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Brownian Stochastic Current: White Noise Approach

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Abstract In this paper, we present a new idea to study the stochastic current within the canonical framework of white noise analysis. We define Wick-type stochastic current by using Wick integral with respect to Brownian motion, firstly. Moreover, we prove that the Brownian stochastic current is considered as a Hida distribution in terms of white noise approach and **S**-transform.

Keywords Brownian motion; stochastic current; Hida distribution; white noise approach.

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1. Introduction

The concept of current comes from geometric measure theory. The simplest is the functional

$$\varphi \to \int_0^T \langle \varphi(\gamma(t)), \gamma(t)' \rangle_{R^d} \mathrm{d}t,$$

where $\varphi : R^d \to R^d$ and $\gamma(t)$ is a rectifiable curve. A functional $\xi(x)$ is defined by

$$\xi(x) = \int_0^T \delta(x - \gamma(t))\gamma(t)' \mathrm{d}t,$$

where $\delta(x)$ is a Dirac function. If we want to simulate this current, we need replace the deterministic curve $\gamma(t)$ with stochastic process X_t. At the same time, the stochastic integral must be properly interpreted. Recently, attentions have been paid to the research on stochastic current. In general, stochastic current is defined by

$$\varphi \to I(\varphi) = \int_0^T \langle \varphi(X_t), \mathrm{d}X_t \rangle,$$

where φ is a vector function on R^d belonging to some Banach spaces V, X_t is a stochastic process and the integral is some version of a stochastic integral defined through regularization. Stochastic current is a continuous version of the mapping, i.e., stochastic current is regarded as a stochastic element of the dual space of V (see [1]).

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The problem of stochastic current is motivated by the study of fluidodynamical models. In [2], in the study of the energy of a vortex filament naturally appear some stochastic double integrals related to Wiener process

$$\int_{[0,T]^2} f(X_s - X_t)\mathrm{d}X_s\mathrm{d}X_t,$$

where $f(x) = K_\alpha(x)$ is the kernel of the pseudo-differential operator $(1-\Delta)^{-\alpha}$.

Some results about stochastic currents of Gaussian processes have been obtained in recent years. One direction of research is the regularity for differential stochastic integral. For example, Flandoli et. al have studied the existence and regularity of stochastic currents through Malliavin calculus, where the integrals are defined as Skorohod integrals with respect to the Brownian motion and fractional Brownian motion, respectively. In [3] authors have shown the Sobolev regularity of the stochastic current, which is associated with the pathwise integral.

Because there exists non-adaptable stochastic integral, when we define Brownian stochastic current, we have to look for some methods to deal with this integral easily and give a proper explanation. It seems to be desirable to define the Wick-type Brownian stochastic current rather than other stochastic current. For this purpose, motivated by [4-8], we define the Brownian stochastic current via Wick integral and verify that it is a Hida distribution by using white noise analysis and **S**-transform. Let us compare our results with the analogous ones from the case of Brownian motion in [4]. Note that Flandoli et. al [4] considered the Skorohod-type stochastic current as a distribution in the Sobolev spaces of negative order through Malliavin calculus.

The rest of paper is organized as follows. In Section 2, we provide some background materials in white noise analysis. In Section 3, we firstly define the Wick-type stochastic current of Brownian motion. Lastly, we prove that Brownian stochastic current is a Hida distribution in white noise analysis framework.

2. White noise analysis

In this section we briefly recall some notions and facts in white noise analysis, and refer to [5, 7–9] for details.

The starting point of white noise analysis is the real Gelfand triple $S(R) \subset L^2(R, R^d) \subset S^*(R)$, where $S(R)$ and $S^*(R)$ are the Schwartz spaces of test functions and tempered distributions, respectively.

