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The Uniqueness of Skeleton Presentation of Complete Bipartite Graph $K_{m,n}$

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Abstract Kobayashi discussed some kinds of standard embeddings into 3-manifolds of spatial graphs. He introduced the concept of book presentation, which is a standard embedding of spatial graphs with good properties, and proved that the book presentation of minimum sheets of K_n is unique up to the sheet translation and the ambient isotopy. In this present paper we give the definition of skeleton presentation of spatial graphs, and prove that the skeleton presentation of minimum floors of a complete bipartite graph $K_{m,n}$ is unique up to ambient isotopy.

Keywords complete bipartite graph; skeleton presentation; floor; ambient isotopy.

MR(2010) Subject Classification 57M15; 57M25

1. Introduction

Kobayashi discussed some kinds of standard embeddings into 3-manifolds of spatial graphs in [1] and [2]. He introduced the concept of book presentation, which is a standard embedding of spatial graphs with good properties. In [1] Kobayashi conjectured that the book presentation of minimum sheets of K_n is unique up to the sheet translation and the ambient isotopy. In [3] Yin et al. discussed some properties of book presentation of spatial graphs, and proved that the book presentation of minimum sheets of a complete graph K_{2m} with even vertices is unique up to sheet translation and ambient isotopy.

In this paper we give the definition of skeleton presentation of spatial graphs, and prove that the skeleton presentation of minimum floors of a complete bipartite graph $K_{m,n}$ is unique up to ambient isotopy.

In Section 2, we will preview some definitions and give some new definitions, and in Section 3, we will give the main results.

2. Preliminary

A graph G is denoted by (V, E), where V is a set. The element in V is called a vertex. Let E be a subset of $V \times V$. Then each of its elements is called an edge.

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The set of vertices is denoted by $V(G)$, and the set of edges is denoted by $E(G)$, G is called a finite graph if $V(G)$ and $E(G)$ are both finite sets, otherwise G is called an infinite graph. Only finite graph will be discussed in this paper. The following Definition 1 to Definition 3 can be found in [3] and [4].

Definition 1 *The graph $K_{i,j}$ is the graph given by taking two sets of vertices, the first set having i vertices and the second set having j vertices. All of the vertices in any one of the sets are connected by edges to all of the vertices in the other two sets but to none of the other vertices in their own set. We call $K_{i,j}$ a complete bipartite graph.*

Obviously, a complete bipartite graph $K_{m,n}$ contains $(m\times n)$ edges. For a spatial embedding $f : G \to R^3$ of a graph G, let $SE(G)$ be the set of embedding from G into R^3. An element of $SE(G)$ is called a spatial embedding of the graph or simply a spatial graph.

Definition 2 *Let $f, g \in SE(G)$, $f, g : G \to R^3$, and $I = [0, 1]$ be a unit closed interval. The map $\Phi : G \times I \to R^3 \times I$ is called*

(1) Level preserving, if for any $t \in I$, there exists a map $\Phi_t : G \to R^3$ so that $\Phi(x,t) = (\Phi_t(x), t)$.

(2) Locally flat, if for any point of the image of Φ, there is a neighborhood N s.t., $(N, N \cap \Phi(G \times I))$ is homeomorphic to the standard pairs of disks (D^4, D^2) or $(D^3 \times I, X_n \times I)$, n is non-negative.

(3) Between f and g, if there is a real number, so that for all $x \in G$, $0 \le t \le \theta$, $\Phi(x,t) = (f(x), t)$; and for all $x \in G$, $1-\theta \le t \le 1$, $\Phi(x,t) = (g(x), t)$.

Definition 3 *Let $f, g \in SE(G)$, $f, g : G \to R^3$, and $I = [0, 1]$ be a unit closed interval. f and g are called*

(1) An ambient isotopic, if there is a level preserving and locally flat embedding map $\Phi : G \times I \to R^3 \times I$ between f and g.

(2) Cobordism, if there is a locally flat map $\Phi : G \times I \to R^3 \times I$ between f and g.

(3) Isotopic, if there is a level preserving map $\Phi : G \times I \to R^3 \times I$ between f and g.

