The Signless Laplacian Spectral Radii and Spread of Bicyclic Graphs

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Abstract The signless Laplacian spread of a graph is defined to be the difference between
the largest eigenvalue and the smallest eigenvalue of its signless Laplacian matrix. In this
paper, we determine the first to 11th largest signless Laplacian spectral radii in the class of
bicyclic graphs with n vertices. Moreover, the unique bicyclic graph with the largest or the
second largest signless Laplacian spread among the class of connected bicyclic graphs of order
n is determined, respectively.

Keywords bicyclic graph; signless Laplacian; spread; spectral radius.

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1. Introduction

In this paper, let G = (V, E) be a simple graph with n vertices and m edges. If G is
connected with m = n + c − 1, then G is called a c-cyclic graph. Especially, if c = 0, 1, 2 or
3, then G is called a tree, a unicyclic graph, a bicyclic graph or a tricyclic graph, respectively.
Let Bn be the class of bicyclic graphs with n vertices. Let d(u) = dG(u) denote the degree of
the vertex u of G. Let ∆ = ∆(G) and δ = δ(G) be the maximum degree and minimum degree
of G, respectively. If d(u) = 1, we call u a pendant vertex of G. The neighbor set of a vertex
v is denoted by N(v) = N_G(v). The adjacency matrix of G is the n × n matrix A(G) = (a_{ij}),
where a_{ij} = 1 if and only if v_i and v_j are adjacent in G and a_{ij} = 0 otherwise. Since A(G) is
symmetric, the eigenvalues of A(G) can be arranged as follows: λ_1(G) ≥ · · · ≥ λ_{n−1}(G) ≥ λ_n(G).
The adjacency spread of the graph G is defined as [14]: SA(G) = λ_1(G) − λ_n(G). The spread
of a graph has received much attention. In [24], Petrović determined all connected graphs with
adjacency spread at most 4. In [14,17], Gregory, Liu et al. presented some lower and upper
bounds for the spread of a graph. In [16], Li et al. determined the unique graph with maximal
spread among all unicyclic graphs on n(≥ 18) vertices with a maximum matching of cardinality
k. After then, in [12] and [23], Fan and Petrović et al. determined the maximal spreads among
all unicyclic graphs and all bicyclic graphs of given order n, respectively.

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Let $D(G) = \text{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ denote the diagonal matrix of vertex degrees of a graph $G$ of order $n$. The Laplacian matrix of $G$ is defined as $L(G) = D(G) - A(G)$. Since $L(G)$ is positive semidefinite, its eigenvalues can be arranged as follows: $\mu_1(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$. The Laplacian spread of the graph $G$, denoted by $SL(G)$, is defined to be $SL(G) = \mu_1(G) - \mu_{n-1}(G)$. In [13], Fan et al. identified the maximal and minimal Laplacian spreads among all trees (resp., connected unicyclic graphs) of given order. Fan et al. proved that there exist exactly two bicyclic graphs with maximal Laplacian spread among all bicyclic graphs of given order. In [9], Chen and Wang proved that there exist exactly five types of tricyclic graphs with maximal Laplacian spread among all tricyclic graphs of given order. Bao et al. [1] and Li et al. [15] determined the maximal Laplacian spread among all unicyclic graphs of given order by using different methods, respectively. In [26], You and Liu determined the minimal Laplacian spread among all unicyclic graphs on $n$ vertices. In [20], by using different methods from [1, 13, 15, 26], Liu determined the four trees (resp., the three unicyclic graphs), which share the second to fourth (resp., the second to fourth) largest Laplacian spreads among all the trees (resp., connected unicyclic graphs) of given order. In [11], Fan et al. proved that there exist exactly two bicyclic graphs with maximal Laplacian spread among all bicyclic graphs of given order. In [9], Chen and Wang proved that there exist exactly five types of tricyclic graphs with maximal Laplacian spread among all tricyclic graphs of given order.

The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$. It is well known that $Q(G)$ is also positive semidefinite, and its eigenvalues can be arranged as follows: $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$. The signless Laplacian spread of the graph $G$, denoted by $SLQ(G)$, is defined to be $SLQ(G) = q_1(G) - q_n(G)$. Let $q(G)$ be the signless Laplacian spectral radius of $G$, namely, $q(G) = q_1(G)$. Let $\Phi(G,x)$ be the signless Laplacian characteristic polynomial of $G$, that is, $\Phi(G,x) = \det(xI - Q(G))$, simply $Q$-polynomial of $G$. Research on signless Laplacian matrices has become popular recently. In [2, 3, 5–8], some properties of signless Laplacian spectra of graphs were studied. In [18], Liu and Liu presented some upper and lower bounds for $SLQ(G)$ and determined the unique unicyclic graph with maximal signless Laplacian spread among the class of connected unicyclic graphs of order $n$. In [21], Liu gave the second to fourth largest signless Laplacian spectral radii and the second to fourth largest signless Laplacian spreads of unicyclic graphs with $n$ vertices. In this paper, we determine the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with $n$ vertices. Moreover, the unique bicyclic graph with the largest or the second largest signless Laplacian spread among the class of connected bicyclic graphs of order $n$ is determined, respectively.

