Journal of Mathematical Research with Applications
Sept., 2013, Vol. 33, No. 5, pp. 607–615
DOI:10.3770/j.issn:2095-2651.2013.05.010
Http://jmre.dlut.edu.cn

On Pairwise Semi-Stratifiable Spaces

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Abstract In this paper, some characterizations of pairwise semi-stratifiable spaces are given by means of pairwise g-functions and semi-continuous functions and the pairwise semi-stratifiability of topological ordered C-spaces with semi-stratifiable topology is discussed.

Keywords pairwise semi-stratifiable spaces; bitopological space; semi-continuous functions; topological ordered spaces; C-spaces.

MR(2010) Subject Classification 54E20; 54C10; 26A15

1. Introduction

It is one of the questions in general topology how to characterize the generalized metric spaces [2, 3]. Künzí and Mushaandja [9] obtained some characterizations of some topological ordered spaces and generalized metric spaces. It is well known that stratifiable spaces [1] form one of the more interesting class of generalized metric spaces. This notion has been generalized to bitopological spaces [4], and many properties have been extended. Marín and Romaguera [7] introduced the notion of monotonically normal bitopological spaces (or Pairwise monotonically normal spaces) which is a useful generalization of pairwise stratifiable spaces, and characterized pairwise monotonically normal spaces in terms of a mixed condition of insertion and extension of semi continuous functions. It is well known that a topological space is stratifiable if and only if it is monotonically normal and semi-stratifiable [8]. A bitopological space is pairwise stratifiable if and only if it is pairwise monotonically normal and pairwise semi-stratifiable [7]. In [11], Li gave some characterizations of pairwise stratifiable spaces by means of pairwise g-functions and semi-continuous functions. We know that pairwise semi-stratifiable space is also a useful generalization of pairwise stratifiable spaces. It is a natural question how to characterize pairwise semi-stratifiable spaces in terms of a mixed condition of insertion and extension of semi-continuous functions. In this paper, we give some characterizations of pairwise semi-stratifiable spaces by means of pairwise g-functions and extension of semi-continuous functions. Finally, we also discuss the pairwise semi-stratifiability of topological ordered C-spaces with semi-stratifiable topology and obtain results which is a generalization of Theorem 1 in [9].

Received June 2, 2012; Accepted September 3, 2012
Supported by the National Natural Science Foundation of China (Grant Nos. 10971185; 10971186; 11201414), Fujian Province Support College Research Plan Project (Grant No. JK2011031) and the Natural Science Foundation of Fujian Province (Grant Nos. 2012J05013; 2013J01029).
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For a bitopological space (X,τ_i,τ_j), τ_i, τ_j are topologies on X ($i,j=1,2$ and $i\neq j$), and $\tau_i^c=\{X-O: O\in\tau_i\}$. We write $\mathrm{cl}_{\tau_i} A$ for the closure of A in the topological space (X,τ_i). Similarly, we write $\mathrm{int}_{\tau_i} A$ for the interior of A in (X,τ_i) ($i=1,2$). The set of positive integers is denoted by N. We refer the readers to [10, 12] for undefined terms.

A real-valued function f defined on a topological space (X,τ_i) is τ_i-lower (τ_i-upper) semi-continuous if for each $x\in X$ and each real number r with $f(x)>r$ ($f(x)<r$), there exists a τ_i-open set $U\subset X$, and $x\in X$ such that $f(x')>r$ ($f(x')<r$) for every $x'\in U$. We write $\mathrm{LSC}_{\tau_i}(X)$ ($\mathrm{USC}_{\tau_i}(X)$) for the set of all real-valued τ_i-lower (τ_i-upper) semi-continuous functions on X into $I=[0,1]$. Let f be a real-valued τ_i-lower (τ_i-upper) semi-continuous function defined on a topological space (X,τ_i), then for any real number r, $\{x\in X: f(x)\leqslant r\}\in\tau_i^c$ ($\{x\in X: f(x)\geqslant r\}\in\tau_i^c$).