Definition 4 *Let*

$$\begin{aligned}
\widetilde{P}^+_m &= \{(x^+_m, y^+_m, z^+_m) \in R^3 \mid z^+_m = -m(y^2-y),\ 0 \le y \le 1\},\\
\widetilde{P}^+_{m-1} &= \{(x^+_{m-1}, y^+_{m-1}, z^+_{m-1}) \in R^3 \mid z^+_{m-1} = (1-m)(y^2-y),\ 0 \le y \le 1\},\\
&\cdots\\
\widetilde{P}^+_1 &= \{(x^+_1, y^+_1, z^+_1) \in R^3 \mid z^+_1 = -(y^2-y),\ 0 \le y \le 1\},\\
\widetilde{P}^0_0 &= \{(x^0_0, y^0_0, z^0_0) \in R^3 \mid z^0_0 = 0,\ 0 \le y \le 1\},\\
\widetilde{P}^-_1 &= \{(x^-_1, y^-_1, z^-_1) \in R^3 \mid z^-_1 = y^2-y,\ 0 \le y \le 1\},\\
&\cdots\\
\widetilde{P}^-_{n-1} &= \{(x^-_{n-1}, y^-_{n-1}, z^-_{n-1}) \in R^3 \mid z^-_{n-1} = (n-1)(y^2-y),\ 0 \le y \le 1\},\\
\widetilde{P}^-_n &= \{(x^-_n, y^-_n, z^-_n) \in R^3 \mid z^-_n = n(y^2-y),\ 0 \le y \le 1\}.
\end{aligned}$$

Let $\widetilde{\mathcal{B}}$ be the set of $\widetilde{P}_m^+, \ldots, \widetilde{P}_1^+, \widetilde{P}_0^0, \widetilde{P}_1^-, \ldots, \widetilde{P}_n^-$. We call it a skeleton. Let $A = \{(x, y, z) \in R^3 \mid y = z = 0\}$, $B = \{(x, y, z) \in R^3 \mid y = 1, z = 0\}$. We call A and B the binder of $\widetilde{\mathcal{B}}$. Let $\mathcal{B}_n = \{\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n\}$, where $\widetilde{P}_1, \widetilde{P}_2, \ldots, \widetilde{P}_n$ are different elements of $\widetilde{\mathcal{B}}$. Let $P_i = \widetilde{P}_i - (A \cup B)$. We call P_i the i-th floor of $\mathcal{B}_n$. Thus $\mathcal{B}_n = \bigcup_{i=1}^n \widetilde{P}_i = \bigcup_{i=1}^n P_i \cup A \cup B$ is a skeleton with n floors $\{P_i\}$ and two binders A and B.

Definition 5 *Let $\psi : G \to \mathcal{B}_n$ be an embedding satisfying that*

(1) $\psi(V(G)) \subset (A \cup B)$;

(2) For any edge $e \in E(G)$, $\psi(\mathrm{Int}(e)) \subset P_i$ for some P_i;

(3) For any floor P_i there is at least one edge e of G with $\psi(\mathrm{Int}(e)) \subset P_i$.

Then we call $\widetilde{G} = \psi(G)$ (or the embedding ψ) a skeleton presentation of G with n floors. It is clear that $1 \leq n \leq\mid E(G) \mid$. When n is minimum, we call $\widetilde{G}$ a skeleton presentation of G with minimum floors.

Definition 6 *For a finite bipartite graph $K_{m,n}$, there are two sets of vertices such that each two vertices in the same set do not have any edge between them. Denote the two vertex subsets by $V^1(K_{m,n}) = \{V_1, V_2, \ldots, V_m\}$, $V^2(K_{m,n}) = \{V_1', V_2', \ldots, V_n'\}$. If a skeleton presentation $\psi : K_{m,n} \to \mathcal{B}_s$ satisfies $\psi(V^1) \subset A$, $\psi(V^2) \subset B$, where A and B are the binders of $\mathcal{B}_s$, we call $\widetilde{G} = \psi(G)$ a normal skeleton presentation, or $N.S.P$ for short.*

3. The main Theorem

Lemma 1 *Let $\psi : K_{m,m} \to \mathcal{B}_p$ be an $N.S.P$ of a complete bipartite graph $K_{m,m}$. Then there are m floors at least.*

Proof There are two sets of points, $V^1(K_{m,m}) = \{V_1, V_2, \ldots, V_m\}$, $V^2(K_{m,m}) = \{V_1', V_2', \ldots, V_m'\}$. We denote (V_a, V_b') to be an edge of the graph, where $V_a \in V^1$, $V_b' \in V^2$. And the sum of (V_a, V_b') is denoted by $(a + b)$, and the set of edges with the sum l is denoted by $E(l)$. There are m edges $(V_1, V_m'), (V_2, V_{m-1}'), \ldots, (V_m - 1, V_2'), (V_m, V_1')$ in $E(m + 1)$. And these m edges cannot pairwise lie in one floor as shown in Figure 1.