2. The signless Laplacian spectral radii of bicyclic graphs

In this section, we will determine the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with $n$ vertices. We first introduce some preliminary results.

**Lemma 2.1** ([22, 25]) If $G$ is a graph with at least one edge, then $q(G) \geq \mu_1(G) \geq \Delta + 1$. If $G$ is connected, then the first equality holds if and only if $G$ is bipartite, and the second equality holds if and only if $\Delta = n - 1$.

**Lemma 2.2** ([10]) Let $G$ be a connected graph on $n(n \geq 2)$ vertices. Then $q(G) \leq \max\{d(v) + m(v), v \in V(G)\}$, where $m(v) = \sum_{u \in \mathcal{N}(v)} d(u)/d(v)$. 

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**Lemma 2.3** ([21]) Suppose \( c \geq 1 \) and \( G \) is a c-cyclic graph on \( n \) vertices with \( \Delta \leq n - 3 \). If \( n \geq 2c + 5 \), then \( q(G) \leq n - 1 \).

When \( c = 2 \), let \( G \) be a bicyclic graph with \( n \) vertices and \( \Delta \leq n - 3 \). If \( n \geq 9 \), by Lemma 2.3, then \( q(G) \leq n - 1 \). From Lemma 2.1, we can obtain that if \( \Delta = n - 2 \), then \( q(G) > n - 1 \).

If \( \Delta = n - 1 \), then \( q(G) \geq n \). Thus all bicyclic graphs on \( n \) vertices with \( \Delta = n - 2 \) or \( \Delta = n - 1 \) are shown in Figure 1. Next we will prove bicyclic graphs with the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with \( n \) vertices only are the graphs shown in Figure 1.

![Figure 1 The bicyclic graphs with the first to 11th largest signless Laplacian spectral radii in the class of bicyclic graphs with \( n \) vertices](image)

**Theorem 2.4** Let \( B \) be a bicyclic graph in \( B_n \) and \( n \geq 9 \). Then we have the following results.

1. If \( B \in B_n \), then \( q(B) \leq q(F_1^{n}) \), where the equality holds if and only if \( B \cong F_1^{n} \).
2. If \( B \in B_n \setminus \{F_1^{n}\} \), then \( q(B) \leq q(F_2^{n}) \), where the equality holds if and only if \( B \cong F_2^{n} \).
3. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}\} \), then \( q(B) \leq q(H_5^{n}) \), where the equality holds if and only if \( B \cong H_5^{n} \).
4. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}\} \), then \( q(B) \leq q(H_6^{n}) \), where the equality holds if and only if \( B \cong H_6^{n} \).
5. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}, H_6^{n}\} \), then \( q(B) \leq \min\{q(H_8^{n}), q(H_7^{n})\} \leq \max\{q(H_8^{n}), q(H_7^{n})\} \), where \( q(H_8^{n}) > q(H_7^{n}) \), if \( 9 \leq n \leq 12 \).
6. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}, H_6^{n}, H_7^{n}\} \), then \( q(B) \leq q(H_3^{n}) \), where the equality holds if and only if \( B \cong H_3^{n} \).
7. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}, H_6^{n}, H_7^{n}, H_8^{n}, H_9^{n}, H_{10}^{n}, H_{11}^{n}\} \), then \( q(B) \leq q(H_4^{n}) \), where the equality holds if and only if \( B \cong H_4^{n} \).
8. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}, H_6^{n}, H_7^{n}, H_8^{n}, H_9^{n}, H_{10}^{n}, H_{11}^{n}, H_{12}^{n}\} \), then \( q(B) \leq q(H_3^{n}) \), where the equality holds if and only if \( B \cong H_3^{n} \).
9. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}, H_6^{n}, H_7^{n}, H_8^{n}, H_9^{n}, H_{10}^{n}, H_{11}^{n}, H_{12}^{n}\} \), then \( q(B) \leq q(H_2^{n}) \), where the equality holds if and only if \( B \cong H_2^{n} \).
10. If \( B \in B_n \setminus \{F_1^{n}, F_2^{n}, H_5^{n}, H_6^{n}, H_7^{n}, H_8^{n}, H_9^{n}, H_{10}^{n}, H_{11}^{n}, H_{12}^{n}\} \), then \( q(B) \leq q(H_1^{n}) \), where the equality holds if and only if \( B \cong H_1^{n} \).
In order to prove Theorem 2.4, the following Lemma is needed.

