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Eigenvalue Estimates for Complete Submanifolds in the Hyperbolic Spaces

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Abstract In this paper, we study upper bounds of the first eigenvalue of a complete noncompact submanifold in an $(n+p)$-dimensional hyperbolic space $\mathbb{H}^{n+p}$. In particular, we prove that the first eigenvalue of a complete submanifold in $\mathbb{H}^{n+p}$ with parallel mean curvature vector H and finite $L^q(q \geq n)$ norm of traceless second fundamental form is not more than $\frac{(n-1)^2(1-|H|^2)}{4}$. We also prove that the first eigenvalue of a complete hypersurfaces which has finite index in $\mathbb{H}^{n+1}(n \leq 5)$ with constant mean curvature vector H and finite $L^q(2(1-\sqrt{\frac{2}{n}}) < q < 2(1+\sqrt{\frac{2}{n}}))$ norm of traceless second fundamental form is not more than $\frac{(n-1)^2(1-|H|^2)}{4}$.

Keywords finite L^q norm curvature; first eigenvalue; hyperbolic space; stable hypersurface.

MR(2010) Subject Classification 53C40; 53C21

1. Introduction

Let $\mathbb{H}^{n+p}$ be an $(n+p)$-dimensional hyperbolic space of constant curvature -1. Let M^n be an n-dimensional complete oriented submanifold in $\mathbb{H}^{n+p}$. Fix a point $x \in M$ and choose a local orthonormal frame $\{e_1, e_2, \ldots, e_{n+p}\}$ such that, restricted to M, $\{e_1, e_2, \ldots, e_n\}$ are tangent fields. For each α, $n+1 \leq \alpha \leq n+p$, define a Weingarten transform $A_\alpha\colon T_xM \to T_xM$ by

$$\langle A_\alpha X, Y\rangle = \langle \widetilde{\nabla}_X Y, e_\alpha\rangle,$$

where X, Y are tangent fields and $\widetilde{\nabla}$ is the Riemannian connection on $\mathbb{H}^{n+p}$. We denote by H the mean curvature vector of M, i.e.,

$$H = \frac{1}{n}\sum_{\alpha=n+1}^{n+p} TrA_\alpha e_\alpha.$$

We say that M has parallel mean curvature vector if $\nabla^\perp H = 0$. Note that this condition implies $|H|$ is constant on M^n, and if $p = 1$ then the two conditions are equivalent. It is easy to see that the minimal submanifold has parallel mean curvature vector. For α, $n+1 \leq \alpha \leq n+p$, define a

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bilinear map $\phi_\alpha\colon T_xM \to T_xM$ by

$$\langle \phi_\alpha X, Y\rangle = \langle X, Y\rangle\langle H, e_\alpha\rangle - \langle A_\alpha X, Y\rangle,$$

and define a bilinear map $\phi\colon T_xM \times T_xM \to T_xM^\perp$ by

$$\phi(X,Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi_\alpha X, Y\rangle e_\alpha.$$

It is easy to see that the tensor ϕ is traceless. We have

$$|A|^2 = |\phi|^2 + n|H|^2,$$

where A denotes the second fundamental form of M.

In this paper, we study the upper bounds of the first eigenvalue of the Laplace operator on a complete stable hypersurface in $\mathbb{H}^{n+p}$ with finite L^q norm curvature. To state some results, we recall some notations and definitions.

Definition 1 *Let $i\colon M^n \to N^{n+1}$ be an isometric immersion of an orientable manifold M with constant mean curvature vector. The immersion i is called weakly stable if*

$$\int_M \left[|\nabla f|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2)f^2\right] \geq 0 \tag{1}$$

for any $f \in C_0^\infty(M)$ satisfying $\int_M f = 0$, where ∇f is the gradient of f in the induced metric of M, Ric *is the Ricci tensor of N and ν is the unit normal vector field of M, while i is called strongly stable if (1) holds for any $f \in C_0^\infty(M)$.*

Definition 2 *The first eigenvalue of a Riemannian manifold M, is defined to be*

$$\lambda_1(M) = \inf_f \frac{\int_M |\nabla f|^2}{f^2}, \tag{2}$$

where the infimum is taken over all compactly supported Lipschitz functions on M.