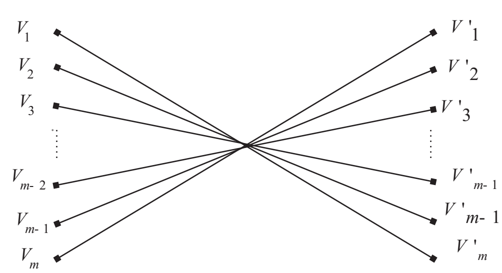

Figure 1 m edges pairwise intersect

So these m edges in $E(m + 1)$ must be in m distinguished floors, respectively. Next we will give a way to show there exist an $N.S.P$ of complete bipartite graph $K_{m,m}$ with m floors.

First, we put edges $(V_1, V_1'), (V_1, V_2'), \ldots, (V_1, V_m'), (V_2, V_m'), \ldots, (V_m, V_m')$ in P_1, as shown in Figure 2.

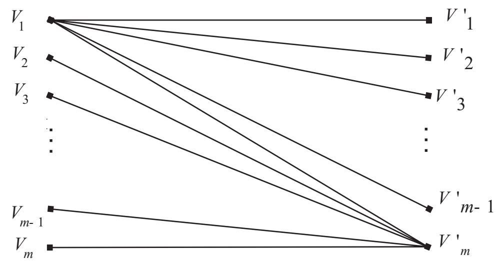

Figure 2 P_1

Second, we put edges $(V_2, V_1'), (V_2, V_2'), \ldots, (V_2, V_{m-1}'), (V_3, V_{m-1}'), \ldots, (V_m, V_{m-1}')$ in P_2.

...

Next, we put edges $(V_{m-1}, V_1'), (V_{m-1}, V_2'), (V_m, V_2')$ in P_{m-1}.

Above all, we put edge (V_m, V_1') in P_m.

So, there is an $N.S.P$ of complete bipartite graph $K_{m,m}$ with m floors and m is the smallest number of floors.

Lemma 2 *Let $\psi : K_{m,n} \to \mathcal{B}_p$ be an $N.S.P$ of a complete bipartite graph $K_{m,n}$. Then there are k floors at least, where k is the smaller number of $\{m, n\}$.*

Proof Without loss of generality, assume $m \leqslant n$. The two sets of vertices are $V^1(K_{m,n}) = \{V_1, V_2, \ldots, V_m\}$, $V^2(K_{m,n}) = \{V_1', V_2', \ldots, V_n'\}$. Consider the edges in $E(n+1)$, where $E(n+1) = \{(V_1, V_n'), (V_2, V_{n-1}'), \ldots, (V_{m-1}, V_{n+m+2}'), (V_m, V_{n-m+1}')\}$.

And these m edges cannot pairwise lie in one floor. So they must be put in m distinguished floors, respectively.

Now we give a way to show there exist an $N.S.P$ of complete bipartite graph $K_{m,n}$ with m floors.

First, we put edges $(V_1, V_1'), (V_1, V_2'), \ldots, (V_1, V_n'), (V_2, V_n'), \ldots, (V_m, V_n')$ in P_1.

Secondly, we put edges $(V_2, V_1'), (V_2, V_2'), \ldots, (V_2, V_{n-1}'), (V_3, V_{n-1}'), \ldots, (V_m, V_{n-1}')$ in P_2.

...

Next, we put edge $(V_{m-1}, V_1'), (V_{m-1}, V_2'), \ldots, (V_{m-1}, V_{n-m+2}'), (V_m, V_{n-m+2}')$ in P_{m-1}.

Above all, we put edge $(V_m, V_1'), (V_m, V_2'), \ldots, (V_m, V_{n-m+1}')$ in P_m.

So, there is an $N.S.P$ of a complete bipartite graph $K_{m,n}$ with m floors and m is the smallest number of floors.