Let (X,τ_i,τ_j) be a bitopological space and $A\subset X$. We write χ_A for the *characteristic function* on A, that is, a function $\chi_A: X\to[0,1]$ defined by

$$\chi_A(x)=\begin{cases}1, & \text{if } x\in A,\\ 0, & \text{if } x\notin A.\end{cases}$$

Then $\chi_A\in\mathrm{USC}_{\tau_i}(X)$ if $A\in\tau_i^c$, and $\chi_A\in\mathrm{LSC}_{\tau_i}(X)$ if $A\in\tau_i$.

Definition 1.1 ([4]) *A bitopological space (X,τ_i,τ_j) is called pairwise stratifiable if for each $A\in\tau_i^c$, one can assign a sequence $\{D(n,A)\}_{n\in N}$ of τ_j-open sets such that*

(1) $A\subset D(n,A)$ for each $n\in N$;

(2) If $A\subset E$, then $D(n,A)\subset D(n,E)$ for each $n\in N$, where $\{D(n,E)\}_{n\in N}$ is the sequence assigned to the $E\in\tau_i^c$;

(3) $A=\bigcap_{n\in N}\mathrm{cl}_{\tau_i} D(n,A)$.

A bitopological space (X,τ_i,τ_j) is called pairwise semi-stratifiable if D satisfies the condition (2) and

(1′) $A=\bigcap_{n\in N} D(n,A)$.

It immediately follows from the Definition 1.1 that a bitopological space (X,τ_i,τ_j) is pairwise semi-stratifiable if and only if there exists an operator $D: N\times\tau_i^c\to\tau_j$ such that $D(n,A)\in\tau_j$ for each $n\in N$ and each $A\in\tau_i^c$, and satisfies the conditions (1′) and (2) in Definition 1.1.

We may assume that the operator D is monotonic with respect to n, that is, $D(n+1,A)\subset D(n,A)$ for each $n\in N$ and each $A\in\tau_i^c$.

The following lemma, included for convenience, is clearly just another way of stating the definition.

Lemma 1.2 *A topological space (X,τ_i,τ_j) is pairwise semi-stratifiable if and only if for each $V\in\tau_i$, one can assign a sequence $\{F(n,V)\}_{n\in N}$ of τ_j-closed sets such that*

(1) $V=\bigcup_{n\in N} F(n,V)$;

(2) If $V\subset G$, then $F(n,V)\subset F(n,G)$ for each $n\in N$, where $F(n,G)$ is the sequence of τ_j-closed sets assigned to the $G\in\tau_i$;

(3) $F(n,V)\subset F(n+1,V)$ for all $V\in\tau_i$ and $n\in N$.

Definition 1.3 ([7]) *A bitopological space (X, τ_i, τ_j) is called pairwise monotonically normal if for each pair (H, K) of disjoint subsets of X such that $H \in \tau_i^c$ and $K \in \tau_j^c$, one can assign a $D(K, H) \in \tau_i$ and $D(H, K) \in \tau_j$ such that*

(i) $H \subset D(H, K) \subset \mathrm{cl}_{\tau_i} D(H, K) \subset X\text{-}K$, $K \subset D(K, H) \subset \mathrm{cl}_{\tau_j} D(K, H) \subset X\text{-}H$, and

(ii) If the pair (H, K) and (H', K') satisfy $H \subset H'$ and $K' \subset K$, then $D(H, K) \subset D(H', K')$, and $D(K', H') \subset D(K, H)$.

The next result is useful in the proof of our main theorems.

Lemma 1.4 ([4]) *A bitopological space (X, τ_i, τ_j) is pairwise stratifiable if and only if it is pairwise monotonically normal and pairwise semi-stratifiable.*

2. Characterizations of pairwise semi-stratifiable spaces

We give a characterization of pairwise semi-stratifiable spaces by the pairwise g-functions. A pairwise g-function on a bitopological space X is a function $g_i : N \times X \to \tau_j$ such that for each $x \in X$ and each $n \in N$, $x \in g_i(n, x)$ and $g_i(n + 1, x) \subset g_i(n, x)$ ($i, j = 1, 2$ and $i \neq j$). In this section, all topologies are T_1.