Let $(L^2) \equiv L^2(S^*(R), \mathrm{d}\mu)$ be the Hilbert space of μ-square integrable functionals on $S^*(R)$. Then by the Wiener-Itô-Segal isomorphism theorem, for each $\Phi \in (L^2)$ this implies the chaos expansion $\Phi(\omega) = \sum_{n=0}^\infty \langle : \omega^{\otimes n} :, F_n\rangle$. Let $\Gamma(A)$ be the second quantization of A, where A is defined by

$$(A\mathbf{g})_i(t) = (-\frac{d^2}{dt^2} + t^2 + 1)g_i(t).$$

For each integer p, let (S_p) be the completion of $\mathrm{Dom}\Gamma(A)^p$ with respect to the Hilbert norm $\|\cdot\|_p = \|\Gamma(A)^p\|_0$. Let $(S) = \bigcap_{p\geq 0}(S_p)$ be the projective limit of $\{(S_p) \mid p \geq 0\}$ and $(S)^* = \bigcup_{p\geq 0}(S_{-p})$ be the inductive limit of $\{(S_{-p}) \mid p \geq 0\}$, respectively.

The second Gelfand triple is: $(S) \subset (L^2) \subset (S)^*$. Elements of (S) (resp., $(S)^*$) are called Hida testing (resp., generalized) functionals. For $f \in S(R)$, $\mathbf{S}$-transform is defined to be the bilinear dual product on $(S) \times (S)^*$ by $\mathbf{S}\Phi(f) = \ll \Phi, e^{-(1/2)\|\xi\|^2} \exp\langle ., f\rangle \gg$.

Definition 2.1 ([5, 9]) *A function $G : S(R) \to \mathbb{C}$ is called a U-functional whenever*

(i) For every $\mathbf{f}_1, \mathbf{f}_2 \in S(R)$ the mapping $G(\lambda\mathbf{f}_1 + \mathbf{f}_2)$ has an entire extension to $\lambda \in \mathbb{C}$;

(ii) There are constants $C_1, C_2 > 0$ such that

$$| G(z\boldsymbol{f}) | \leq C_1 \exp\{C_2 \mid z \mid^2 | A^p \boldsymbol{f} |_2^2\}$$

with $p > 0$, $\forall z \in \mathbb{C}$.

Lemma 2.2 ([5, 9]) *Let $\{G_k\}_{k\in N}$ denote a sequence of U-functional with following properties:*

(i) For all $\boldsymbol{f} \in S(R), \{G_k(\boldsymbol{f})\}_{k\in N}$ is a Cauchy sequence;

(ii) There exist C_i, p such that $| G_k(z\boldsymbol{f}) | \leq C_1 \exp\{C_2 \mid z \mid^2 | A^p \boldsymbol{f} |_2^2\}$ uniformly in R. Then there is a unique $\Phi \in (S)^*$ such that $\mathbf{S}^{-1}G_k$ converges strongly to Φ.

Lemma 2.3 ([5, 9]) *Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space, and Φ_λ be a mapping defined on Ω with values in $(S)^*$. We assume $\mathbf{S}$-transform of Φ_λ:*

(i) is a μ-measurable function of λ for $\boldsymbol{f} \in S(R)$;

(ii) obeys a U-functional estimate

$$| \mathbf{S}\Phi_\lambda(z\boldsymbol{f}) | \leq C_1(\lambda) \exp\{C_2(\lambda) \mid z \mid^2 | A^p \boldsymbol{f} |_2^2\}$$

for some fixed p and for $C_1 \in L^1(\mu)$, $C_2 \in L^\infty(\mu)$. Then Φ_λ is Bochner-integrable in the Hilbert spaces $(S)_{-q}$ for q large enough and

$$\int_\Omega \Phi_\lambda \mathrm{d}\mu(\lambda) \in (S)^*, \quad \mathbf{S}(\int_\Omega \Phi_\lambda \mathrm{d}\mu(\lambda))(\mathbf{f}) = \int_\Omega (\mathbf{S}\Phi_\lambda)(\mathbf{f}) \mathrm{d}\mu(\lambda).$$

3. Stochastic current of Brownian motion

In this section, we define Wick-type Brownian stochastic current, firstly. Next we use developed white noise analysis and the $\mathbf{S}$-transform to deal with the problems of existence of stochastic currents. Under some conditions, we show that Wick-type Brownian stochastic current is a Hida distribution.