Theorem 3 *The $N.S.P$ of the complete bipartite graph $K_{m,m}$ with minimum floors is unique up to ambient isotopy.*

Proof By Lemma 1, the $N.S.P$ of $K(m, m)$ with minimum floors contains m floors denoted by $P_1, P_2, \ldots, P_m$. There are m edges with the sum being $m+1$ in $K(m, m)$, namely $(V_1, V_m'), (V_2, V_{m-1}'),$

$\ldots,(V_m,V_1')$. And they must be in m distinguished floors, respectively. Without loss of generality, assume $(V_1,V_m')\subset P_1,(V_2,V_{m-1}')\subset P_2,\ldots,(V_m,V_1')\subset P_m$, as shown in Figure 3.

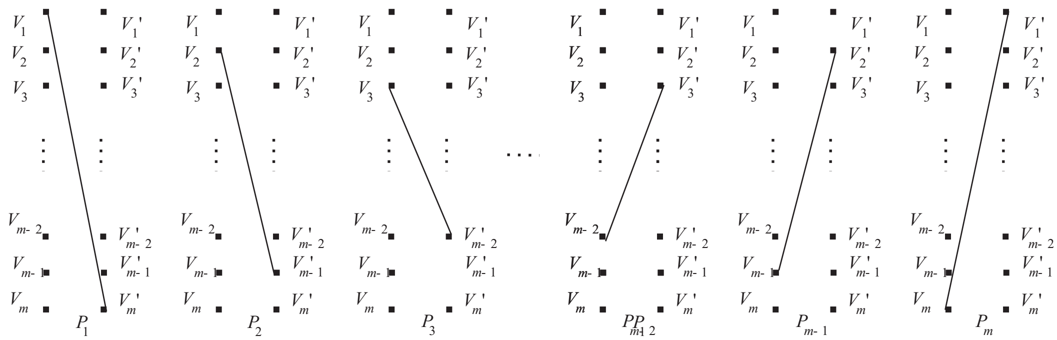

Figure 3 m floors with the sum $m+1$

We will put the edges with the sum k $(2\le k\le m)$ upper of the edges of $E(m+1)$. While we put the edges with the sum k $(m+2\le k\le 2m)$ lower of the edges of $E(m+1)$. We only need consider the case of putting edges with the sum k $(2\le k\le m)$, the other cases are similar. So we divide it into $m-1$ steps and construct all possible $N.S.P$ of $K_{m,m}$.

First, we put the edges $E(m)=\{(V_1,V_{m-1}'),(V_2,V_{m-2}'),\ldots,(V_{m-1},V_1')\}$. We only have the following two kinds of possibilities:

Case $\bar{1}$ If (V_1,V_{m-1}') is put into floor P_1, we have,

$$(V_1,V_{m-1}')\subset P_1\ ;\ (V_2,V_{m-2}')\subset P_2\ ;\ \ldots\ ;\ (V_k,V_{m-k}')\subset P_k\ ;$$
$$(V_{k+1},V_{m-k-1}')\subset P_{k+2}\ ;\ \ldots\ ;\ (V_{m-1},V_1')\subset P_m\ (1\le k\le m-2).$$

Case $\bar{2}$ If (V_1,V_{m-1}') is put into floor P_2, we have

$$(V_1,V_{m-1}')\subset P_2;(V_2,V_{m-2}')\subset P_3;\ldots;(V_{m-1},V_1')\subset P_m.$$

In the above two cases, for Case $\bar{1}$ by an isotopy as

$$P_1\xrightarrow{(V_1,V_{m-1}')}P_2\xrightarrow{(V_2,V_{m-2}')}P_3\xrightarrow{(V_3,V_{m-3}')}\cdots\xrightarrow{(V_{k-1},V_{m-k+1}')}P_k\xrightarrow{(V_k,V_{m-k}')}P_{k+1},$$

we get Case $\bar{2}$.

Secondly, we will prove the following different kinds of possibilities are ambient isotopy by induction. Suppose in the i-th step, the floors with edges of sum $m-i+1$ satisfy,

$$(V_1,V_{m-i}')\subset P_{x_1};(V_2,V_{m-i-1}')\subset P_{x_2};\ldots;(V_{m-i},V_1')\subset P_{x_{m-i}}.$$

Where $\{x_1,x_2,\ldots,x_{m-i}\}$ is a subset of $\{1,2,\ldots,m\}$ and $x_1<x_2<\cdots<x_{m-i}$. $E(m-i)=\{(V_1,V_{m-i-1}'),(V_2,V_{m-i-2}'),\ldots,(V_{m-i-1},V_1')\}$.