Theorem 2.1 *For a bitopological space (X, τ_i, τ_j), the following conditions are equivalent:*

(1) X is pairwise semi-stratifiable;

(2) There exists a pairwise g-function $g_i : N \times \tau_i^c \to \tau_j$ ($i, j = 1, 2$ and $i \neq j$) which satisfies that:

(i) If $F \in \tau_i^c$, then $F = \bigcap_{n \in N} g_i(n, F)$;

(ii) If $F, H \in \tau_i^c$ and $F \subset H$, then $g_i(n, F) \subset g_i(n, H)$ for each $n \in N$, where $\{g_i(n, H)\}_{n \in N}$ is the sequence assigned to the set H.

(3) There exists a pairwise g-function $g_i : N \times X \to \tau_j$ such that if $x \in g_i(n, x_n)$, then $\{x_n\}\tau_i$-convergent to x.

Proof (1)$\Rightarrow$ (2). Let (X, τ_1, τ_2) be pairwise semi-stratifiable space and D an operator on X which satisfies the conditions (1) and (2) in Definition 1.1. For each $x \in X$, let $g_i(n, x) = \bigcap_{k \leqslant n} D(k, \{x\})$. Then $g_i : N \times X \to \tau_j$ is a g-function and $g_i(n, F) = \bigcup\{g_i(n, y) : y \in F\}$. The g_i satisfies the required conditions.

(2)$\Rightarrow$ (1). Suppose (2) holds. For each $F \in \tau_i^c$ and each $n \in N$, define $D(n, F) = \bigcup\{g_i(n, t) : t \in F\}$, then $D(n, F) \in \tau_j$. The function D defined in this way is a pairwise semi-stratifiable operator in (X, τ_i, τ_j).

(1)$\Rightarrow$ (3). Assume $g_i : N \times X \to \tau_j$ is defined as in (1)$\Rightarrow$ (2).

Let $x, x_n \in X$ and $x \in g_i(n, x_n)$ for each $n \in N$. If $\{x_n\}$ is not τ_i-convergent to x, then there exists a subsequence $\{x_{n_m}\}$ of $\{x_n\}$ such that $F = \{x_{n_m} : m \in N\}$ is a τ_i-closed subset of X and $x \notin F$. By Definition 1.1, $F = \bigcap_{n \in N} D(n, F)$ and

$$\bigcap_{n \in N} D(n, F) = \bigcap_{n \in N} \bigcup\{g_i(n, y) : y \in F\} = \bigcap_{n \in N} g_i(n, F),$$

and therefore, $x \notin \bigcup\{g_i(n_0,y) : y \in F\} = g(n_0,F)$ for each $n_0 \in N$. Thus $x \notin g_i(k,F)$ for each $k \geqslant n_0$. We can choose an $n_m \in N$ and $n_m \geqslant n_0$, $x \notin g_i(n_m,y)$ for each $y \in F$ and $x \notin g_i(n_m,x_{n_m})$, which is a contradiction.

(3)⇒ (1). Suppose there exists pairwise g-function $g_i : N \times X \to \tau_j$ that satisfies the condition (3) in the theorem. Let $F \subset X$ be τ_i-closed, and define

$$D(n,F) = g_i(n,F) = \bigcup_{n\in N}\{g_i(n,t) : t \in F\}.$$

Then $D(n,F) \in \tau_j$ for each $n \in N$. Thus D is an operation on X which satisfies the conditions $(1')$ and (2) in Definition 1.1. In fact, it is clear that (2) holds. For $(1')$, we need only prove that $\bigcap_{n\in N} D(n,F) \subset F$.

If $\bigcap_{n\in N} D(n,F)$-$F \neq \emptyset$, there exists $x \in X$ such that $x \notin F$ and $x \in g_i(n,x_n)$ for some $x_n \in F$ and each $n \in N$. By the condition (3) in the theorem, we have the sequence $\{x_n\}$ is τ_i-convergent to x, and therefore $x \in \mathrm{cl}_{\tau_i}F = F$, which is a contradiction. □

Theorem 2.2 *A bitopological space (X,τ_i,τ_j) is pairwise semi-stratifiable if and only if for each partially ordered set $(\mathbb{H},<)$ and a map $H : N \times \mathbb{H} \to \tau_i^c$ such that*

(1) $H(n+1,h) \subset H(n,h)$ for each $h \in \mathbb{H}$ and each $n \in N$;

(2) For arbitrary $h_1,h_2 \in H$, if $h_1 \leqslant h_2$, then $H(n,h_2) \subset H(n,h_1)$ for each $n \in N$,

there is a map $G : N \times \mathbb{H} \to \tau_j$, such that (1) and (2) hold for G, and there hold the following (a)–(b)

(a) $H(n,h) \subset G(n,h)$ for all $h \in \mathbb{H}$ and all $n \in N$;

(b) $\bigcap_{n\in N} H(n,h) = \bigcap_{n\in N} G(n,h)$ for all $h \in \mathbb{H}$.

