*

$$x_{n+1}=(1-\beta_n)x_n+\beta_nTx_n,\quad n\geq 1$$

is called Mann iterative sequence of T.

Lemma 4.3 ([12]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be nonnegative real sequences satisfying*

$$a_{n+1}\leq(1-t_n)a_n+b_n+c_n,\quad n\geq 1,$$

where $t_n\in[0,1]$, $\sum_{n=1}^\infty t_n=\infty$, $b_n=o(t_n)$ and $\sum_{n=1}^\infty c_n<\infty$. Then $\lim_{n\to\infty}a_n=0$.

Theorem 4.4 *Let $\Omega,\partial\Omega$ be defined as above and $B=\{f|f\in H^\alpha_{\partial\Omega},\|f\|_\alpha\leq M,0<\alpha<1\}$. Then T' has a unique fixed point in B.*

Proof We prove the theorem in three steps.

(1) B is a Bananch space because $B\subset H^\alpha_{\partial\Omega}$ and B is a closed subspace.

(2) By the remark of Theorem 3.1, we know $T[f]\in H^\alpha_{\partial\Omega}$. Hence $T'[f]\in H^\alpha_{\partial\Omega}$.

Again from (3.7), we can obtain

$$\|T[f]\|_\alpha\leq A\|f\|_\alpha.$$

Hence

$$\|T'[f]\|_\alpha = \|\lambda T[f]\|_\alpha < \beta\|f\|_\alpha \le M.$$

So $T' : B \to B$.

(3) By (3.7), we can get

$$\|T[f]\|_\alpha \le A\|f\|_\alpha.$$

So for any $f_1, f_2 \in B$, we have

$$\begin{aligned}\|T'[f_1] - T'[f_2]\|_\alpha &= |\lambda|\|T[f_1] - T[f_2]\|_\alpha \le |\lambda|A\|f_1 - f_2\|_\alpha \\ &< \beta\|f_1 - f_2\|_\alpha.\end{aligned}$$

Namely, T' is a contraction mapping on B. Therefore, we can obtain that T' has a unique fixed point on B by the contract mapping principle. □

Remark Let $\Omega, \partial\Omega$ be defined as above and $B = \{f|f \in H^\alpha_{\partial\Omega}, \|f\|_\alpha \le M, 0 < \alpha < 1\}$. Then there exists a unique $f \in B$ satisfying $T[f] = \frac{1}{\lambda}f$.

Theorem 4.5 *Let $\{\beta_n\}$ be a sequence contained in $[0,1]$, $\sum_{n=1}^\infty \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$. Then for any given $f_1 \in B$, we can define an iterative sequence $\{f_n\}$ as follows*

$$f_{n+1} = (1-\beta_n)f_n + \beta_n T'[f_n], \quad n \ge 1.$$

Then $\{f_n\}$ strongly converges to the unique fixed point of T'.

Proof From Theorem 4.4, we know T' has a unique fixed point f, namely $T'[f] = f$. Thus we can obtain

$$\begin{aligned}\|f_{n+1} - f\|_\alpha &= \|(1-\beta_n)f_n + \beta_n T'[f_n] - f\|_\alpha \\ &= \|(1-\beta_n)(f_n - f) + \beta_n(T'[f_n] - f)\|_\alpha \\ &= \|(1-\beta_n)(f_n - f) + \beta_n(T'[f_n] - T^{'}[f])\|_\alpha \\ &\le (1-\beta_n)\|f_n - f\|_\alpha + \beta_n\beta\|f_n - f\|_\alpha \\ &= [1 - \beta_n(1-\beta)]\|f_n - f\|_\alpha.\end{aligned}$$

Let $t_n = \beta_n(1-\beta)$, $b_n = 0$ and $c_n = 0$. Then we know $t_n \in [0,1]$, $\sum_{n=1}^\infty t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=1}^\infty c_n < \infty$. So by Lemma 4.3, we can get that $\{f_n\}$ strongly converges to the unique fixed point of T'. □

Remark Let $\{\beta_n\}$ be a sequence contained in $[0,1]$, $\sum_{n=1}^\infty \beta_n = \infty$ and $\lim_{n\to\infty} \beta_n = 0$. For any given $f_1 \in B$, define the same sequence as in Theorem 4.5. Then the sequence $\{\frac{1}{\lambda}f_n\}$ strongly converges to the solution of the equation $T[f] = \frac{1}{\lambda}f$.

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