1) If we put these $m-i-1$ edges into the $m-i$ floors $\{P_{x_1}\ ,\ P_{x_2}\ ,\ \ldots\ ,\ P_{x_{m-i}}\}$. (V_1,V_{m-i-1}') can be put in two ways, one is in floor P_{x_1}, and the other is in floor P_{x_2}.

Case 1 If (V_1, V'_{m-i-1}) is fixed in P_{x_1}, we have

$$(V_1, V'_{m-i-1}) \subset P_{x_1}; (V_2, V'_{m-i-2}) \subset P_{x_2}; \ldots;\ (V_k, V'_{m-i-k}) \subset P_{x_k};$$

$$(V_{k+1}, V'_{m-i-k-1}) \subset P_{x_{k+2}}; \ldots; (V_{m-i-1}, V'_1) \subset P_{x_{m-i}}\ (1 \le s \le m-i-2).$$

Case 2 If (V_1, V'_{m-i-1}) is fixed in P_{x_2}, we have,

$$(V_1, V'_{m-i-1}) \subset P_{x_2}; (V_2, V'_{m-i-2}) \subset P_{x_3}; \ldots; (V_{m-i-1}, V'_1) \subset P_{x_{m-i}}.$$

In the above two cases, for Case 1 by an isotopy as

$$P_{x_1} \xrightarrow{(V_1,V'_{m-i-1})} P_{x_2} \xrightarrow{(V_2,V'_{m-i-2})} P_{x_3} \xrightarrow{(V_3,V'_{m-i-3})} \cdots \xrightarrow{(V_{k-1},V'_{m-i-k+1})} P_{x_k} \xrightarrow{(V_k,V'_{m-i-k})} P_{x_{k+1}},$$

we get Case 2.

2) If an edge (V_k, V'_{m-i-k}) is not in the floor $P_{x_1}, P_{x_2}, \ldots, P_{x_{m-i}}$. Suppose $(V_k, V'_{m-i-k}) \in P_{x_a}$. Then $x_a \in \{1, 2, \ldots, m\}$ and $x_a \notin \{x_1, x_2, \ldots, x_{m-i}\}$. Let us consider the two sets of edges as follows

$$E^1 = \{(V_1, V'_{m-i-1}), (V_2, V'_{m-i-2}), \ldots, (V_{k-1}, V'_{m-i-k+1})\},$$
$$E^2 = \{(V_{k+1}, V'_{m-i-k-1}), (V_{k+2}, V'_{m-i-k-2}), \ldots, (V_{m-i-1}, V'_1)\}.$$

The edges in E^1 can be put in k floors $P_{x_1}, P_{x_2}, \ldots, P_{x_k}$, and the edges in E^2 can be put in floors $P_{x_{k+1}}, P_{x_{k+2}}, \ldots, P_{x_{m-i}}$.

(a) If (V_1, V'_{m-i-1}) is fixed in P_{x_1}, we get

$$(V_1, V'_{m-i-1}) \subset P_{x_1}; (V_2, V'_{m-i-2}) \subset P_{x_2}; \ldots; (V_s, V'_{m-i-s}) \subset P_{x_s};$$

$$(V_{s+1}, V'_{m-i-s-1}) \subset P_{x_{s+2}}; \ldots; (V_{k-1}, V'_{m-i-k+1}) \subset P_{x_k}\ (1 \le s \le k-2).$$

(b) If (V_1, V'_{m-i-1}) is fixed in P_{x_2}, we get

$$(V_1, V'_{m-i-1}) \subset P_{x_2}; (V_2, V'_{m-i-2}) \subset P_{x_3}; \ldots; (V_{k-1}, V'_{m-i-k+1}) \subset P_{x_k}.$$

(a') If $(V_{k+1}, V'_{m-i-k-1})$ is fixed in $P_{x_{k+1}}$, we get

$$(V_{k+1}, V'_{m-i-k-1}) \subset P_{x_{k+1}}; (V_{k+2}, V'_{m-i-k-2}) \subset P_{x_{k+2}}; \ldots; (V_t, V'_{m-i-t}) \subset P_{x_t};$$

$$(V_{t+1}, V'_{m-i-t-1}) \subset P_{x_{t+2}}; \ldots; (V_{m-i-1}, V'_1) \subset P_{x_{m-i}}\quad (k+1 \le t \le m-i-2).$$

(b') If $(V_{k+1}, V'_{m-i-k-1})$ is fixed in $P_{x_{k+2}}$, we get

$$(V_{k+1}, V'_{m-i-k-1}) \subset P_{x_{k+1}}; (V_{k+2}, V'_{m-i-k-2}) \subset P_{x_{k+2}}; \ldots; (V_{m-i-1}, V'_1) \subset P_{x_{m-i}}.$$

Then we can put the edges into the skeleton in the following ways

Case 3, bb' ; Case 4, ab' ; Case 5, ba' ; Case 6, aa'.