Proof Necessity. Let (X,τ_i,τ_j) be a semi-pairwise stratifiable space and operator F be defined by Lemma 1.2. We show that the map $G : N \times \mathbb{H} \to \tau_j$ defined by

$$G(n,h) = X - F(n, X - H(n,h)),$$

satisfies the conditions of the theorem. By the properties of F and H, one can easily verify that the conditions (1) and (2) hold for G. Since $H(n,h) \in \tau_i^c$ for each $h \in \mathbb{H}$ and all $n \in N$, we have $X - H(n,h) \in \tau_i$. By the condition (1) in Lemma 1.2, the equality $\bigcup_{n\in N} F(n,V) = V$ holds for all $V \in \tau_i$, and therefore, we have $F(n,V) \subset V$ for each $n \in N$. Then $X - H(n,h) \supset F(n, X - H(n,h))$, and so $H(n,h) \subset G(n,h)$ for each $h \in \mathbb{H}$ and each $n \in N$.

So we need only to show that $\bigcap_{n\in N} H(n,h) \supset \bigcap_{n\in N} G(n,h)$ for each $h \in \mathbb{H}$. If $x \notin \bigcap_{n\in N} H(n,h)$, then $x \notin H(m_0,h)$ for some $m_0 \in N$. Consequently, $x \in F(n_0, X - H(m_0,h))$ for some $n_0 \in N$ by $X - H(m_0,h) = \bigcup_{n\in N} F(n, X - H(m_0,h))$. Let $m = \max\{n_0,m_0\}$. Then

$$x \in F(n_0, X - H(m_0,h)) \subset F(m, X - H(m_0,h)) \subset F(m, X - H(m,h)).$$

Thus $x \in F(m, X - H(m,h))$. But $F(m, X - H(m,h)) \cap G(m,h) = \emptyset$, hence $x \notin G(m,h)$, so $x \notin \bigcap_{n\in N} G(n,h)$, which proves the necessity.

Sufficiency. For each $U \in \tau_i$, we consider the map $H : N \times \tau_i \to \tau_i^c$ defined by $H(n,U) = X - U$. One can easily verify that H satisfies the conditions (1) and (2) above. So there is a map

$G: N\times\tau_i\to\tau_j$ such that the conditions (1) and (2) hold for G. Moreover, $H(n,U)\subset G(n,U)$ for all $n\in N$ and all $U\in\tau_i$, and $\bigcap_{n\in N}H(n,U)=\bigcap_{n\in N}G(n,U)$. Let $F(n,U)=X-G(n,U)$. Then the map $F: N\times\tau_i\to\tau_j^c$ satisfies the conditions in Lemma 1.2. In fact, it is clear that the condition (2) holds. Since $H(n,U)\subset G(n,U)$, we have

$$F(n,U)=X-G(n,U)\subset X-H(n,U)=U$$

for each $n\in N$ and

$$\begin{aligned}U&=X-(X-U)=X-\bigcap_{n\in N}H(n,U)=X-\bigcap_{n\in N}G(n,U)\\&=\bigcup_{n\in N}(X-G(n,U))=\bigcup_{n\in N}F(n,U).\end{aligned}$$

Therefore the condition (1) holds. So X is a pairwise semi-stratfiable space. □

Theorem 2.3 *A bitopological space (X,τ_i,τ_j) is pairwise semi-stratifiable if and only if for each function $f\in \mathrm{LSC}_{\tau_i}(X)$, one assigns a function $h(f)\in \mathrm{USC}_{\tau_j}(X)$ such that*

(1) $0\leqslant h(f)\leqslant f$ and $0<h(f)(x)<f(x)$ whenever $f(x)>0$.

(2) $h(f)\leqslant h(f')$ whenever $f\leqslant f'$.