**Lemma 2.5** ([19, 21]) Let \( G \) be a graph on \( n - k \) (\( 1 \leq k \leq n - 2 \)) vertices with \( V(G) = \{v_n, v_{n-1}, \ldots, v_k\} \). If \( G' \) is obtained from \( G \) by attaching \( k \) new pendant vertices, say, \( v_1, v_2, \ldots, v_k \) to \( v_k+1 \), then \( \Phi(Q(G'), x) = (x-1)^k \cdot \det(xI_{n-k} - Q(G) - M_{n-k}) \), where \( a_{11}(Q(G)) \) is corresponding to the vertex \( v_{k+1} \), and \( M_{n-k} = \text{diag}\{k + (k/(x-1)), 0, \ldots, 0\} \).

**Example** Let \( F_n^1 \) be the bicyclic graph as shown in Figure 1. By Lemma 2.5, we have

\[
\Phi(Q(F_n^1), x) = (x-1)^{n-4} \det(M),
\]

where \( M = \begin{bmatrix}
x - (n-1) - \frac{n-4}{x-1} & -1 & -1 & -1 \\
-1 & x - 2 & -1 & 0 \\
-1 & -1 & x - 3 & -1 \\
-1 & 0 & -1 & x - 2
\end{bmatrix}.
\]

By using “Matlab”, it easily follows that \( \Phi(Q(F_n^1), x) = (x-1)^{n-4}(x-2)(x^3 - (n+4)x^2 + 4nx - 8) \). With the similar method, by Lemma 2.5 and using “Matlab”, we have the following results.

- \( \Phi(Q(F_n^2), x) = (x-1)^{n-4}(x-3)(x^3 - (n+3)x^2 + 3nx - 8) \).
- \( \Phi(Q(H_n^1), x) = (x-1)^{n-5}(x-2)(x^4 - (n+5)x^3 + 6nx^2 - 6nx + 8) \).
- \( \Phi(Q(H_n^2), x) = (x-1)^{n-6}(x-2)^2(x^3 - (n+4)x^2 + 5(n-2)x - 3n) \).
- \( \Phi(Q(H_n^3), x) = (x-1)^{n-6}(x-2)(x^5 - (n+6)x^4 + (7n+7)x^3 - (14n-2)x^2 + 16 + 6n)x - 8) \).
- \( \Phi(Q(H_n^4), x) = (x-1)^{n-6}(x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+10)x^3 + (31n+10)x^2 - (11n+32)x + 16) \).
- \( \Phi(Q(H_n^5), x) = (x-1)^{n-6}(x-2)(x^5 - (n+6)x^4 + (7n+4)x^3 - (11n+2)x^2 + 4n + 16)x - 8) \).
- \( \Phi(Q(H_n^6), x) = (x-1)^{n-5}(x^5 - (n+6)x^4 + (7n+7)x^3 - (13n+3)x^2 + 4n + 24)x - 8) \).
- \( \Phi(Q(H_n^7), x) = (x-1)^{n-4}(x-2)(x^3 - (n+3)x^2 + 4n - 4)x - 8) \).
- \( \Phi(Q(H_n^8), x) = (x-1)^{n-6}(x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+12)x^3 + 30n + 29)x^2 - (9n+64)x + 24) \).
- \( \Phi(Q(H_n^9), x) = (x-1)^{n-6}(x-3)(x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + 3n + 24)x - 8) \).

Because the \( Q \)-polynomial of a graph has only real roots, we only deal with the polynomials with real roots in this paper. If \( f(x) \) is a polynomial in the variable \( x \), the degree of \( f(x) \) is denoted by \( \partial(f) \), and the maximum root of the equation \( f(x) = 0 \) by \( q_1(f) \). The following Lemma provides an effective method to compare the largest roots of two polynomials.

**Lemma 2.6** ([4]) Let \( f(x), g(x) \) be two monic polynomials with real roots, and \( \partial(f) \geq \partial(g) \). If
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\[ f(x) = q(x)g(x) + r(x), \]
where \( q(x) \) is also a monic polynomial, and \( \partial(r) \leq \partial(g) \), \( q_1(g) > q_1(q) \), then

1. When \( r(x) = 0 \), then \( q_1(f) = q_1(g) \);
2. When \( r(x) > 0 \) for any \( x \) satisfying \( x \geq q_1(g) \), then \( q_1(f) < q_1(g) \);
3. When \( r(q_1(g)) < 0 \), then \( q_1(f) > q_1(g) \).