For Case 3, by an isotopy as, $P_{x_a} \xrightarrow{(V_k,V'_{m-i-k})} P_{x_{k+1}}$, we get Case 2.

For Case 4, by an isotopy as,

$$P_{x_1} \xrightarrow{(V_1,V'_{m-i-1})} P_{x_2} \xrightarrow{(V_2,V'_{m-i-2})} P_{x_3} \xrightarrow{(V_3,V'_{m-i-3})} \cdots \xrightarrow{(V_{s-1},V'_{m-i-s+1})} P_{x_s} \xrightarrow{(V_s,V'_{m-i-s})} P_{x_{s+1}},$$

$$P_{x_a} \xrightarrow{(V_k,V'_{m-i-k})} P_{x_{k+1}},$$

we get Case 2.

For Case 5, by isotopy as,

$$P_{x_{k+1}} \xrightarrow{(V_{k+1},V'_{m-i-k-1})} P_{x_{k+2}} \xrightarrow{(V_{k+2},V'_{m-i-k-2})} P_{x_{k+3}} \xrightarrow{(V_{k+3},V'_{m-i-k-3})} \cdots$$
$$\xrightarrow{(V_{t-1},V'_{m-i-t+1})} P_{x_t} \xrightarrow{(V_t,V'_{m-i-t})} P_{x_{t+1}},$$
$$P_{x_a} \xrightarrow{(V_k,V'_{m-i-k})} P_{x_{k+1}},$$

we get Case 2.

For Case 6, by isotopy as,

$$P_{x_1} \xrightarrow{(V_1,V'_{m-i-1})} P_{x_2} \xrightarrow{(V_2,V'_{m-i-2})} P_{x_3} \xrightarrow{(V_3,V'_{m-i-3})} \cdots \xrightarrow{(V_{s-1},V'_{m-i-s+1})} P_{x_s} \xrightarrow{(V_s,V'_{m-i-s})} P_{x_{s+1}},$$
$$P_{x_{k+1}} \xrightarrow{(V_{k+1},V'_{m-i-k-1})} P_{x_{k+2}} \xrightarrow{(V_{k+2},V'_{m-i-k-2})} P_{x_{k+3}} \xrightarrow{(V_{k+3},V'_{m-i-k-3})} \cdots$$
$$\xrightarrow{(V_{t-1},V'_{m-i-t+1})} P_{x_t} \xrightarrow{(V_t,V'_{m-i-t})} P_{x_{t+1}},$$
$$P_{x_a} \xrightarrow{(V_k,V'_{m-i-k})} P_{x_{k+1}},$$

we get Case 2.

Similarly, if the number of edges not in $P_{x_1}, P_{x_2}, \ldots, P_{x_{m-i}}$ is more than one, we can get the similar conclusion. The induction will end in the $(m-1)$ steps.

Finally, there are $m-1$ steps, so without loss of generality, denote the character string by $\{y_1, y_2, \ldots, y_{m-1} \mid y_i \in R\}$. Consider the $(i+1)$-th step in the construction before.

Now we choose $N.S.P$ which are different only on the $(i+1)$-th step. We can consider only 6 cases as what has discussed before.

The cases are denoted by

Case 1, $\{a_1, \ldots, a_{i+1}, \ldots, a_{m-1}\}$;

Case 2, $\{b_1, \ldots, b_{i+1}, \ldots, b_{m-1}\}$;

Case 3, $\{c_1, \ldots, c_{i+1}, \ldots, c_{m-1}\}$;

Case 4, $\{d_1, \ldots, d_{i+1}, \ldots, d_{m-1}\}$;

Case 5, $\{e_1, \ldots, e_{i+1}, \ldots, e_{m-1}\}$;

Case 6, $\{f_1, \ldots, f_{i+1}, \ldots, f_{m-1}\}$,

where $a_j = b_j = c_j = d_j = e_j = f_j$ when $j \neq i+1$, $a_{i+1} \neq b_{i+1} \neq c_{i+1} \neq d_{i+1} \neq e_{i+1} \neq f_{i+1}$. The edges in floor P_{x_j} whose summation are smaller than or equal to $m-i$ are denoted by $S_j = (E(m-i) \cup E(m-i-1) \cup \cdots \cup E(2)) \cap P_{x_j}$.

