Implicative Ideals and Fuzzy Implicative Ideals of a Distributive Implication Groupoid

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Abstract We introduce the notions of implicative ideals and fuzzy implicative ideals of a distributive implication groupoid. Some properties of these ideals will be investigated. In particular, the necessary and sufficient conditions for an ideal (fuzzy ideal) to be an implicative ideal (fuzzy implicative ideal) is given. By using the concept of level sets, we will characterize the fuzzy implicative ideals of a distributive implication groupoid. Finally, an extension property for fuzzy implicative ideals is given.

Keywords implication groupoids; distributive implication groupoids; fuzzy ideals; implicative ideals; fuzzy implicative ideals.

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1. Introduction

In the 50-ties, L-Henkin and T-Skolem first introduced the notion of Hilbert algebra as an algebraic counterpart of intuitionistic logic. In particular, Hilbert algebras were extensively studied by Busneag [2] and Jun [11]. They have shown that the filters of a Hilbert algebra formed the deductive systems, because there exist various modifications of Hilbert algebras, we use the one which is cited in [2]. In fact, a Hilbert algebra is an algebra \( H = (H, *, 1) \) of type \((2, 0)\) satisfying the following axioms.

\[
\begin{align*}
(H1) & \quad x * (y * x) = 1; \\
(H2) & \quad (x * (y * z)) * ((x * y) * (x * z)) = 1; \\
(H3) & \quad x * y = 1 \text{ and } y * x = 1
\end{align*}
\]

imply \( x = y \).

We notice that the properties of ideals and congruences in Hilbert algebras were first considered by Chajda and Halas in [5]. Later, Chajda and Halas [6] further introduced the concept of implication groupoids which is a generalization of the implication reduct of intuitionistic logic. In other words, by using Hilbert algebra, they studied the connections among the deals, deductive systems and the congruence kernels whenever the implication groupoid is distributive. The concept of fuzzy ideals, fuzzy deductive systems in Hilbert algebras have lately been studied by Dudek, Jun and others [8–12]. They also discussed the relationship between the fuzzy ideals and their fuzzy deductive systems. Many interesting results have been obtained. On the other hand,
the ideals, filters and their properties of an implicative semigroup have already been studied and investigated by Shum, Sambassiva Rao, Ramana Murty, Chan and Shum in [14–16] and [7].

In this paper, we discuss the concept of implicative ideals and the fuzzy implicative ideals of a distributive implication groupoid. Some necessary and sufficient conditions for ideals (fuzzy ideals) of an implicative ideal (fuzzy implicative ideal) will be given. By using the notion of level sets, we will characterize the fuzzy implicative ideals of a distributive implication groupoid. Finally, we give an extension property of fuzzy implicative ideals.

2. Preliminaries

In this section, we first recall some definitions and basic results which were given in [3, 6].

**Definition 2.1** We call an algebra \((A, *, 1)\) of type \((2,0)\) a distributive implication groupoid if \(A\) satisfies the following identities:

1. \(x * x = 1\);
2. \(1 * x = x\);
3. \(x * (y * z) = (x * y) * (x * z)\) (left distributivity) for all \(x, y, z \in A\).

We note that an algebra \((A, *, 1)\) satisfying the first two conditions in the above definition is called an implication groupoid [6].

The following is an example of an implication groupoid.

**Example 2.2** Let \(A = \{1, a, b, c\}\) in which \(*\) is defined by

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Table 1 Definition of \(*\) in Example 2.2

Thus \((A, *, 1)\) is a distributive implication groupoid.

In every distributive implication groupoid, one can introduce the so called induced relation \(\leq\) by setting \(x \leq y\) if and only \(x * y = 1\).

**Lemma 2.3** Let \((A, *, 1)\) be a distributive implication groupoid. Then \(A\) satisfies the identities

\[x * 1 = 1\] and \[x * (y * x) = 1\].

Moreover, the induced relation \(\leq\) is a quasi-order on \(A\) and the following relationships are satisfied

(i) \(x \leq 1\); (ii) \(x \leq y * x\); (iii) \(x * ((x * y) * y) = 1\); (iv) \(1 \leq x\) implies \(x = 1\); (v) \(y * z \leq (x * y) * (x * z)\); (vi) \(x \leq y\) implies \(y * z \leq x * z\); (vii) \(x * (y * z) \leq y * (x * z)\); (viii) \(x * y \leq (y * z) * (x * z)\).

We give the following basic definition.
Definition 2.4 Let $A = (A, *, 1)$ be a distributive implication groupoid. A subset $I \subseteq A$ is called an ideal of $A$ if

$I_1$ $1 \in I$; $I_2$ $x \in I$ and $x * y \in I$ imply $y \in I$.

The ideals of a distributive implication groupoid are characterized in the following theorem.

Theorem 2.5 Let $I$ be a subset of a distributive implication groupoid $A$. Then $I$ is an ideal of $A$ if and only if for any $a, b \in I$ and $x \in A$, $a * (b * x) = 1$ implies $x \in I$.

We give the following definition of level sets.

Definition 2.6 Let $X$ be a set. A fuzzy set in $X$ is a function $\mu : X \rightarrow [0, 1]$. Let $\mu$ be a fuzzy set in a set $X$. For $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X | \mu(x) \geq \alpha\}$ is called a level subset of $\mu$.

For the fuzzy ideals, we give the following definition.

Definition 2.7 A fuzzy set $\mu$ in $A$ is called a fuzzy ideal of $A$ if it satisfies the following conditions:

$I_1$ $\mu(1) \geq \mu(x)$;
$I_2$ $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$, for all $x, y \in A$.

The fuzzy ideals of a distributive implication groupoid are characterized in the following theorem.