**Proof of Theorem 2.4** Note that \( F_n^1 \) and \( F_n^2 \) are the only two bicyclic graphs with \( \Delta = n-1 \), and \( H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7, H_n^8 \) are all the bicyclic graphs with \( \Delta = n-2 \). Assume that \( B \in B_n \setminus \{ F_n^1, F_n^2, H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7, H_n^8 \} \). By Lemmas 2.1 and 2.3, we have

\[ \max\{q(F_n^1), q(F_n^2)\} \geq n > n-1 > q(B) \], because \( \Delta(B) \leq n-3 \).

In order to finish the proof of Theorem 2.4, we only need to prove that

\[ q(F_n^1) > q(F_n^2) > q(H_n^5) > q(H_n^8) > \max\{q(H_n^7), q(H_n^6)\} > \min\{q(H_n^7), q(H_n^6)\} \]

where

\[ \begin{cases} q(H_n^7) > q(H_n^6), & \text{if } 9 \leq n \leq 12, \\ q(H_n^6) < q(H_n^7), & \text{if } n > 12. \end{cases} \]

**Claim 1** \( q(F_n^1) > q(F_n^2) \).

For the \( Q \)-polynomials of \( F_n^1 \) and \( F_n^2 \), by Lemma 2.1, we have \( q_1(F_n^1) \geq n, q_1(F_n^2) \geq n \). Let \( g(x) = x^3 - (n + 4)x^2 + 4nx - 8 \) and \( f(x) = x^3 - (n + 3)x^2 + 3nx - 8 \). It is easy to see that

\[ f(x) = g(x) + x(x - n). \]

Let \( r(x) = x - n \). We can obtain \( q_1(F_n^1) \neq n \) and \( q_1(F_n^2) \neq n \). So when \( x \geq q_1(g) > n \) and \( n \geq 9, r(x) > 0 \). By Lemma 2.6, we obtain \( q_1(F_n^2) < q_1(F_n^1) \).

**Claim 2** \( q(F_n^1) > q(H_n^5) > q(H_n^8) \).

For the \( Q \)-polynomials of \( F_n^1 \) and \( H_n^5 \), let \( g(x) = x^3 - (n + 3)x^2 + 3nx - 8 \) and \( f(x) = x^3 - (n + 6)x^2 + (7n + 4)x^3 - (11n + 2)x^2 + (4n + 16)x - 8 \). It is easy to see that

\[ f(x) = g(x)(x - 1)(x - 2) + (n - 7)x^3 + 12x^2 - (2n + 8)x + 8. \]

Let \( r(x) = (n - 7)x^3 + 12x^2 - (2n + 8)x + 8 \). If \( x \geq q_1(g) > n - 1, n \geq 9 \), then \( r'(x) = 3(n - 7)x^2 + 24x - (2n + 8) > 0 \). Since \( r(n - 1) > 0 \), \( r(x) > 0 \). By Lemma 2.6, we obtain \( q_1(H_n^5) < q_1(F_n^2) \).

For the \( Q \)-polynomials of \( F_n^2 \) and \( H_n^8 \), let \( g(x) = x^3 - (n + 3)x^2 + 3nx - 8 \) and \( f(x) = x^6 - (n + 8)x^5 + (9n + 18)x^4 - (27n + 12)x^3 + (30n + 29)x^2 - (9n + 64)x + 24 \). It is easy to see that

\[ f(x) = g(x)(x - 1)^2(x - 3) + x((n - 4)x^3 - (5n - 20)x^2 + (6n - 20)x - 8) \]

Let \( r(x) = (n - 4)x^3 - (5n - 20)x^2 + (6n - 20)x - 8 \). If \( x \geq q_1(g) > n - 1 \) and \( n \geq 9 \), then \( r(x) > 0 \). By Lemma 2.6, we deduce that \( q_1(H_n^8) < q_1(F_n^2) \).

For the graphs \( H_n^5 \) and \( H_n^8 \). If \( x \geq n - 1 \) and \( n \geq 9 \), then

\[ \Phi(Q(H_n^8), x) - \Phi(Q(H_n^5), x) = (x - 1)^{n-6}(2x^4 - (2n - 2)x^3 + (4n + 9)x^2 - (24 + n)x + 8) > 0. \]
We obtain \( q_1(H_n^8) > q_1(H_n^3) \).

**Claim 3** \( q(H_n^8) > \max\{q(H_n^7), q(H_n^9)\} > \min\{q(H_n^7), q(H_n^9)\} > q(H_n^1) \), where

\[
\begin{align*}
q(H_n^7) &> q(H_n^9), \quad 9 \leq n \leq 12, \\
q(H_n^7) &< q(H_n^9), \quad n > 12.
\end{align*}
\]

For the graphs \( H_n^7 \) and \( H_n^9 \), if \( x \geq n - 1 \) and \( n \geq 9 \), then
\[
\Phi(Q(H_n^7), x) - \Phi(Q(H_n^9), x) = (x-1)^{n-6}(x^4 - (n+4)x^3 + (6n-9)x^2 - (7n-16)x + 8) > 0.
\]

We obtain \( q_1(H_n^8) > q_1(H_n^7) \).