(1) Changing $\{a_1, \ldots, a_{i+1}, \ldots, a_{m-1}\}$ by isotopy as

$$P_{x_1} \xrightarrow{S_1} P_{x_2} \xrightarrow{S_2} P_{x_3} \xrightarrow{S_3} \cdots \xrightarrow{S_{k-1}} P_{x_k} \xrightarrow{S_k} P_{x_{k+1}} \xrightarrow{S_{k+1}} P_{x_1},$$

we will get $\{b_1, \ldots, b_{i+1}, \ldots, b_{m-1}\}$.

(2) Changing $\{c_1, \ldots, c_{i+1}, \ldots, c_{m-1}\}$ by isotopy as

$$P_{x_a} \xrightarrow{S_a} P_{x_{k+1}},$$

we will get $\{b_1, \ldots, b_{i+1}, \ldots, b_{m-1}\}$.

(3) Changing $\{d_1, \dots, d_{i+1}, \dots, d_{m-1}\}$ by isotopy as

$$P_{x_1} \xrightarrow{S_1} P_{x_2} \xrightarrow{S_2} P_{x_3} \xrightarrow{S_3} \cdots \xrightarrow{S_{s-1}} P_{x_s} \xrightarrow{S_s} P_{x_{s+1}} \xrightarrow{S_{s+1}} P_{x_1}, \quad P_{x_a} \xrightarrow{S_a} P_{x_{k+1}},$$

we will get $\{b_1, \dots, b_{i+1}, \dots, b_{m-1}\}$.

(4) Changing $\{e_1, \dots, e_{i+1}, \dots, e_{m-1}\}$ by isotopy as

$$P_{x_{k+1}} \xrightarrow{S_{k+1}} P_{x_{k+2}} \xrightarrow{S_{k+2}} P_{x_{k+3}} \xrightarrow{S_{k+3}} \cdots \xrightarrow{S_{t-1}} P_{x_t} \xrightarrow{S_t} P_{x_{t+1}} \xrightarrow{S_{t+1}} P_{x_{k+1}}, \quad P_{x_a} \xrightarrow{S_a} P_{x_k},$$

we will get $\{b_1, \dots, b_{i+1}, \dots, b_{m-1}\}$.

(5) Changing $\{f_1, \dots, f_{i+1}, \dots, f_{m-1}\}$ by isotopy as

$$P_{x_1} \xrightarrow{S_1} P_{x_2} \xrightarrow{S_2} P_{x_3} \xrightarrow{S_3} \cdots \xrightarrow{S_{s-1}} P_{x_s} \xrightarrow{S_s} P_{x_{s+1}} \xrightarrow{S_{s+1}} P_{x_1},$$
$$P_{x_{k+1}} \xrightarrow{S_{k+1}} P_{x_{k+2}} \xrightarrow{S_{k+2}} P_{x_{k+3}} \xrightarrow{S_{k+3}} \cdots \xrightarrow{S_{t-1}} P_{x_t} \xrightarrow{S_t} P_{x_{t+1}} \xrightarrow{S_{t+1}} P_{x_{k+1}}, \quad P_{x_a} \xrightarrow{S_a} P_{x_k},$$

we will get $\{b_1, \dots, b_{i+1}, \dots, b_{m-1}\}$.

Similarly, for the other cases with more than one edge not in $P_{x_1}, P_{x_2}, \dots, P_{x_{m-i}}$, we will get the same conclusion. For any two cases, if they are different in k steps by induction, they are equivalent to each other by k steps as before.

Theorem 4 *The N.S.P of the complete bipartite graph $K_{m,n}$ with minimum floors is unique up to ambient isotopy.*

Proof Without loss of generality, assume $m \leqslant n$. By Lemma 2, the *N.S.P* of $K(m,n)$ with minimum floors contains m floors denoted by $P_1, P_2, \dots, P_m$. Assume $(V_1, V'_n) \subset P_1, (V_2, V'_{n-1}) \subset P_2, \dots, (V_m, V'_{n-m+1}) \subset P_m$, as shown in Figure 3.