If M is a complete noncompact Riemannian manifold, by the Domain Monotonicity Principle, $\lambda_1(M) = \lim_{R\to+\infty} \lambda_1(B_p(R))$, where $B_p(R) \subset M$ is some geodesic ball with radius R and center p. It is easy to see that $\lambda_1(M) \geq 0$. According to Schoen and Yau [12], it is an important question to find conditions which will imply that $\lambda_1(M) > 0$. In this direction, Mckean [11] proved that if M is an n dimensional complete simply connected manifold with sectional curvature bounded above by $-k^2$ for some non-zero constant k, then $\lambda_1(M) \geq \frac{(n-1)^2k^2}{4}$. It was proved by Cheung and Leung [4] that for an n-dimensional complete submanifold M in $\mathbb{H}^{n+p}$ with bounded mean curvature $|H| \leq \alpha < \frac{n-1}{n}$, then $\lambda_1(M) \geq \frac{(n-1-n\alpha)^2}{4}$. The result due to Castillon [3] is that the first eigenvalue of a complete hypersurface in $\mathbb{H}^{n+1}$ with constan mean curvature $|H| < 1$ and finite L^n norm of traceless second fundamental form is not less than $\frac{(n-1)^2(1-|H|^2)}{4}$. Candel [2] proved that the first eigenvalue of a complete simply connected stable minimal surface in $\mathbb{H}^3$ satisfies $\frac{1}{4} \leq \lambda_1(M) \leq \frac{4}{3}$. Recently, Seo [13] showed that the first eigenvalue of a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$ with finite L^2 norm of the second fundamental form is not more than n^2.

Throughout this article, we always assume that M is a complete, non-compact, connected Riemannian manifold without boundary. In this case, we will simply say that M is a complete manifold.

Our main results in this paper are stated as follows.

Theorem 1 *Let M^n be a complete submanifold with parallel mean curvature vector H in $\mathbb{H}^{n+p}$. For $q \geq n$, if*

$$\int_M |\phi|^q < +\infty,$$

then $\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2)}{4}$.

Remark We do not assume $|H| \leq 1$ in Theorem 1, because for a complete submanifold with parallel mean curvature vector H in $\mathbb{H}^{n+p}$ with $\int_M |\phi|^q < +\infty$ $(q \geq n)$, by Theorem 6.2 of [1] we have $|H| \leq 1$.

Theorem 2 *Let M^n $(n \leq 5)$ be a complete weakly stable hypersurface in $\mathbb{H}^{n+1}$ with constant mean curvature vector H. For $(1-\sqrt{\frac{2}{n}}) < d < (1+\sqrt{\frac{2}{n}})$, if*

$$\int_M |\phi|^{2d} < +\infty,$$

then $\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2)}{4}$.

Remark When M is a complete hypersurface in $\mathbb{H}^{n+1}$ with constant mean curvature and finite index, the assertions of Theorem 2 and Proposition 1 still hold for a result of [6].

Corollary 1 *Let M^n $(n \leq 5)$ be a complete weakly stable hypersurface in $\mathbb{H}^{n+1}$ with constant mean curvature vector H. If*

$$\int_M |\phi|^d < +\infty, \quad d = 1, 2, 3,$$

then $\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2)}{4}$.

By Cheung and Leung's result and Theorem 2, we get the following corollary.

Corollary 2 *Let M^n $(n \leq 5)$ be a complete weakly stable minimal hypersurface in $\mathbb{H}^{n+1}$. For $2(1-\sqrt{\frac{2}{n}}) < d < 2(1+\sqrt{\frac{2}{n}})$, if*

$$\int_M |A|^d < +\infty,$$

then $\lambda_1(M) = \frac{(n-1)^2}{4}$.

Corollary 3 *Let M^2 be a complete weakly stable minimal hypersurface in $\mathbb{H}^3$. For any positive number p, if*

$$\int_M |A|^p < +\infty,$$

then $\lambda_1(M) = \frac{1}{4}$.

Theorem 3 *Let M^n $(n \geq 6)$ be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$. For $2(1-\sqrt{\frac{2}{n}}) < d \leq 2(1-\frac{2}{n})$, if*

$$\int_M |A|^d < +\infty,$$

then $\lambda_1(M) \leq n(n-2)$.