For the \( Q \)-polynomials of \( H_n^8 \) and \( H_n^8 \), let \( g(x) = x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + (3n+24)x - 8 \) and \( f(x) = x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+12)x^3 + (30n+29)x^2 - (9n+64)x + 24 \).

It is easy to see that
\[
f(x) = g(x)(x-3) + x(-x^3 + (n+2)x^2 - (3n+1)x + 16).
\]

Let \( r(x) = -x^3 + (n+2)x^2 - (3n+1)x + 16 \). If \( x \geq 1 \) and \( n \geq 9 \), then \( r(q_1(g)) < 0 \).

By Lemma 2.6, we obtain \( q_1(H_n^8) < q_1(H_n^8) \).

For the \( Q \)-polynomials of \( H_n^7 \) and \( H_n^9 \), let \( g(x) = x^4 - (n+5)x^3 + 6nx^2 - 6nx + 8 \) and \( f(x) = x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + (3n+24)x - 8 \). It is easy to see that
\[
f(x) = g(x)(x-1) + x(x^3 - (n+1)x^2 + (2n-2)x - 3n + 16).
\]

Let \( r(x) = x^3 - (n+1)x^2 + (2n-2)x - 3n + 16 \). If \( x \geq 1 \) and \( n \geq 9 \), then \( r(x) > 0 \).

By Lemma 2.6, we obtain \( q_1(H_n^7) > q_1(H_n^7) \).

For the graphs \( H_n^7 \) and \( H_n^9 \), if \( x \geq n - 1 \) and \( n \geq 9 \), then
\[
\Phi(Q(H_n^7), x) - \Phi(Q(H_n^9), x) = (x-1)^{n-5}(x-1)(n+1)x^2 + (n+1)x + 2n - 4 > 0.
\]

We obtain \( q_1(H_n^7) > q_1(H_n^7) \).

For the \( Q \)-polynomials of \( H_n^7 \) and \( H_n^9 \), let \( g(x) = x^5 - (n+3)x^2 + (4n-4)x - 8 \) and \( f(x) = x^5 - (n+5)x^4 + (6n+4)x^3 - (10n+2)x^2 + (3n+24)x - 8 \). It is easy to see that
\[
f(x) = g(x)(x-1)^2 + x(x^2 - (n-1)x - (n-12)).
\]

Let \( r(x) = x^2 - (n-1)x - (n-12) \). If \( x > 1 \) and \( n \geq 9 \), then \( r(x) > 0 \). By Lemma 2.6, we obtain \( q_1(H_n^7) > q_1(H_n^9) \). If \( n > 12 \), then \( r(n-1) < 0 \). We can calculate the largest root of \( q(x) \) between \( n-1 \) and the largest root of \( r(x) \), then \( r(q_1(g(x))) < 0 \). By Lemma 2.6, \( q_1(H_n^9) \geq q_1(H_n^7) \).

**Claim 4** \( q(H_n^7) > q(H_n^9) > q(H_n^7) \).

For the \( Q \)-polynomials of \( H_n^7 \) and \( H_n^9 \), let \( g(x) = x^4 - (n+5)x^3 + 6nx^2 - 6nx + 8 \) and \( f(x) = x^6 - (n+8)x^5 + (9n+18)x^4 - (27n+10)x^3 + (31n+10)x^2 - (11n+32)x + 16 \). It is easy to see that
\[
f(x) = g(x)(x-2)(x-1) + x(x^3 - nx^2 + (n+2)x + n - 8).
\]
Let $r(x) = x^3 - nx^2 + (n+2)x + n - 8$. If $x \geq q_1(g) > n - 1$ and $n \geq 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^2) > q_1(H_n^3)$.

For the $Q$-polynomials of $H_n^2$ and $H_n^3$, let $g(x) = x^n - (n+2)x^{n-1} - (n+4)x^{n-2} - (n+6)x^{n-3} - (n+8)x^{n-4} - (2n+4)x^{n-5} - (2n+6)x^{n-6} + (n+10)x^{n-7} - (n+12)x^{n-8} + (n+14)x^{n-9} + (n+16)x^{n-10}$ and $f(x) = (x-2)(x^3 - (n+5)x^2 + (6n+4)x + n + 8)$. It is easy to see that $f(x) = g(x) + x(x^3 + (n+2)x^2 - (6n+4)x + n + 8)$.

Let $r(x) = x^3 + (n+2)x^2 - (6n+4)x + n + 8$. If $x \geq q_1(g) > n - 1$ and $n \geq 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^3) > q_1(H_n^2)$.