Definition 3.1 *Let $\varphi : R^d \to R^d$ be defined on the set of all smooth compact support vector fields. Then $\varphi \to I(\varphi) \equiv \int_0^T \langle\varphi(B_t), \diamond dB_t\rangle$ is a functional in space of those vector fields. Wick-type Brownian stochastic current is given by*

$$\xi(x) = \int_0^T \delta(x - B_t) \diamond W_t \mathrm{d}t, \tag{1}$$

where $W_t = \frac{dB_t}{dt}$ and $\diamond$ denotes Wick integral.

The Wick-type stochastic current given in Definiton 3.1 is restricted to Brownian motion, which is a case of the stochastic current in our introduction. The stochastic current in our paper is defined as Wick integral with respect to Brownian motion.

Theorem 3.2 *For every positive integer d and $\varepsilon > 0$ Brownian stochastic current*

$$\xi_\varepsilon(x) = \int_0^T p_\varepsilon(x - B_t) \diamond W_t \mathrm{d}t,$$

where $p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}} exp\{-\frac{x^2}{2\varepsilon}\}$, is a Hida distribution. Moreover, for each $\mathbf{f} \in S(R)$, $\mathbf{S}$-transform of $\xi_\varepsilon(x)$ is given by

$$\mathbf{S}(\xi_\varepsilon(x))(\mathbf{f}) = \int_0^T (\frac{1}{2\pi(\varepsilon + t)})^{\frac{d}{2}} \exp\{\frac{(x - \int_0^t \mathbf{f}(s)ds)^2}{2(\varepsilon + t)}\} \mathbf{f}(t)\mathrm{d}t. \tag{2}$$

Proof Put

$$\Phi_\varepsilon(\mathbf{w}) \equiv (\frac{1}{2\pi\varepsilon})^{\frac{d}{2}} \exp\{-\frac{(x - B_t)^2}{2\varepsilon}\}.$$

For arbitrary $\mathbf{f} \in S(R)$, we need to verify that the $\mathbf{S}$-transform of the integrand

$$\begin{aligned} &\mathbf{S}(\Phi_\varepsilon(\mathbf{w}) \diamond W_t)(\mathbf{f}) = \mathbf{S}(\Phi_\varepsilon(\mathbf{w}))(\mathbf{f})\mathbf{S}(W_t)(\mathbf{f}) \\ &= \prod_{i=1}^d (\frac{1}{2\pi(\varepsilon + t)})^{\frac{1}{2}} \exp\{-\frac{(x - \int_0^t f_i(s)\mathrm{d}s)^2}{2(\varepsilon + t)}\} f_i(t) \end{aligned}$$

obeys the conditions of Lemma 2.3 with regard to Lebesgue measure $\mathrm{d}\lambda$ on R^d. Measurability is evident. Next we will prove that the bound condition is also satisfied.

For all complex $z \in \mathbb{C}$, we have

$$\begin{aligned} &\mid \mathbf{S}(\Phi_\varepsilon(\mathbf{w}) \diamond W_t)(z\mathbf{f}) \mid \\ &\le (\frac{1}{2\pi(\varepsilon + t)})^{\frac{d}{2}} \exp\{\frac{x^2 + \mid \int_0^t z\mathbf{f}(s)\mathrm{d}s \mid^2}{\varepsilon + t}\} \mid z\mathbf{f}(t) \mid \\ &\le (\frac{1}{2\pi(\varepsilon + t)})^{\frac{d}{2}} \exp\{\frac{x^2 + \mid z \mid^2 t^2 \sum_{i=1}^d \sup_{x \in R} \mid f_i(x) \mid^2}{\varepsilon + t}\} \mid z \mid \sum_{i=1}^d \sup_{x \in R} \mid f_i(x) \mid, \end{aligned}$$

where exponential part in the last inequality is integrable, and $\frac{t^2}{t+\varepsilon}$ is bounded on $[0, T]$. Thus, according to Lemma 2.3 the result is obtained. □

Theorem 3.3 *Set $t > 0$. Then Bochner integral*

$$\delta(x - B(t)) \equiv (\frac{1}{2\pi})^d \int_{R^d} \exp\{i\lambda(x - B_t)\}\mathrm{d}\lambda \tag{3}$$

and

$$\xi(x) = \int_0^T \delta(x - B_t) \diamond W_t \mathrm{d}t \tag{4}$$

are both Hida distributions. Moreover, when ε tends to 0, $\xi_\varepsilon(x)$ converges to $\xi(x)$ in $(S)^*$.