Proof Suppose that (X,τ_i,τ_j) is pairwise semi-stratifiable. For each $n\in N$ and $f\in \mathrm{LSC}_{\tau_i}(X)$, let

$$H(n,f)=\{x\in X: f(x)\leqslant 1/2^{n-1}\}.$$

Then $H(n,h)\in\tau_i^c(X)$. So the equality defines a map

$$H: N\times \mathrm{LSC}_{\tau_i}(X)\to\tau_i^c$$

and it is easy to verify that H satisfies the conditions (1) and (2) in Theorem 2.2. Since (X,τ_i,τ_j) is a pairwise semi-stratifiable space, there exists a map

$$G: N\times \mathrm{LSC}_{\tau_i}(X)\to\tau_j$$

such that conditions (1), (2), (a) and (b) hold in Theorem 2.2 for G. By the condition (b), we have

$$\bigcap_{n\in N}H(n,f)=\bigcap_{n\in N}G(n,f)=\{x\in X: f(x)=0\}. \qquad (*)$$

Now let $\alpha(n,f)=\chi_{G(n,f)}$. Since $G(n,f)\in\tau_j$, we have $\alpha(n,f)\in \mathrm{LSC}_{\tau_j}(X)$. Let

$$h(f)(x)=1-\sum_{n=1}^{\infty}\frac{1}{2^n}\alpha(n,f)(x),\ \text{for all } x\in X.$$

By Theorem 2.4 in [14], we have $\sum_{n=1}^{\infty}\frac{1}{2^n}\alpha(n,f)\in \mathrm{LSC}_{\tau_j}(X)$. So $h(f)\in \mathrm{USC}_{\tau_j}(X)$.

We shall show that the map h defined above satisfies the necessary conditions (1) and (2) in the theorem. Suppose that $x\in X$. If $f(x)=0$, then $x\in H(n,f)\subset G(n,f)$ and so $\alpha(n,f)(x)=1$ for all $n\in N$ by $(*)$. Therefore,

$$h(f)(x)=1-\sum_{n=1}^{\infty}\frac{1}{2^n}\alpha(n,f)(x)=1-\sum_{n=1}^{\infty}\frac{1}{2^n}=0.$$

If $f(x)>0$, then $x\notin\bigcap_{n\in N}G(n,f)$ by $(*)$. Let $k=\min\{n:x\notin G(n,f)\}$. Then $x\in G(n,f)$ and so $\alpha(n,f)(x)=0$ for all $n<k$. But $x\notin G(k,f)$, and so $x\notin H(n,f)$ by $H(n,f)\subset G(n,f)$. This implies that $f(x)>1/2^{k-1}$. Hence, we have

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\alpha(n,f)(x)=\sum_{n=1}^{k-1}\frac{1}{2^n}\alpha(n,f)(x)+\sum_{n=k}^{\infty}\frac{1}{2^n}\alpha(n,f)(x)$$
$$=1-\frac{1}{2^{k-1}}+\sum_{n=k}^{\infty}\frac{1}{2^n}\alpha(n,f)(x).$$

Consequently,

$$1-\frac{1}{2^{k-1}}\leqslant\sum_{n=1}^{\infty}\frac{1}{2^n}\alpha(n,f)(x)<\sum_{n=1}^{\infty}\frac{1}{2^n}=1.$$

By the definition of h, we have $0<h(f)(x)\leq\frac{1}{2^{k-1}}<f(x)$. Therefore, (1) holds.

It remains to show that $h(f_1)\leqslant h(f_2)$ whenever $f_1\leqslant f_2$. Suppose that $f_1\leqslant f_2$. Then $G(n,f_2)\subset G(n,f_1)$, and so $\chi_{\alpha(n,f_2)}\leqslant\chi_{\alpha(n,f_1)}$, and therefore, we have $\alpha(n,f_2)\leqslant\alpha(n,f_1)$ for each $n\in N$. By the definition of h, one easily sees that $h(f_1)\leqslant h(f_2)$. Therefore, (2) holds.