**Claim 5** $q(H_n^3) > q(H_n^2) > q(H_n^1)$.

For the $Q$-polynomials of $H_n^2$ and $H_n^3$, let $g(x) = x^5 - (n+6)x^4 + (7n+7)x^3 - (14n-2)x^2 + (6n+16)x - 8$ and $f(x) = x^5 - (n+6)x^4 + (7n+7)x^3 - (13n+3)x^2 + (4n+24)x - 8$. It is easy to see that $f(x) = g(x)x((n-5)x - 2n + 8)$. Let $r(x) = (n-5)x - 2n + 8$. If $x \geq q_1(g) > n - 1$ and $n \geq 9$, then $r(x) > 0$. By Lemma 2.6, we obtain $q_1(H_n^1) > q_1(H_n^2)$.

For the $Q$-polynomials of $H_n^1$ and $H_n^3$, let $g(x) = x^3 - (n+4)x^2 + (5n-2)x - 3n$ and $f(x) = x^3 - (n+6)x^2 + (7n+7)x - (14n-2)x^2 + (6n+16)x - 8$. It is easy to see that $f(x) = g(x)x(x \geq 0). ByLemma 2.6, we obtain $q_1(H_n^3) > q_1(H_n^1)$.

For the $Q$-polynomials of $H_n^2$ and $H_n^3$, let $g(x) = x^3 - (n+6)x^2 + (7n+7)x^3 - (13n+3)x^2 + (4n+24)x - 8$ and $f(x) = (x-2)x(x \geq 0). By Lemma 2.6, we obtain $q_1(H_n^1) > q_1(H_n^3)$.

Hence, by Claims 1–5, Theorem 2.4 is proved. □

### 3. The (second) largest signless Laplacian spreads of bicyclic graphs

In this section, we determine the unique bicyclic graph with the largest or the second largest signless Laplacian spreads among the class of connected bicyclic graphs of order $n$, respectively.

**Theorem 3.1** If $n \geq 9$ and $B \in B_n \setminus \{F_n^1, F_n^2\}$, then $SQ(F_n^1) > SQ(F_n^2) > SQ(B)$.

To prove Theorem 3.1, we need to introduce a lemma as follows.

**Lemma 3.2** Suppose $B$ is a bicyclic graph on $n$ vertices with $\Delta \leq n - 3$. If $n \geq 9$, then $SQ(B) \leq n - 1$.

**Proof** Note that $q_n(B) \geq 0$ and $SQ(B) = q_1(B) - q_n(B) \leq q_1(B)$. We only need to prove
$q_1(B) \leq \max\{d(v) + m(v) : v \in V\} \leq n - 1$. By Lemma 2.2, suppose $d(u) + m(u) = \max\{d(v) + m(v) : v \in V\}$. We consider the following three cases.

**Case 1** If $d(u) = 1$, suppose $v \in N(u)$. Then, $d(u) + m(u) = 1 + d(v) \leq 1 + \Delta \leq n - 2 < n - 1$.

**Case 2** If $d(u) = 2$, suppose $N(u) = \{w, v\}$. Note that $B$ is a bicyclic graph. Then, $|N(v) \cap N(w)| \leq 3$ and $|N(v) \cup N(w)| \leq n$. Therefore,

$$d(u) + m(u) = 2 + \frac{d(v) + d(w)}{2} \leq 2 + \frac{n + 3}{2} \leq n - 1.$$

**Case 3** Suppose $3 \leq d(u) \leq n - 3$. Note that $B$ has $n + 1$ edges. So we have

$$d(u) + m(u) \leq d(u) + \frac{2(n + 1) - d(u) - 2}{d(u)} = d(u) - 1 + \frac{2n}{d(u)}.$$ 

Next we will prove that $d(u) - 1 + \frac{2n}{d(u)} \leq n - 1$, equivalently, $d(u)(n - d(u)) \geq 2n$. Let $g(x) = (n - x)x$, where $3 \leq x \leq n - 3$. Since $g'(x) = n - 2x$ and $3 \leq x \leq n - 3$, we have $g(x) \geq g(3) = n - 3 = 3n - 9 \geq 2n$. Then the result follows. □

**Proof of Theorem 3.1** Note that $F_n^1$ and $F_n^2$ are the only two bicyclic graphs with $\Delta = n - 1$, and $H_2^1, H_3^1, H_4^1, H_5^1, H_6^1, H_7^1, H_8^1, H_9^1$ are all the bicyclic graphs with $\Delta = n - 2$. If $B \in B_n$ is a bicyclic graph on $n(\geq 9)$ vertices with $\Delta(B) \leq n - 3$, by Lemma 3.2, then $SQ(B) \leq n - 1$. In order to finish the proof of Theorem 3.1, we will prove firstly the following six claims.