2. Proofs of main theorems

In [9, 14], it was proved that the following estimate holds for Ricci curvature of a submanifold M in $\mathbb{H}^{n+p}$.

$$\mathrm{Ric} \geq \frac{n-1}{n}\Big(-n+2n|H|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\sqrt{|A|^2-n|H|^2} - |A|^2\Big).$$

Applying the above inequality to the traceless second fundamental form $|\phi|$ and using the identity $|A|^2 = |\phi|^2 + n|H|^2$, we get

$$\mathrm{Ric} \geq -(n-1) + (n-1)|H|^2 - \frac{(n-2)\sqrt{n(n-1)}|\phi||H|}{n} - \frac{(n-1)|\phi|^2}{n}. \tag{3}$$

Proof of Theorem 1 By Proposition 6.1 and Theorem 6.2 of [1], $|H| \leq 1$ and for all $\epsilon > 0$ there exists a compact set Ω such that $|\phi| < \epsilon$ in $M \setminus \Omega$. By (3), we obtain that $Ric(x) \geq -(n-1)(1-|H|^2+\epsilon')$ for any $x \in M \setminus B_p(R_0)$, where ϵ' depends only on $\epsilon, |H|$ and n. Then from the proof of Heintze-Karcher's comparison theorem [8], we have

$$V(r) \leq C(n)\exp[(n-1)\sqrt{1-|H|^2+\epsilon'}r].$$

If $\lambda_1(M) > \frac{(n-1)^2(1-|H|^2+\epsilon')}{4}$, then it follows from Theorem 1.4 of [10]

$$V(r) \geq C\exp(2\sqrt{\lambda_1(M)}r) > C\exp[(n-1)\sqrt{1-|H|^2+\epsilon'}r],$$

leading to a contradiction. So we have $\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2+\epsilon')}{4}$. By the arbitrariness of ϵ, we get $\lambda_1(M) \leq \frac{(n-1)^2(1-|H|^2)}{4}$. $\square$

By Castillon's result and Theorem 1, we get the following corollary.

Corollary 4 *Let M be a complete hypersurface in $\mathbb{H}^{n+1}$ with constant mean curvature vector H. If*

$$\int_M |\phi|^n < +\infty,$$

then

$$\lambda_1(M) = \frac{(n-1)^2(1-|H|^2)}{4}.$$

Remark By Corollary 4, we have $\lambda_1(\mathbb{H}^n) = \frac{(n-1)^2}{4}$, which has been proved by Mckean in [11].

Before we prove Theorems 2 and 3, we need the following Proposition 1. Although Proposition 1 was proved in [7], for completeness, we still include it.

Proposition 1 ([7]) *Let M be a complete weakly stable hypersurface in $\mathbb{H}^{n+1}$ with constant* *mean curvature. For* $2(1-\sqrt{\frac{2}{n}})<d<2(1+\sqrt{\frac{2}{n}})$, *if*

$$\int_M |\phi|^d < +\infty,$$

then

$$\int_M |\phi|^{d+2} < +\infty.$$

Proof If M is a hypersurface with constant mean curvature vector H in $\mathbb{H}^{n+1}$, Cheung and Zhou [5] got the following Simon's inequality:

$$|\phi|\Delta|\phi| \geq \frac{2}{n}|\nabla|\phi||^2 - |\phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\phi|^3 + n(|H|^2-1)|\phi|^2. \tag{4}$$

By computation of (4), we obtain that

$$|\phi|^\alpha\Delta|\phi|^\alpha \geq (1-\frac{n-2}{n\alpha})|\nabla|\phi|^\alpha|^2 - \alpha|\phi|^{2\alpha+2} - \alpha\frac{n(n-2)}{\sqrt{n(n-1)}}H|\phi|^{2\alpha+1} + \alpha n(H^2-1)|\phi|^{2\alpha},$$

where α is a positive constant. Take $a=\frac{n(n-2)}{\sqrt{n(n-1)}}H$ and $b=n(H^2-1)$, then the above inequality can be rewritten as

$$|\phi|^\alpha\Delta|\phi|^\alpha \geq (1-\frac{n-2}{n\alpha})|\nabla|\phi|^\alpha|^2 - \alpha|\phi|^{2\alpha+2} - \alpha a|\phi|^{2\alpha+1} + \alpha b|\phi|^{2\alpha}. \tag{5}$$

From the definition of weak stability, the index of M is at most 1. We know from a result of [6] that if M has finite index, then it is strongly stable outside a compact set, i.e., we have a compact set $D\subset M$ such that

$$\int_{M\setminus D}|\nabla f|^2 \geq \int_{M\setminus D}(|\phi|^2+n(H^2+c))f^2 = \int_{M\setminus D}(|\phi|^2+b)f^2 \tag{6}$$

for all smooth functions f compactly supported in $M\setminus D$.