Conversely, suppose there is a map $h:\mathrm{LSC}_{\tau_i}(X)\to\mathrm{USC}_{\tau_j}(X)$ that satisfies the conditions (1) and (2) given in the theorem. For any fixed $V\in\tau_i$, we consider the function $f_V=\chi_V$. Then $f_V\in\mathrm{LSC}_{\tau_i}(X)$ and so $h(f_V)\in\mathrm{USC}_{\tau_j}(X)$. For each $n\in N$, let

$$F(n,V)=\{x\in X:h(f_V)(x)\geqslant 1/2^n\}.$$

Then the equality above defines a map $F:N\times\tau_i\to\tau_j^c$. We shall show that the map F satisfies the conditions (1) through (3) in Lemma 1.2.

For each $n\in N$ and $x\in X$, if $x\notin V$, then $f_V(x)=0$. Also $x\notin F(n,V)$ for all $n\in N$ by the condition $0\leqslant h(f_V)\leqslant f_V$ in the theorem. Hence $F(n,V)\subset V$ for all $n\in N$ and so $\bigcup_{n\in N}F(n,V)\subset V$. Conversely, for each $x\in V$, we have $f_V(x)=\chi_V(x)=1>0$, and so $h(f_V)(x)>0$ by the condition (1) given in the theorem. Hence there is $m\in N$ such that $h(f_V)(x)\geqslant\frac{1}{2^m}$, which implies that $x\in F(m,V)$. Therefore, $V\subset\bigcup_{n\in N}F(n,V)$. Hence the condition (1) in Lemma 1.2 holds.

If $U,V\in\tau_i$, and $U\subset V$, then $f_U\subset f_V$. Also $h(f_U)\leqslant h(f_V)$ by the condition (2) given in the theorem. Thus $F(n,U)\subset F(n,V)$ for all $n\in N$. Therefore, the condition (2) holds in Lemma 1.2. Therefore, X is a pairwise semi-stratifiable space. $\square$

Corollary 2.4 *A bitopological space (X,τ_i,τ_j) is pairwise semi-stratifiable if and only if for each pair (A,U) of subsets of X, $A\in\tau_j^c$, $U\in\tau_i$ and $A\subset U$, there is an $h_{A,U}\in\mathrm{LSC}_{\tau_j}(X)$ such that $A=h_{A,U}^{-1}(0)$, $X-U=h_{A,U}^{-1}(1)$, and $h_{A,U}\geqslant h_{B,V}$ whenever $B\in\tau_i^c$, $A\subset B$ and $V\in\tau_j$, $U\subset V$.*

Proof Necessity. Suppose that (X,τ_i,τ_j) is a pairwise semi-stratifiable space. Then, by Theorem 2.3, there is a map $\phi:\mathrm{LSC}_{\tau_i}(X)\to\mathrm{USC}_{\tau_j}(X)$ that satisfies the conditions (1) and (2) in Theorem 2.3. For each pair (A,U) of subsets of X, $A\in\tau_j^c$, $U\in\tau_i$ and $A\subset U$. Let $f_A=1-\chi_A$, $g_U=\phi(\chi_U)$. Since $A\in\tau_j^c$ and $U\in\tau_i$, we have $f_A\in\mathrm{LSC}_{\tau_j}(X)$ and $\chi_U\in\mathrm{LSC}_{\tau_i}(X)$. Therefore

$f_A \in \mathrm{LSC}_{\tau_j}(X)$ and $g_U \in \mathrm{USC}_{\tau_j}(X)$. Define $h_{A,U} : X \to [0,1]$ by $h_{A,U} = \frac{f_A}{1+h(g_U)}$ for all $x \in X$. Thus $h_{A,U} \in \mathrm{LSC}_{\tau_j}(X)$ by Proposition 1.2 in [14]. It is easy to verify that $h_{A,U} \geq h_{B,V}$ whenever $A \subset B$ and $U \subset V$.

From the definition of $h_{A,U}$, one can see that $h_{A,U}(x) = 0$ if and only if $x \in A$, which implies that $A = h^{-1}_{A,U}(0)$. Similarly, we can verify that $X - U = f^{-1}_{A,U}(1)$.