**Claim 1** If $n \geq 9$, then $SQ(F_n^1) > n - 0.3 > n - 1$.

Indeed by Lemma 2.5, we have

$$\Phi(Q(F_n^1), x) = (x - 1)^{n-4}(x - 2)\varphi_1(x),$$

where $\varphi_1(x) = x^3 - (n + 4)x^2 + 4nx - 8$. Since $n \geq 9$, $\varphi_1(0) = -8 < 0, \varphi_1(0.3) = 1.11n - 8.333 > 0, \varphi_1(3) = 3n - 17 > 0, \varphi_1(4) = -8 < 0, \varphi_1(n) = -8 < 0$, and $\varphi_1(n + 1) = (n + 1)(n - 3) - 8 > 0$, then it follows that $0 < q_n(F_n^1) < 0.3$ and $n < q_1(F_n^1) < n + 1$. Thus,

$$SQ(Q(F_n^1)) = q_1(F_n^1) - q_n(F_n^1) > n - 0.3.$$ 

Then we have $SQ(F_n^1) > n - 0.3 > n - 1$.

**Claim 2** If $n \geq 9$, then $SQ(F_n^2) > n - 0.35 > n - 1$.

Indeed by Lemma 2.5, we have

$$\Phi(Q(F_n^2), x) = (x - 1)^{n-4}(x - 3)\varphi_2(x)$$

where $\varphi_2(x) = x^3 - (n + 3)x^2 + 3nx - 8$. Since $n \geq 9$, $\varphi_2(0) = -8 < 0, \varphi_2(0.35) = 0.9275n - 8.32463 > 0, \varphi_2(1) = 2n - 10 > 0, \varphi_2(3) = -8 < 0, \varphi_2(n) = -8 < 0$ and $\varphi_2(n + 1) = n^2 - n - 10 > 0$, then it follows that $0 < q_n(F_n^2) < 0.35$ and $n < q_1(F_n^2) < n + 1$. Thus,

$$SQ(Q(F_n^2)) = q_1(F_n^2) - q_n(F_n^2) > n - 0.35.$$ 

Then we have $SQ(F_n^2) > n - 0.35 > n - 1$.

**Claim 3** If $n \geq 9$, then $SQ(H_n^5) < n - 0.35 < \min\{SQ(F_n^1), SQ(F_n^2)\}$. 
By Lemma 2.5, we have
\[ \Phi(Q(H_n^\alpha), x) = (x^2 + 6) x^3 - 11 n + 2 x^2 + (4 n + 16) x - 8. \]
where \( \varphi_3(x) = x^5 - (n + 6) x^4 + (7 n + 4) x^3 - (11 n + 2) x^2 + (4 n + 16) x - 8. \) Since \( n \geq 9, \varphi_3(0) = -8 < 0, \varphi_3(0.4) = 0.2624 n - 1.80736 > 0, \varphi_3(0.6) = -0.1776 n + 1.04416 < 0, \varphi_3(1.6) = 0.3584 n + 0.02816 > 0, \varphi_3(4) = 32 n - 232 > 0, \varphi_3(5) = -5 n - 103 < 0, \varphi_3(n - 1) = -9 n^2 + 38 n - 37 < 0, \varphi_3(n - 0.35) = \frac{12}{7} n^4 - \frac{273}{7} n^3 + \frac{13621}{28} n^2 + \frac{84457}{7000} n - \frac{45157277}{4200000} > 0 \) and \( \varphi_3(n - 0.3) = \frac{7}{10} n^4 - \frac{287}{700} n^3 + \frac{436}{125} n^2 + \frac{82907}{10000} n - \frac{1313903}{100000} > 0, \) and by Claims 1 and 2, then \( SQ(H_n^\alpha) = q_1(H_n^\alpha) - q_0(H_n^\alpha) < q_1(H_n^\alpha) - q_0(H_n^\alpha) < q_1(H_n^\alpha) < q_1(H_n^\alpha) < n - 0.35 < \min\{SQ(F_1^\alpha), SQ(F_2^\alpha)\}. \)

