On Sums of Powers of Odd Integers

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Abstract In this paper, by using superposition method, we aim to show that $\sum_{i=1}^{n} (2i - 1)^{2k-1}$ is the product of $n^2$ and a rational polynomial in $n^2$ with degree $k-1$, and that $\sum_{i=1}^{n} (2i - 1)^{2k}$ is the product of $n(2n - 1)(2n + 1)$ and a rational polynomial in $(2n - 1)(2n + 1)$ with degree $k-1$. Moreover, recurrence formulas to compute the coefficients of the corresponding rational polynomials are also obtained.

Keywords odd number; sums of powers; binomial theorem; superposition method.

MR(2010) Subject Classification 11A25; 11B57; 11B75

1. Introduction

The study on sums of powers of integers $\sum_{i=1}^{n} i^m$ has a long history. In recent years, it has been extensively studied and many good results have been obtained [2–6, 8, 9, 11–13, 16], some of which were improved in our former paper [10] along with [3–6, 16]. It is well-known that one of the unusual and effective methods used to express the sums of powers of integers used the so-called Bernoulli numbers, while Edwards [8] traced the knowledge of the formula to Johann Faulhaber who showed us how to obtain the coefficients by matrix inversion. By submitting problem E3204 to the Math Monthly [9], Gessel responded to the same stimulus on a bivariate generating function for Faulhaber’s coefficients. Faulhaber’s work, including more generally r-fold sums of powers, was nicely exposed by Knuth [11]. For the study of polynomial relations between sums of powers functions, the readers can refer to Beardon [2]. Recall that in [10] we have proved that

$$\sum_{i=1}^{n} i^{2k+1} = n^2(n+1)^2 \sum_{i=1}^{k} a_i n^{k-i}(n+1)^{k-i}, \quad k = 1, 2, \ldots, \quad (1.1)$$

where

$$\begin{align*}
a_1 &= \frac{1}{2(k+1)}, \\
a_i &= \frac{\sum_{r=1}^{i-1} a_r (\frac{k-r+2}{2i-2r+1})}{k-i+2}, \quad i = 2, 3, \ldots, k. \quad (1.2)
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\end{align*}$$

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In fact, \( a_i \) (\( i = 2, 3, \ldots, k \)) can also be rewritten as

\[
\begin{cases}
    a_i = -\sum_{r=1}^{i-1} b_r \left( \frac{k-r+2}{2i-2r+1} \right), & i \leq \left\lfloor \frac{k+2}{2} \right\rfloor, \\
    a_i = -\sum_{r=2i-k-1}^{i-1} b_r \left( \frac{k-r+2}{2i-2r+1} \right), & i > \left\lfloor \frac{k+2}{2} \right\rfloor,
\end{cases}
\]

Therefore, we have

\[
\begin{align*}
a_2 &= -\frac{k-1}{2 \cdot 3!}, \\
\frac{a_3}{3!} &= \frac{k(k-2)(7k-1)}{3!}, \\
\frac{a_4}{3!} &= -\frac{k(k-1)(k-3)}{3!} (31k^2 - 27k - 10), \\
\frac{a_5}{10!} &= 3k(k-1)(k-2)(k-4) \left( \frac{127k^3 - 310k^2 + 37k + 90}{10!} \right), \\
\frac{a_6}{11!} &= -\frac{k(k-1)(k-2)(k-3)(k-5)}{11!} \left( \frac{2555k^4 - 12674k^3 + 14161k^2 + 2486k - 3864}{11!} \right), \\
\frac{a_7}{15!} &= \frac{k(k-1)(k-2)(k-3)(k-4)(k-6)}{15!} \left( \frac{141477k^5 - 1197437k^4 + 31092673k^3 - 22432587k^2 - 7706534k + 6399960}{15!} \right), \\
\end{align*}
\]

\[
\begin{align*}
a_{k-1} &= -\frac{4a_{k-2} + a_{k-3}}{3}, \\
a_k &= -\frac{a_{k-1}}{2}.
\end{align*}
\]