Let $q\geq 0$ and $f\in C_0^\infty(M\setminus D)$. Multiplying (5) by $|\phi|^{2q\alpha}f^2$ and integrating over $M\setminus D$, we obtain

$$\begin{aligned}
&(1-\frac{n-2}{n\alpha})\int_{M\setminus D}|\nabla|\phi|^\alpha|^2|\phi|^{2q\alpha}f^2\\
&\leq \alpha\int_{M\setminus D}|\phi|^{2(q+1)\alpha}f^2|\phi|^2 + \alpha a\int_{M\setminus D}|\phi|^{2(q+1)\alpha}f^2|\phi| -\\
&\quad \alpha b\int_{M\setminus D}|\phi|^{2(q+1)\alpha}f^2 + \int_{M\setminus D}|\phi|^{(2q+1)\alpha}f^2\Delta|\phi|^\alpha\\
&= \alpha\int_{M\setminus D}|\phi|^{2(q+1)\alpha}f^2|\phi|^2 + \alpha a\int_{M\setminus D}|\phi|^{2(q+1)\alpha}f^2|\phi| - \alpha b\int_{M\setminus D}|\phi|^{2(q+1)\alpha}f^2 -\\
&\quad (2q+1)\int_{M\setminus D}|\nabla|\phi|^\alpha|^2|\phi|^{2q\alpha}f^2 - 2\int_{M\setminus D}|\phi|^{(2q+1)\alpha}f\langle\nabla f,\nabla|\phi|^\alpha\rangle,
\end{aligned}$$

which gives

$$(2(q+1)-\frac{n-2}{n\alpha})\int_{M\setminus D}|\nabla|\phi|^\alpha|^2|\phi|^{2q\alpha}f^2$$

$$\leq \alpha \int_{M\setminus D} |\phi|^{2(q+1)\alpha} f^2 |\phi|^2 + \alpha a \int_{M\setminus D} |\phi|^{2(q+1)\alpha} f^2 |\phi| -$$
$$\alpha b \int_{M\setminus D} |\phi|^{2(q+1)\alpha} f^2 - 2 \int_{M\setminus D} |\phi|^{(2q+1)\alpha} f \langle \nabla f, \nabla |\phi|^\alpha \rangle. \tag{7}$$

Using the Cauchy-Schwarz inequality, we can rewrite (7) as

$$(2(q+1) - \frac{n-2}{n\alpha} - \epsilon) \int_{M\setminus D} |\nabla |\phi|^\alpha|^2 |\phi|^{2q\alpha} f^2$$
$$\leq \alpha \int_{M\setminus D} |\phi|^{2(q+1)\alpha} f^2 |\phi|^2 + \alpha a \int_{M\setminus D} |\phi|^{2(q+1)\alpha} f^2 |\phi| -$$
$$\alpha b \int_{M\setminus D} |\phi|^{2(q+1)\alpha} f^2 + \frac{1}{\epsilon} \int_{M\setminus D} |\phi|^{2(q+1)\alpha} |\nabla f|^2,$$

for some positive constant ϵ.

On the other hand, replacing f by $|\phi|^{(1+q)\alpha} f$ in the inequality (6) and using the Cauchy-Schwarz inequality, we have

$$\int_{M\setminus D} (|\phi|^2 + b) f^2 |\phi|^{2(1+q)\alpha} \leq \int_{M\setminus D} |\nabla (|\phi|^{(1+q)\alpha} f)|^2$$
$$\leq (1+q)(1+q+\epsilon) \int_{M\setminus D} |\nabla |\phi|^\alpha|^2 |\phi|^{2q\alpha} f^2 +$$
$$(1 + \frac{1+q}{\epsilon}) \int_{M\setminus D} |\phi|^{2(q+1)\alpha} |\nabla f|^2. \tag{9}$$