**Claim 4**
If \( n \geq 9, \) then \( SQ(H_n^\alpha) < n - 0.35 < \min\{SQ(F_1^\alpha), SQ(F_2^\alpha)\}. \)

Similarly, for the graph \( H_n^\alpha, \) by Lemma 2.5, we have that
\[ \Phi(Q(H_n^\alpha), x) = (x^2 + 6) x^3 - 11 n + 2 x^2 + (4 n + 16) x - 24. \] Since \( n \geq 9, \varphi_4(0) = 24 > 0, \varphi_4(0.4) = -0.30784 n + 2.65498 > 0, \varphi_4(0.5) = 0.16525 n - 1.35938 > 0, \varphi_4(1) = 2 n - 12 > 0, \varphi_4(2) = -2 n + 12 < 0, \varphi_4(3) = 12 > 0, \varphi_4(4) = -4 n - 24 < 0, \varphi_4(n - 1) = -9 n^3 + 77 n^2 - 200 n + 156 < 0, \varphi_4(n - 0.35) = \frac{12}{7} n^4 - \frac{556}{7} n^3 + \frac{1409}{100} n^2 + \frac{32873}{4000} n - \frac{6299969}{1000000} > 0 \) and \( \varphi_4(n - 0.3) = \frac{7}{10} n^4 - \frac{147}{20} n^3 + \frac{1809}{100} n^2 + \frac{1649}{100} n^2 + \frac{32873}{4000} n - \frac{6299969}{1000000} > 0, \) and by Claims 1 and 2, then \( SQ(H_n^\alpha) = q_1(H_n^\alpha) - q_0(H_n^\alpha) < q_1(H_n^\alpha) < n - 0.35 < \min\{SQ(F_1^\alpha), SQ(F_2^\alpha)\}. \)

**Claim 5**
If \( n \geq 9, B \in B_n \) and \( \Delta(B) = n - 2, \) then \( SQ(B) < n - 0.35 < \min\{SQ(F_1^\alpha), SQ(F_2^\alpha)\}. \)

Indeed if \( n \geq 9, B \in B_n \) and \( \Delta(B) = n - 2, \) then \( B \) is one of the bicyclic graphs \( H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7, H_n^8 \) and \( H_n^9. \) By Theorem 2.4, Claims 1, 2, 3 and 4, we have \( SQ(B) = q_1(B) - q_0(B) < q_1(B) < n - 0.35 < \min\{SQ(F_1^\alpha), SQ(F_2^\alpha)\}. \)

**Claim 6**
If \( n \geq 9, \) then \( SQ(F_2^\alpha) < SQ(F_1^\alpha). \)

Indeed by Lemma 2.5, we have \( \Phi(Q(F_1^\alpha), x) = (x - 1)^{n - 4}(x - 2) \varphi_1(x) \) and \( \Phi(Q(F_2^\alpha), x) = (x - 1)^{n - 4}(x - 3) \varphi_2(x), \) where \( \varphi_1(x) = x^3 - (n + 4) x^2 + 4 n x - 8 \) and \( \varphi_2(x) = x^3 - (n + 3) x^2 + 3 n x - 8. \)

By Theorem 2.4, we have \( q(F_1^\alpha) > q(F_2^\alpha). \) Next we will prove \( q_0(F_1^\alpha) < q_0(F_2^\alpha) \) as follows.

According to the proof of Theorem 2.4, suppose
\[ F(x) = \Phi(Q(F_2^\alpha), x) - \Phi(Q(F_1^\alpha), x) = (x - 1)^{n - 4} \psi(x), \]
where \( \psi(x) = x^2 - n x + 8. \)

When \( 0 < x < \frac{n - \sqrt{n^2 - 32}}{2}, \) \( n \geq 9, \psi(x) > 0. \) For \( \varphi_1(x) = x^3 - (n + 4) x^2 + 4 n x - 8 \) and \( \varphi_2(x) = x^3 - (n + 3) x^2 + 3 n x - 8. \) \( \psi(x) = -8 < 0 \) and \( \varphi_1(n - 32) = 4 \sqrt{n^2 - 32} - 4 n + 24 > 0, \varphi_2(x) = -8 < 0 \) and \( \varphi_2(n - 32) = 4 \sqrt{n^2 - 32} - 4 n + 16 > 0, \) then \( 0 < q_0(F_1^\alpha) < \frac{n - \sqrt{n^2 - 32}}{2} \) and \( 0 < q_0(F_2^\alpha) < \frac{n - \sqrt{n^2 - 32}}{2}. \)

**Case 1**
When \( n \) is odd, \( 0 < x < \frac{n - \sqrt{n^2 - 32}}{2} \leq 1 \) and \( n \geq 9, \) then \( (x - 1)^{n - 4} < 0 \) and \( \psi(x) > 0, \) then \( F(x) < 0. \) We can obtain \( q_0(F_1^\alpha) < q_0(F_2^\alpha). \)

**Case 2**
When \( n \) is even, \( 0 < x < \frac{n - \sqrt{n^2 - 32}}{2} \leq 1 \) and \( n \geq 9, \) then \( (x - 1)^{n - 4} > 0 \) and \( \psi(x) > 0, \) then \( F(x) > 0. \) We also obtain \( q_0(F_1^\alpha) < q_0(F_2^\alpha). \)
By combining with the above arguments, \( SQ(F_n^1) = q_1(F_n^1) - q_n(F_n^1) > q_1(F_n^2) - q_n(F_n^2) = SQ(F_n^2) \).

By Lemma 3.2, and Claims 1–6, Theorem 3.1 is proved. \(\square\)

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