Sufficiency. For each $U \in \tau_i$, let $f_U = 1 - h_{\emptyset,U}$ where $\emptyset$ is the empty set. Then $f_U \in \mathrm{USC}_{\tau_j}(X)$, and $f_U \geqslant h_V$ when $U \subset V$. It is easy to verify that $f_U(x) = 0$ if and only if $x \notin U$. For each $n \in N$, let $F(n,U) = \{x \in X : f_U(x) \geqslant \frac{1}{2^n}\}$, and hence $F(n,U) \in \tau_j^c$ by $f_U \in \mathrm{USC}_{\tau_j}(X)$. To prove that (X, τ_1, τ_2) is pairwise semi-stratifiable, it suffices to show that F satisfies the conditions (1) through (3) in Lemma 1.2. It can be proved in the same manner as in proof of the sufficiency in Theorem 2.3. □

Corollary 2.5 *A bitopological space (X, τ_i, τ_j) is pairwise semi-stratifiable space if and only if for each $U \in \tau_i$, there is an $f_U : X \to [0,1]$ and $f_U \in \mathrm{USC}_{\tau_j}(X)$ such that $X - U = h_U^{-1}(0)$, and $f_U \geqslant h_V$ whenever $U \subset V$.*

Proof Suppose that (X, τ_i, τ_j) is a pairwise semi-stratifiable space, and let $f_U = 1 - f_{\emptyset,U}$, where $f_{\emptyset,U}$ is the function given in Corollary 2.4. Then $f_U \in \mathrm{USC}_{\tau_j}(X)$, and $f_U \leqslant h_V$ when $U \subset V$. Since $X - U = f^{-1}_{\emptyset,U}(1)$ and $f_U = 1 - f_{\emptyset,U}$, it can be checked that $x \in X - U$ if and only if $x \in f_U^{-1}(0)$, which implies that $X - U = f_U^{-1}(0)$.

The sufficiency can be proved in the same manner as the proof of the sufficiency in Corollary 2.4. □

3. Topological ordered C-spaces and pairwise semi-stratifiable spaces

A topological ordered space $(X, \tau, \leqslant)$ is a set X endowed with a topology τ and a partial order $\leqslant$. A subset A of X is said to be an upper set of X if $x \leqslant y$ and $x \in A$ imply that $y \in A$. Similarly, we say that a subset A of X is a lower set of X if $y \leqslant x$ and $x \in A$ imply that $y \in A$. We let $\tau^\flat$ denote the collection of τ-open lower sets of X and $\tau^\sharp$ denote the collection of τ-open upper sets of X. It can be easily verified that the $\tau^\flat$ and $\tau^\sharp$ are topologies on X, and that $(X, \tau^\flat, \tau^\sharp)$ is a bitopological space (in the following often more briefly called bispace [5]). For any subset A of X, $i(A)$ (resp., $d(A)$) will denote the intersection of all upper (lower) sets of X containing A. If $A = i(A)$ (resp., $d(A) = A$), we say A is an upper (a lower) set. Following Priestley [6], we shall call a topological ordered space $(X, \tau, \leqslant)$ a C-space if $d(F)$ and $i(F)$ are closed whenever F is a closed subset of X. Similarly, a topological ordered space $(X, \tau, \leqslant)$ is called an I-space if $d(G)$ and $i(G)$ are open whenever G is an open subset of X.

Theorem 3.1 *Let $(X, \tau, \leqslant)$ be a topological ordered C-space with a semi-stratifiable topology τ. Then $(X, \tau^\flat, \tau^\natural)$ is pairwise semi-stratifiable.*

Proof Since (X, τ) is semi-stratifiable, for each closed set F of X there is a sequence $(F_n)_{n \in N}$ of open sets such that:

(i) $\bigcap_{n\in N} F_n = F$ and

(ii) For all pairs H, F of closed sets such that $H \subset F$, we have $H_n \subset F_n$ whenever $n \in N$, where $\{H_n\}_{n\in N}$ is the sequence of open sets assigned to the H.