Proof Recalling some facts in [5], we can show that

$$\delta(x - B(t)) \equiv (\frac{1}{2\pi})^d \int_{R^d} \exp\{i\lambda(x - B_t)\}\mathrm{d}\lambda \tag{5}$$

is a Hida distribution.

In fact, let us introduce the following notation

$$\Phi(\mathbf{w}) \equiv \exp\{i\lambda(x - B_t)\}.$$

We will check that the **S**-transform of $\Phi(\mathbf{w})$ satisfies the condition of its applicability with respect to Lebesgue on R^d. From the definition of **S**-transform, we now calculate

$$\begin{aligned}\mathbf{S}(\Phi(\mathbf{w}))(\mathbf{f}) &= \mathbf{S}(e^{i\lambda(x-B_t)})(\mathbf{f}) = \exp\{i\lambda x\}E(\exp\{-i\lambda\langle \mathbf{w}+\mathbf{f},\mathbf{I}_{[0,t]}\rangle\}) \\ &= \exp\{i\lambda x - \frac{1}{2}\lambda^2 t - i\lambda\int_0^t \mathbf{f}(s)\mathrm{d}s\}. \end{aligned} \tag{6}$$

The measurability is obvious and the boundedness condition can be verified as follows

$$\begin{aligned}&\mid \mathbf{S}(\delta(x-B_t)(z\mathbf{f}) \mid \\ &= (\frac{1}{2\pi})^d \mid \int_{R^d} \exp\{-\frac{1}{2}\lambda^2 t + i\lambda(x - z\int_0^t \mathbf{f}(s)\mathrm{d}s)\}\mathrm{d}\lambda \mid \\ &\le (\frac{1}{2\pi})^d \mid \int_{R^d} \exp\{-\frac{1}{4}\mid\lambda\mid^2 t\}\exp\{-\frac{1}{4}\mid\lambda\mid^2 t + \mid\lambda\mid\mid x - \mid z\mid \int_0^t \mathbf{f}(s)\mathrm{d}s \mid\}\mathrm{d}\lambda \mid \\ &\le (\frac{1}{2\pi})^d \mid \int_{R^d} \exp\{-\frac{1}{4}\mid\lambda\mid^2 t\}\exp\{-(\frac{\mid\lambda\mid\sqrt{t}}{2} - \frac{1}{\sqrt{t}}\mid x - \mid z\mid\int_0^t \mathbf{f}(s)\mathrm{d}s\mid)^2\}\cdot \\ &\quad \exp\{\frac{1}{t}\mid x - \mid z\mid \int_0^t \mathbf{f}(s)\mathrm{d}s\mid^2\}\mathrm{d}\lambda \mid \\ &\le (\frac{1}{2\pi})^d \mid \int_{R^d} \exp\{-\frac{1}{4}\mid\lambda\mid^2 t\}\exp\{2(x^2 + \mid z\mid^2 t^2(\sum_{i=1}^d \sup_{x\in R}\mid f_i(x)\mid)^2)\frac{1}{t}\}\mathrm{d}\lambda \mid,\end{aligned}$$

for all $z \in \mathbb{C}$.

Secondly, by using similar method, we apply Lemma 2.3 again to verify

$$\xi(x) = \int_0^T \delta(x-B_t)\diamond W_t \mathrm{d}t \tag{7}$$

is also a Hida distribution.