If $(2(q+1) - \frac{n-2}{n\alpha} - \epsilon) > 0$, subtracting (9) $\times$ $(2(q+1) - \frac{n-2}{n\alpha} - \epsilon)$ from (8) $\times$ $(1+q)(1+q+\epsilon)$ and using the Cauchy-Schwarz inequality yields

$$E \int_M |\phi|^2 f^2 |\phi|^{2(1+q)\alpha} \leq F \int_M f^2 |\phi|^{2(1+q)\alpha} + G \int_M |\phi|^{2(q+1)\alpha} |\nabla f|^2, \tag{10}$$

where

$$E = 2(q+1) - \frac{n-2}{n\alpha} - \epsilon - (1+q)(1+q+\epsilon)\alpha - \frac{\epsilon}{2}|a|(1+q)(1+q+\epsilon)\alpha,$$
$$F = \frac{1}{2\epsilon}|a|(1+q)(1+q+\epsilon)\alpha - b(1+q)(1+q+\epsilon)\alpha - b[2(q+1) - \frac{n-2}{n\alpha} - \epsilon].$$

Let $(1+q)\alpha = \frac{d}{2}$. Thus $(1 - \sqrt{\frac{2}{n}}) < (1+q)\alpha < (1 + \sqrt{\frac{2}{n}})$. It is easy to see that $\big(2(q+1) - \frac{n-2}{n\alpha}\big) > 0$ and $2(q+1) - \frac{n-2}{n\alpha} - (1+q)^2\alpha > 0$, and then we can choose $\epsilon > 0$ sufficiently small so that $(2(q+1) - \frac{n-2}{n\alpha} - \epsilon) > 0$ and $E > 0$. So from (10) we have a positive constant C_1 such that

$$\int_{M\setminus D} |\phi|^{d+2} f^2 \leq C_1 \Big(\int_{M\setminus D} |\phi|^d f^2 + \int_{M\setminus D} |\phi|^d |\nabla f|^2 \Big). \tag{11}$$

We can choose R_0 such that D is contained in some geodesic ball $B_p(R_0)$. For $R > R_0 + 1$, let us choose f satisfying the properties that

$$f(x) = \begin{cases} 0 & \text{on} \quad B_p(R_0), \\ 1 & \text{on} \quad B_p(R) \setminus B_p(R_0), \\ 0 & \text{on} \quad M \setminus B_p(2R), \end{cases}$$

and $|\nabla f| \le C_2$, where C_2 is a constant. Since $\int_{M\setminus D}|\phi|^d < +\infty$ and R can be arbitrarily large, from (11) we conclude that $\int_{M\setminus D}|\phi|^{d+2} < +\infty$. Hence we obtain that

$$\int_M |\phi|^d < +\infty \Rightarrow \int_M |\phi|^{d+2} < +\infty. \tag{12}$$

□

Proof of Theorem 2 First, we prove that $\int_M |\phi|^5 < +\infty$. It is easy to see that $3 \in (2(1-\sqrt{\frac{2}{n}}), 2(1+\sqrt{\frac{2}{n}}))$ for $n \le 7$.

1) When $d=3$, by Proposition 1 we get $\int_M |\phi|^5 < +\infty$ since $\int_M |\phi|^3 < +\infty$.

2) When $2-2\sqrt{\frac{2}{n}} < d < 3$, there exist two numbers $p=\frac{2}{d-1} > 1$ and $q=\frac{2}{3-d} > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Since $\int_M |\phi|^d < +\infty$, by Proposition 1 we get $\int_M |\phi|^{d+2} < +\infty$. By the Hölder inequality, we obtain

$$\int_M |\phi|^3 \le \Big(\int_M (|\phi|^{\frac{d}{p}})^p\Big)^{\frac{1}{p}}\Big(\int_M (|\phi|^{\frac{d+2}{q}})^q\Big)^{\frac{1}{q}} < +\infty.$$

By 1), we get $\int_M |\phi|^5 < +\infty$.

3) When $3 < d < 2+2\sqrt{\frac{2}{n}}$, there exist two numbers $p=\frac{2}{d-3} > 1, q=\frac{2}{5-d} > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Since $\int_M |\phi|^d < +\infty$, by Proposition 1 we get $\int_M |\phi|^{d+2} < +\infty$. By the Hölder inequality, we obtain

$$\int_M |\phi|^5 \le \Big(\int_M (|\phi|^{\frac{d}{p}})^p\Big)^{\frac{1}{p}}\Big(\int_M (|\phi|^{\frac{d+2}{q}})^q\Big)^{\frac{1}{q}} < +\infty.$$