Let F be a closed lower set of X. Put $D_{\tau^\flat}(n, F) = X - d(X - F_n)$. Note that $D_{\tau^\flat}(n, F) \in \tau^\natural$ for each $n \in N$. Furthermore $F \subset D_{\tau^\flat}(n, F) \subset F_n$ for each $n \in N$. Therefore $F \subset \bigcap_{n\in N} D_{\tau^\flat}(n, F) \subset \bigcap_{n\in N} F_n = F$. Thus $F = \bigcap_{n\in N} D_{\tau^\flat}(n, F)$. Moreover if F and H are closed lower sets such that $F \subset H$, then $D_{\tau^\flat}(n, F) \subset D_{\tau^\flat}(n, H)$ for every $n \in N$. Indeed, given $n \in N$, we have $F_n \subset H_n$, thus $X - H_n \subset X - F_n$ and $d(X - H_n) \subset d(X - F_n)$. Consequently, $D_{\tau^\flat}(n, F) = X - d(X - F_n) \subset X - d(X - H_n) = D_{\tau^\flat}(n, H)$ for each $n \in N$.

In the same way, let F be a closed upper set of X, we define $D_{\tau^\natural}(n, F) = X - i(X - F_n)$. By Definition 1.1, we conclude that the bitopological space $(X, \tau^\flat, \tau^\natural)$ is pairwise semi-stratifiable. $\square$

Theorem 3.2 *Let $(X, \tau, \leqslant)$ be a topological ordered C-space with a monotonically normal topology τ. Then $(X, \tau^\flat, \tau^\natural)$ is pairwise monotonically normal.*

Proof Since (X, τ) is monotonically normal, for each pair (H, K) of disjoint closed subset of X, there exists open set $D_1(H, K)$ of X such that (i) $H \subset D_1(H, K) \subset \mathrm{cl}_\tau D_1(H, K) \subset X - K$ and (ii) if the pair (H, K) and (H', K') satisfy $H \subset H'$ and $K' \subset K$, then $D_1(H, K) \subset D_1(H', K')$.

Let (H, K) be a pair of disjoint subsets of X such that H is a closed lower set and K is a closed upper set. Put

$$D(H, K) = X - i(X - D_1(H, K)), \quad D(K, H) = X - d(X - D_1(K, H)).$$

Then $D(H, K)$ is an open upper set and $D(K, H)$ is an open lower set. Since $H \subset D_1(H, K)$, we have $X - H \supset X - D_1(H, K)$ and $X - H$ is an open upper set. Furthermore $X - H = i(X - H) \supset i(X - D_1(H, K))$ and $H \subset X - i(X - D_1(H, K)) = D(H, K)$ by the fact that X is a C-space. Therefore, we have $d(\mathrm{cl}_\tau D(H, K)) \subset X - K$ and so $\mathrm{cl}_{\tau^\flat} D_1(H, K) \subset X - K$. We conclude that

$$H \subset D(H, K) \subset \mathrm{cl}_{\tau^\flat} D(H, K) = \mathrm{cl}_{\tau^\flat}(X - D_1(H, K)) \subset X - K.$$

By symmetry of the situation we conclude that

$$K \subset D(K, H) \subset \mathrm{cl}_{\tau^\natural} D(K, H) \subset X - H.$$

Moreover if the pairs (H, K) and (H', K') satisfy $H \subset H'$ and $K' \subset K$, then $D_1(H, K) \subset D_1(H', K')$ and $D_1(K', H') \subset D_1(K, H)$. We have

$$i(X - D_1(H, K)) \supset i(X - D_1(H', K'))$$

and

$$d(X - D_1(K', H')) \supset d(X - D_1(H, K)).$$

Consequently,

$$D(H, K) = X - i(X - D_1(H, K)) \subset X - i(X - D_1(H', K')) = D(H', K')$$

and

$$D(K', H') = X - d(X - D_1(K', H')) \subset X - i(X - D_1(K, H)) = D(K, H).$$

Therefore, the bitopological space $(X, \tau^\flat, \tau^\natural)$ is pairwise monotonically normal. $\square$

Corollary 3.3 ([9]) *Let $(X, \tau, \leqslant)$ be a topological ordered C-space with a stratifiable topology τ. Then $(X, \tau^\flat, \tau^\natural)$ is pairwise stratifiable.*

Proof Since a space is stratifiable if and only if it is a monotonically normal and semi-stratifiable space [8], X is a monotonically normal and semi-stratifiable space. From Proposition 1 in [7], Theorems 3.1 and 3.2, we conclude that $(X, \tau^\flat, \tau^\natural)$ is pairwise stratifiable. $\square$

Acknowledgements We would like to thank the anonymous reviewers for their valuable suggestions, which have helped to improve the quality of the paper.

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