In [10], we have also proved that

\[
\sum_{i=1}^{n} i^{2k} = n(n+1)(2n+1) \sum_{i=1}^{k} b_i n^{k-i}(n+1)^{k-i}, \quad k = 1, 2, \ldots,
\]

where

\[
\begin{align*}
b_1 &= \frac{1}{(4k+2)}; \\
b_i &= -\sum_{r=1}^{i-1} b_r \left( \frac{k-r+2}{2i-2r+1} \right), \quad i = 2, 3, \ldots, k,
\end{align*}
\]

and \( b_i \) (\( i = 2, 3, \ldots, k \)) can be rewritten as

\[
\begin{align*}
b_1 &= -\sum_{r=1}^{i-1} b_r \left( \frac{k-r+2}{2i-2r+1} \right), \quad i \leq \left\lfloor \frac{k+2}{2} \right\rfloor, \\
b_i &= -\sum_{r=2i-k-1}^{i-1} b_r \left( \frac{k-r+2}{2i-2r+1} \right), \quad i > \left\lfloor \frac{k+2}{2} \right\rfloor.
\end{align*}
\]

However, to the best of our knowledge, the results concerning sums of powers of odd integers
are very few [1, 14, 15]. In this case, it follows straightforward that

\[\sum_{i=1}^{n} (2i-1)^m = \sum_{i=1}^{2n} i^m - 2\sum_{i=1}^{n} i^m.\] (1.5)

Thus, using (1.1), (1.2) and (1.5), we obtain that,

\[\begin{aligned}
&\sum_{i=1}^{n} (2i-1) = n^2, \\
&\sum_{i=1}^{n} (2i-1)^3 = n^2(2n^2 - 1), \\
&\sum_{i=1}^{n} (2i-1)^5 = \frac{n^2(16n^4 - 20n^2 + 7)}{3}.
\end{aligned}\] (1.6)

In an analogous way, (1.3) together with (1.4) and (1.5) yields that,

\[\begin{aligned}
&\sum_{i=1}^{n} (2i-1)^2 = \frac{n(2n-1)(2n+1)}{3}, \\
&\sum_{i=1}^{n} (2i-1)^4 = n(2n-1)(2n+1)[\frac{(2n-1)(2n+1)}{5} - \frac{4}{15}], \\
&\sum_{i=1}^{n} (2i-1)^6 = n(2n-1)(2n+1)[\frac{(2n-1)^2(2n+1)}{7} - \frac{4(2n-1)(2n+1)}{7} + \frac{16}{21}].
\end{aligned}\] (1.7)

The relations in (1.6) and (1.7) motivate us to conjecture that for \(k = 1, 2, \ldots,\)

\[\begin{aligned}
&\sum_{i=1}^{n} (2i-1)^{2k-1} = n^2 \sum_{i=1}^{k} c_i n^{2k-2i}, \quad (1.8) \\
&\sum_{i=1}^{n} (2i-1)^{2k} = n(2n-1)(2n+1) \sum_{i=1}^{k} d_i (2n-1)^{k-i} (2n+1)^{k-i}, \quad (1.9)
\end{aligned}\]

where \(c_i\) and \(d_i\) (\(i = 1, 2, \ldots, k\)) are undetermined constants.

In fact, \(\sum_{i=1}^{n} (2i-1)^{2k-1}\) is a rational polynomial in \(n^2\) from Faulhaber's theorem on sums of odd powers [1, 7, 11]. However, no explicit calculation formula on \(\sum_{i=1}^{n} (2i-1)^{2k-1}\) has been given. In this paper, we aim to establish two concise calculation formulae on sums of powers of odd integers by using superposition method. In other words, we will show that our conjectures (1.8) and (1.9) are indeed valid. In addition, we obtain recurrence formulas to compute the coefficients of rational polynomials in formulas above. Furthermore, we also give the relations among \(a_i, b_i, c_i\) and \(d_i\) (\(i = 1, 2, \ldots, k\)). Finally, we put forward a conjecture on signs of the coefficients in formulas (1.8) and (1.9).

2. The summation theorems
Theorem 2.1 For any \( n \in N^+ \), (1.8) holds if and only if

\[
\begin{aligned}
c_1 &= \frac{4^{k-1}}{k}, \\
c_i &= -4^{i-1} P(2k-1, 2i-3)\alpha_i, \quad i = 2, 3, \ldots, k,
\end{aligned}
\]

where \( \alpha_2 = \frac{1}{3}, \alpha_i = \frac{\alpha_{i-1}}{(2i-3)!} = \cdots = \frac{\alpha_1}{3!} \) (\( i = 3, 4, \ldots, k \)).

Theorem 2.2 Let \( P(n, r) := n(n-1) \cdots (n-r+1) \) \((n, r \in N^+, r \leq n)\) and \( \alpha_1 = 2 \). Then

\[
c_i = -4^{k-i} P(2k-1, 2i-3)\alpha_i, \quad i = 2, 3, \ldots, k,
\]

where \( \alpha_2 = \frac{1}{3}, \alpha_i = \frac{\alpha_{i-1}}{(2i-3)!} = \cdots = \frac{\alpha_1}{3!} \) (\( i = 3, 4, \ldots, k \)).