By Theorem 1 and $\int_M |\phi|^5 < +\infty$, we obtain $\lambda_1(M) \le \frac{(n-1)^2(1-|H|^2)}{4}$. □

Theorem 4 *Let M^n be a complete stable minimal hypersurface in $\mathbb{H}^{n+1}$. For $(1-\sqrt{\frac{2}{n}}) < d < (1+\sqrt{\frac{2}{n}})$, if*

$$\int_M |A|^{2d} < +\infty,$$

then

$$\lambda_1(M) \le I \triangleq \frac{4n^2d^2}{\sqrt{[2nd^2-2nd+(n-2)]^2+4nd^2[2nd-nd^2-(n-2)]}-[2nd^2-2nd+(n-2)]}.$$

Remark When $d=1$, Theorem 4 is reduced to Theorem 2.2 in [13]. It is easy to see that $d=\frac{n-2}{n}$ can minimize I.

Proof Now, $\phi = A$ for $H=0$ in (5). Let $q \ge 0$ and $f \in C_0^\infty(M)$. Multiplying (5) by $|A|^{2q\alpha}f^2$ and integrating over M, we obtain as (8)

$$\Big(2(q+1)-\frac{n-2}{n\alpha}-\epsilon\Big)\int_M |\nabla|A|^\alpha|^2|A|^{2q\alpha}f^2 \le \alpha\int_M |A|^{2(q+1)\alpha}f^2(|A|^2+n)+\frac{1}{\epsilon}\int_M |A|^{2(q+1)\alpha}|\nabla f|^2. \tag{13}$$

On the other hand, replacing f by $|A|^{(1+q)\alpha}f$ in the stability inequality (1) and using the

Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_M (|A|^2-n)f^2|A|^{2(1+q)\alpha} \leq &\int_M |\nabla(|A|^{(1+q)\alpha}f)|^2\\ =&(1+q)^2\int_M |\nabla|A|^\alpha|^2|A|^{2q\alpha}f^2+\int_M |A|^{2(q+1)\alpha}|\nabla f|^2+\\ &2(1+q)\int_M |A|^{(2q+1)\alpha}f\langle\nabla f,\nabla|A|^\alpha\rangle\\ \leq&(1+q)(1+q+\epsilon)\int_M |\nabla|A|^\alpha|^2|A|^{2q\alpha}f^2+\\ &(1+\frac{1+q}{\epsilon})\int_M |A|^{2(q+1)\alpha}|\nabla f|^2. \end{aligned} \tag{14}$$

Replacing f by $|A|^{(1+q)\alpha}f$ in (2) and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\lambda_1(M)\int_M f^2|A|^{2(1+q)\alpha} \leq &\int_M |\nabla(|A|^{(1+q)\alpha}f)|^2\\ \leq&(1+q)(1+q+\epsilon)\int_M |\nabla|A|^\alpha|^2|A|^{2q\alpha}f^2+\\ &(1+\frac{1+q}{\epsilon})\int_M |A|^{2(q+1)\alpha}|\nabla f|^2. \end{aligned} \tag{15}$$

From (14) and (15), we obtain

$$\begin{aligned}\int_M (|A|^2+n)f^2|A|^{2(1+q)\alpha} \leq &(1+q)(1+q+\epsilon)(1+\frac{2n}{\lambda_1(M)})\int_M |\nabla|A|^\alpha|^2|A|^{2q\alpha}f^2+\\ &(1+\frac{1+q}{\epsilon})(1+\frac{2n}{\lambda_1(M)})\int_M |A|^{2(q+1)\alpha}|\nabla f|^2. \end{aligned}$$

Combining with (13), we have

$$a\int_M (|A|^2+n)f^2|A|^{2(1+q)\alpha} \leq b\int_M |A|^{2(q+1)\alpha}|\nabla f|^2, \tag{16}$$

where

$$\begin{aligned} a=&2(q+1)-\frac{n-2}{n\alpha}-\epsilon-(1+\frac{2n}{\lambda_1(M)})(1+q)(1+q+\epsilon)\alpha,\\ b=&\frac{(2+q)(1+q+\epsilon)}{\epsilon}(1+\frac{2n}{\lambda_1(M)}). \end{aligned}$$