In fact, it follows from (5)–(7)

$$\begin{aligned}\mathbf{S}(\delta(x-B_t)\diamond W_t)(\mathbf{f}) &= (\frac{1}{2\pi})^d \int_{R^d} \mathbf{S}(\Phi(\mathbf{w}))(\mathbf{f})\mathbf{S}(W_t)(\mathbf{f})\mathrm{d}\lambda \\ &= (\frac{1}{2\pi})^d \int_{R^d} \exp\{-\frac{1}{2}\lambda^2 t + i\lambda(x - \int_0^t \mathbf{f}(s)\mathrm{d}s)\}\mathrm{d}\lambda \mathbf{f}(t).\end{aligned}$$

For all $z \in \mathbb{C}$, the bound is obtained as follows:

$$\begin{aligned}&\mid \mathbf{S}(\delta(x-B_t)\diamond W_t)(z\mathbf{f}) \mid \\ &= (\frac{1}{2\pi})^d \mid z\mathbf{f}(t) \mid\mid \int_{R^d} \exp\{-\frac{1}{2}\lambda^2 t + i\lambda(x - z\int_0^t \mathbf{f}(s)\mathrm{d}s)\}\mathrm{d}\lambda \mid \\ &\le (\frac{1}{2\pi})^d \mid z\mathbf{f}(t) \mid\mid \int_{R^d} \exp\{-\frac{1}{4}\mid\lambda\mid^2 t\}\cdot \\ &\quad \exp\{-(\frac{\mid\lambda\mid\sqrt{t}}{2} - \frac{1}{\sqrt{t}}\mid x - \mid z\mid\int_0^t \mathbf{f}(s)\mathrm{d}s\mid)^2\}\cdot \\ &\quad \exp\{\frac{1}{t}\mid x - \mid z\mid\int_0^t \mathbf{f}(s)\mathrm{d}s\mid^2\}\mathrm{d}\lambda \mid \\ &\le (\frac{1}{2\pi})^d \mid z\mid \sum_{i=1}^d \sup_{x\in R}\mid f_i(x)\mid\mid \int_{R^d} \exp\{-\frac{1}{4}\mid\lambda\mid^2 t\}\cdot\end{aligned}$$

$$\exp\{2(x^2+\mid z\mid^2 t^2(\sum_{i=1}^{d}\sup_{x\in R}\mid f_i(x)\mid)^2)\frac{1}{t}\}d\lambda\mid. \tag{8}$$

The first exponential part in (8) is integrable on R^d with respect to λ, while the second part is a constant with respect to λ.

Hence, according to Lemma 2.3, Brownian stochastic current

$$\xi(x)=\int_0^T\delta(x-B_t)\diamond W_t\mathrm{d}t$$

is a Hida distribution, and the following equalities are established

$$\begin{aligned}\mathbf{S}(\xi(x))(\mathbf{f})&=\int_0^T\mathbf{S}(\delta(x-B_t))(\mathbf{f})\mathbf{S}(W_t)(\mathbf{f})\mathrm{d}t\\&=(\frac{1}{2\pi})^d\int_0^T\int_{R^d}\exp\{-\frac{1}{2}\lambda^2t+i\lambda(x-\int_0^t\mathbf{f}(s)\mathrm{d}s)\}\mathbf{f}(t)\mathrm{d}\lambda\mathrm{d}t.\end{aligned}$$

Lastly, when ε tends to 0, by dominated convergence theorem $\mathbf{S}(\xi_\varepsilon(x))(\mathbf{f})$ converges to $\mathbf{S}(\xi(x))(\mathbf{f})$. Applying Lemma 2.2, we obtain the required convergence. In other words, when ε tends to 0, $\xi_\varepsilon(x)$ converges to $\xi(x)$ in $(S)^*$. $\square$

From what we stated above, we can draw the conclusion that our results are different from those in [4], where the Brownian stochastic current was used to define Skorohod integral via Malliavin calculus. Here we are interested in Hida generalized functionals space and the condition of Brownian stochastic current belonging to this space. On the other hand, we give Wick-type Brownian stochastic current for the first time, which admits us to use the developed white noise theory. In particular, the **S**-transform and the space of generalized white noise functionals are playing significant roles.

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