Theorem 2.3 For any \( n \in N^+ \), (1.9) holds if and only if

\[
\begin{aligned}
d_1 &= \frac{1}{2k+1}, \\
d_i &= -4^i \sum_{r=1}^{i-1} d_r \frac{(k-r+2)}{4(k-r+2)}, \quad i = 2, 3, \ldots, k,
\end{aligned}
\]

and \( d_i \) \((i = 2, 3, \ldots, k)\) can also be rewritten as

\[
\begin{aligned}
d_i &= -4^i \sum_{r=1}^{i-1} d_r \frac{(k-r+2)}{4(k-r+2)}, \quad i \leq \lfloor \frac{k+2}{2} \rfloor, \\
d_i &= -4^i \sum_{r=2i-k-1}^{i-1} d_r \frac{(k-r+2)}{4(k-r+2)}, \quad i > \lfloor \frac{k+2}{2} \rfloor.
\end{aligned}
\]

From (1.2) and (1.4) we get immediately the following corollary.

Corollary 2.1 It holds that \( b_i = \frac{k-i+2}{2k+1} a_i \) \((i = 1, 2, \ldots, k)\), where \( a_i \) and \( b_i \) are given by (1.2) and (1.4), respectively.

From which and (1.4), (2.3) we have the following corollary.

Corollary 2.2 It holds that \( d_i = 2^{2i-1}b_i = 2^{2i-1} \frac{k-i+2}{2k+1} a_i \) \((i = 1, 2, \ldots, k)\), where \( a_i, b_i \) and \( d_i \) are as presented in (1.2), (1.4) and (2.3), respectively.

To end this section, we put forward the following conjecture on the signs of the coefficients \( c_i \) and \( d_i \) in Theorems 2.1 and 2.3.

Conjecture It holds that \( \text{sgn } c_i = (-1)^{i-1}, \text{sgn } d_i = (-1)^{i-1}, \) \( i = 1, 2, \ldots, k \).

3. Proofs of the summation theorems

In this section, we present the proofs of our results stated in the above section, by using superposition method.

Proof of Theorem 2.1 Let \( b_m = (2m-1)^{2k-1} \) and \( s_0 = 0 \), \( s_m = \sum_{i=1}^{k} c_i m^{2k-2i+2} \) \((m = 1, 2, \ldots, n)\). Clearly, (1.8) holds if and only if \( b_m = s_m - s_{m-1} \). On the other hand, since

\[
c_i m^{2k-2i+2} - c_i (m-1)^{2k-2i+2} = \frac{c_i}{2^{2k-2i+2}} [(2m)^{2k-2i+2} - (2m-2)^{2k-2i+2}]
\]
we have

\[
s_m - s_{m-1} = \sum_{i=1}^{k} \frac{c_i}{2^{2k-2r+1}} \sum_{r=1}^{k-i+1} (2^{k-2r+2})(2m-1)^{2k-2r+3},
\]

Hence, \(s_m - s_{m-1} = (2m - 1)^{2k-1}\) holds if and only if \(c_i (i = 1, 2, \ldots, k)\) satisfy the following system of linear equations.

\[
\begin{cases}
  c_1(\frac{2k}{1}) = 2^{2k-1}, \\
  c_1(\frac{2k}{3}) + c_2 2^{2k-2} = 0, \\
  \vdots \\
  c_1(\frac{2k}{2i-1}) + c_2 2^{2k-2(\frac{2k}{2i-3})} + \cdots + c_i 2^{2i-2}(\frac{2k-2i+2}{1}) = 0, \\
  \vdots \\
  c_1(\frac{2k}{2k-1}) + c_2 2^{2k-2(\frac{2k}{2k-3})} + \cdots + c_k 2^{2k-2(\frac{1}{1})} = 0.
\end{cases}
\]

(3.1)

The equivalence between (2.1) and (3.1) follows from direct calculation. The proof is completed.

\(\square\)

**Proof of Theorem 2.2** We derive the result by using the second induction method.

From (2.1), \(c_2 = -\frac{4^{k-2}(2k-1)}{3}\), so that for \(i = 2\), (2.2) is true.