Take $(1+q)\alpha=d$. If $\lambda_1(M)>I$, then $2(q+1)-\frac{n-2}{n\alpha}-(1+\frac{2n}{\lambda_1(M)})(1+q)^2\alpha>0$. So we can choose $\epsilon>0$ sufficiently small so that $a>0$. It follows from (16) that the following inequality holds:

$$\int_M (|A|^2+n)f^2|A|^{2(1+q)\alpha} \leq C_3\int_M |A|^{2d}|\nabla f|^2, \tag{17}$$

where C_3 is a constant that depends on n,ϵ and q. Let f be a smooth function on $[0,\infty)$ such that $f\geq 0, f=1$ on $[0,R]$ and $f=0$ in $[2R,\infty)$ with $|f'|\leq\frac{2}{R}$. Then considering $f\circ r$, where r is the function in the definition of $B(R)$, we have from (17)

$$\int_{B(R)} (|A|^2+n)f^2|A|^{2(1+q)\alpha} \leq \frac{4C_1}{R^2}\int_{B(2R)\setminus B(R)} |A|^{2d}. \tag{18}$$

Let $R \to +\infty$. By assumption that $\int_M |A|^{2d} < +\infty$, from (18), we conclude $|A| = 0$, i.e., $\int_M |A|^n = 0 < +\infty$. By Corollary 4, $\lambda_1(M) = \frac{(n-1)^2}{4}$. Contradiction. We obtain $\lambda_1(M) \leq I$.□

Proof of Theorem 3 Taking $d = \frac{n-2}{n}$, we have $G = n(n-2)$. By Theorem 4, $\lambda_1(M) \leq n(n-2)$. When $(1 - \sqrt{\frac{2}{n}}) < d < (1 - \frac{2}{n})$, by Proposition 1, we know that $\int_M |\phi|^{d+2} < +\infty$. Since $\int_M |\phi|^d < +\infty$ and $\int_M |\phi|^{d+2} < +\infty$, using the Hölder inequality, we have $\int_M |\phi|^{2\frac{n-2}{n}} < +\infty$. Hence we complete the proof of Theorem 3. □

References

[1] P. BÉRARD, W. SANTOS. *Curvature estimates and stability properties of CMC-submanifolds in space forms*. Mat. Contemp., 1999, **17**: 77–97.

[2] A. CANDEL. *Eigenvalue estimates for minimal surfaces in hyperbolic space*. Trans. Amer. Math. Soc., 2007, **359**(8): 3567–3575.

[3] P. CASTILLON. *Spectral properties of constant mean curvature submanifolds in hyperbolic space*. Ann. Global Anal. Geom., 1999, **17**(6): 563–580.

[4] L. F. CHEUNG, P. F. LEUNG. *Eigenvalue estimates for submanifolds with bounded mean curvature in the hyperbolic space*. Math. Z., 2001, **236**(3): 525–530.

[5] L. F. CHEUNG, Detang ZHOU. *Stable constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$ and $\mathbb{H}^{n+1}$*. Bull. Braz. Math. Soc. (N.S.), 2005, **36**(1): 99–114.

[6] D. FISCHER-COLBRIE. *On complete minimal surfaces with finite Morse index in three manifolds*. Invent. Math., 1985, **82**(1): 121–132.

[7] Haiping FU. *Bernstein type theorems for complete submanifolds in space forms*. Math. Nachr., 2012, **285**(2-3): 236–244.

[8] E. HEINTZE, H. KARCHER. *A general comparison theorem with applications to volume estimates for submanifolds*. Ann. Sci. École Norm. Sup. (4), 1978, **11**(4): 451–470.

[9] P. F. LEUNG. *An estimate on the ricci curvature of a submanifolds and some applications*. Proc. Amer. Math. Soc., 1992, **114**(4): 1051–1061.

[10] P. LI, Jiaping WANG. *Complete manifolds with positive spectrum*. J. Differential Geom., 2001, **58**(3): 501–534.

[11] H. P. MCKEAN. *An upper bound to the spectrum of Δ on a manifold of negative curvature*. J. Differential Geometry, 1970, **4**: 359–366.

[12] R. SCHOEN, S. T. YAU. *Lectures On Differential Geometry*. International Press, Cambridge, MA, 1994.

[13] K. SEO. *Stable minimal hypersurfaces in the hyperbolic space*. J. Korean Math. Soc., 2011, **48**(2): 253–266.

[14] K. SHIOHAMA, Hongwei XU. *The topological sphere theorem for complete submanifolds*. Compositio Math., 1997, **107**(2): 221–232.