The sum could be $2, 3, \dots, m+n$, then we have the following three sets of edges $R = \{E(k) \mid 2 \leqslant k \leqslant m\}$, $S = \{E(k) \mid m+1 \leqslant k \leqslant n\}$, $T = \{E(k) \mid n+2 \leqslant k \leqslant m+n\}$.

Firstly, put the edges in S, that is,

$$m+1 \leqslant k \leqslant n, \; E(k) = \{(V_1, V'_{k-1}), (V_2, V'_{k-2}), \dots, (V_m, V'_{k-m})\}.$$

We get

$$(V_m, V'_{k-m}) \subset P_m; (V_{m-1}, V'_{k-m+1}) \subset P_{m-1}; \cdots ; (V_1, V'_{k-1}) \subset P_1.$$

Next, put the edges in R and T. We will put the edge in R upper of the edges in $E(m+1)$ as shown in Figure 4, and put the edges in T lower of the edges in $E(n+1)$ as shown in Figure 5.

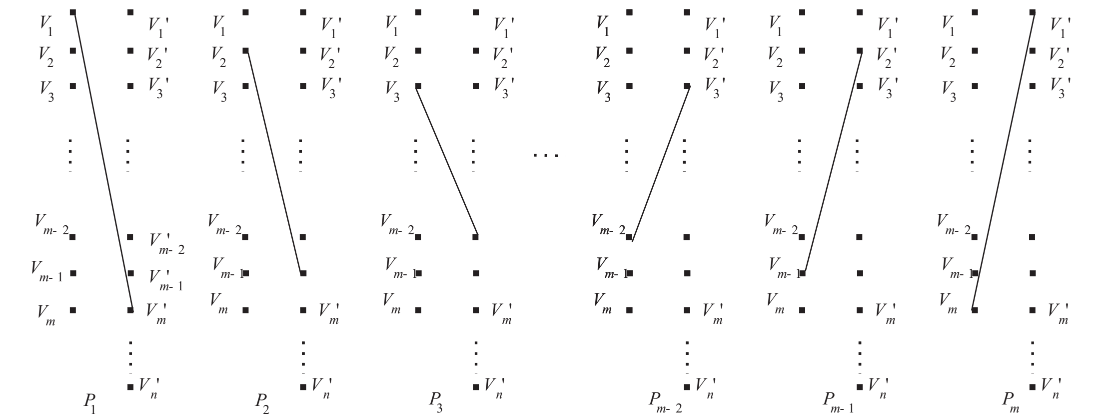

Figure 4 m floors with the sum $m+1$

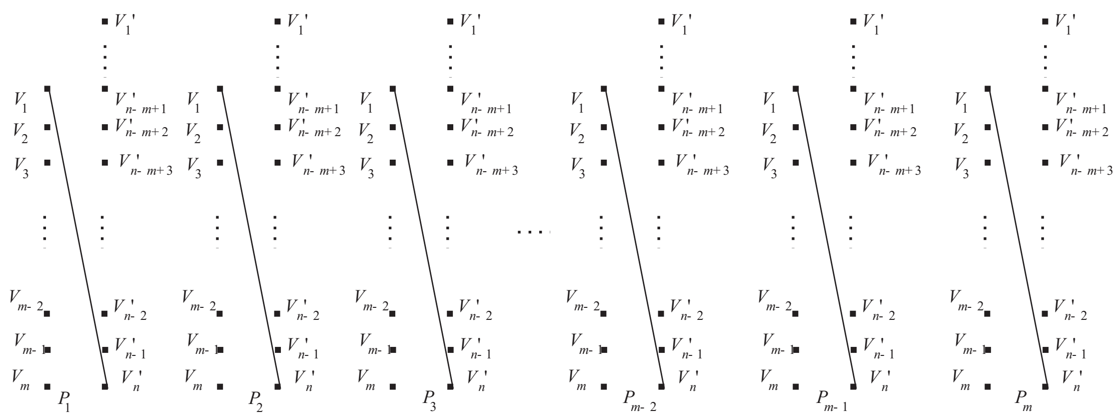

Figure 5 m floors with the sum $n+1$

Then with the same process as in Theorem 3.1, we get the $N.S.P$ of the complete bipartite graph $K_{m,n}$ with minimum floors being unique up to ambient isotopy.

Corollary 5 *The $N.S.P$ of $K_{a,b,c}$ and $K_{a,b,c,d}$ with minimum floors is unique up to ambient isotopy.*

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