Assume that (2.2) is true for \(i \leq m\) \((2 \leq m < k)\). Then from (2.1) we have

\[
c_{m+1} = -\sum_{r=1}^{m} \frac{4^r c_r (2k-2r+2)(2m-1)^{2k-2r+3}}{2^{2m+3}(k-m)}
\]

\[
= -4^r c_r (\frac{2k}{2m+1}) + \sum_{r=2}^{m} \frac{4^r P(2k-1, 2r-3)\alpha_r (2k-2r+2)}{2^{2m+3}(k-m)}
\]

\[
= -4^{k-m-1} P(2k-1, 2m-1)\alpha_m + \sum_{r=2}^{m} \frac{4^{k-m-1} P(2k-1, 2m-1)\alpha_r}{(2m+1)!}
\]

\[
= -4^{k-m-1} P(2k-1, 2m-1)\alpha_{m+1},
\]

hence (2.2) is true for \(i = m + 1\). The proof is completed. \(\square\)

**Proof of Theorem 2.3** Let \(b_m = (2m - 1)^{2k}\) and \(s_0 = 0, s_m = \sum_{i=1}^{k} d_i m (2m-1)^{k-i+1}(2m+1)^{k-i+1}\) \((m = 1, 2, \ldots, n)\). Clearly, (1.9) holds if and only if \(b_m = s_m - s_{m-1}\). For simplicity, we let \(p = k - i + 1\), then it holds that

\[
d_i m (2m-1)^p(2m+1)^p - d_i (m-1)(2m-3)^p(2m-1)^p
\]

\[
= d_i (2m-1)^p[(2m-1+1)(2m-1+2)^p - (2m-1)(2m-2)^p]
\]

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\]

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\[ s \sum_{r=1}^{m} 2^{2r-2} (2^{p}_{2r-1} + (2^{p}_{2r-2}))(2m-1)^{2r-2}. \]

Hence, we have

\[
s_m - s_{m-1} = \sum_{i=1}^{k} d_i \sum_{r=1}^{m} 2^{2r-2} (2^{p}_{2r-1} + (2^{p}_{2r-2}))(2m-1)^{2r-2} + \]

\[
\sum_{r=1}^{m} d_r 2^{2-2r} (2^{k-r+1}_{2r-1} + (k-r+1)) (2m-1)^{2k-2r+2} + \]

\[
\sum_{i=1}^{k} d_i 2^{2i-2r} (2^{k-r+1}_{2i-2r+1} + (k-r+1)) (2m-1)^{2k-2i+2} + \]

\[
\sum_{r=1}^{m} \left[ \sum_{i=1}^{k} d_i 2^{2i-2r} (2^{k-r+1}_{2i-2r+1} + (k-r+1)) \right] + \]

\[
d_{2k-1} 2^{k-2} (2m-1)^{2k-2} = \sum_{p=\lfloor \frac{k}{2} \rfloor + 1}^{k} \sum_{r=1}^{k-p+1} d_r 2^{k-2p-2r+2} (2^{k-r+1}_{2k-2p-2r+3} + (k-r+1)) (2m-1)^{2p} + \]

\[
\sum_{i=1}^{k} d_i 2^{k-2p} (2^{l}_{2i-2p+1} + (l-r+1)) (2m-1)^{2p} + \]

\[
\sum_{r=1}^{m} \left[ \sum_{i=p}^{\lfloor \frac{k}{2} \rfloor} d_{k-i+2} 2^{2i-2p} (2^{l}_{2i-2p+1} + (l-r+1)) + d_{k-2p+1} 2^{2p} (2m-1)^{2p} \right]. \]

Hence, \( s_m - s_{m-1} = (2m-1)^{2k} \) holds if and only if \( d_i \) \((i = 1, 2, \ldots, k)\) satisfy the following system of linear equations.

\[
\begin{align*}
&d_1 (2^1) + 1 = 1, \\
&d_1 2^2 (2^3) + (2^1) + d_2 (2^{k-1}) + 1 = 0, \\
&\ldots \\
&d_1 2^{2i-2} (2^{l}_{2i-2} + (2^{k-2}) + d_2 2^{2i-4} (2^{l}_{2i-4} + (2^{k-4}) + \ldots + d_{i} (2^{k-2} + 1) + 1 = 0, \\
&\ldots \\
&d_{k-1} 2^2 (2^1) + d_k (2^1) + 1 = 0.
\end{align*}
\]

Along the same line as in Theorem 2.1, we can show the equivalence between (2.3) and (3.2).

The proof is completed. □
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References