Theorem 2.8 Let $\mu$ be a fuzzy set in a distributive implication groupoid $A$. Then $\mu$ is a fuzzy ideal of $A$ if and only if for every $\alpha \in [0, 1]$, the level subset $\mu_\alpha$ is an ideal of $A$, when $\mu_\alpha \neq \emptyset$.

3. Implicative ideals

In this section, we introduce the concept of implicative ideals and study their properties. A necessary and sufficient condition for an ideal to be an implicative ideal is given. In what follows, $A$ is used to denote a distributive implication groupoid unless otherwise specified.

We give below the definition of an implicative ideal.

Definition 3.1 A subset $I$ of $A$ is called an implicative ideal of $A$ if it satisfies the following conditions.

$I_1$ $1 \in I$;
$I_2$ $z * ((x * y) * x) \in I$ and $z \in I$ implies that $x \in I$ for all $x, y, z \in A$.

In Example 2.2, the subset $I = \{1, a, c\}$ of $A$ is an implicative ideal of $A$.

In a distributive implication groupoid, the following theorem is obvious.

Theorem 3.2 In a distributive implication groupoid, every implicative ideal is an ideal.

Proof Putting $y = 1$ in ($I_2$). Then $z * x \in I$ and $z \in I$ implies that $x \in I$. □

It is noted that the converse of the above theorem is not true.

Example 3.3 Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by
Then \((A, \ast, 1)\) is a distributive implication groupoid. The subset \(I = \{1, b, e, f, g\}\) of \(A\) is an ideal of \(A\) but not an implicative ideal of \(A\).

In the following, we give a necessary and sufficient condition for an ideal to become an implicative ideal.

**Theorem 3.4** Let \(I\) be an ideal of \(A\). Then the following are equivalent.

(i) \(I\) is implicative ideal;

(ii) \((x \ast y) \ast x \in I \Rightarrow x \in I\).

**Proof** (i)\(\Rightarrow\) (ii). Assume (i). Let \((x \ast y) \ast x \in I\). Since \(1 \in I\) and \(1 \ast ((x \ast y) \ast x) = (x \ast y) \ast x \in I\), we have \(x \in I\).

(ii)\(\Rightarrow\) (i). Assume (ii). Let \(z \ast ((x \ast y) \ast x) \in I\) and \(z \in I\). Then \((x \ast y) \ast x \in I\) and hence \(x \in I\).

**Theorem 3.5** Let \(I\) be a subset of a distributive implication groupoid \(A\) containing \(1\). Then \(I\) is an implicative ideal of \(A\) if and only if for any \(a, b \in I\) and \(x, y \in A\), \(a \ast (b \ast ((x \ast y) \ast x)) = 1\) implies \(x \in I\).

**Proof** Let \(I\) be an implicative ideal of \(A\). Assume \(a, b \in I\) and \(x, y \in A\) such that \(a \ast (b \ast ((x \ast y) \ast x)) = 1\). Then, by Theorem 2.5, \((x \ast y) \ast x \in I\). Therefore, \(x \in I\).

Conversely, assume that the condition holds. Let \(a, b \in I\) and \(x \in A\) such that \(a \ast (b \ast x) = 1\). Then \(a \ast (b \ast ((x \ast 1) \ast x)) = a \ast (b \ast x) = 1\) and hence \(x \in I\). So that \(I\) is an ideal of \(A\). Let \((x \ast y) \ast x \in I\). Then \(((x \ast y) \ast x) \ast (1 \ast ((x \ast y) \ast x)) = 1\) and hence \(x \in I\). Therefore, \(I\) is an implicative ideal of \(A\).

We now give an extension property for the implicative ideals.

**Theorem 3.6** Let \(I\) be an implicative ideal of \(A\). If \(J\) is an ideal of \(A\) containing \(I\), then \(J\) is an implicative ideal of \(A\).

**Proof** Let \(J\) be an ideal of \(A\) containing \(I\) and \(z \ast ((x \ast y) \ast x) \in J, z \in J\). Then \((x \ast y) \ast x \in J\). Now

\[ [(x \ast y) \ast x] \ast [(x \ast y) \ast x] = 1 \in I \]
implies that
\[((x*y)*x)*((x*y)*y))*((x*y)*x)*x\] * ... \[\in I.

Then by Theorem 3.6, we get
\[((x*y)*x)*x \in I \subseteq J.

Since \(J\) is an ideal of \(A\), we have \(x \in J\). Therefore, \(J\) is an implicative ideal of \(A\). \(\square\)

4. Fuzzy implicative ideals

In this section, we introduce the concept of fuzzy implicative ideal in a distributive implication groupoid and study its properties. By using the notion of level set, we characterize the fuzzy implicative ideal of a distributive implication groupoid.

We first mention the definition of a fuzzy implicative ideal.

**Definition 4.1** A fuzzy subset \(\mu\) of a distributive implication groupoid \(A\) is said to be a fuzzy implicative ideal if it satisfies the following conditions.

- (FI\(_1\)) \(\mu(1) \geq \mu(x)\);
- (FI\(_2\)) \(\mu(x) \geq \min\{\mu(z * ((x * y) * x)), \mu(z)\}\) for all \(x, y, z \in A\).

**Theorem 4.2** In a distributive implication groupoid \(A\). Every fuzzy implicative ideal is a fuzzy ideal.

**Proof** Let \(\mu\) be a fuzzy implicative ideal of \(A\) and \(x, y, z \in A\). Now put \(y = x\) in (FI\(_2\)). Then we have the following equality.

\[\mu(x) \geq \min\{\mu(z * ((x * x) * x)), \mu(z)\} = \min\{\mu(z * x), \mu(z)\}.\]

This shows that \(\mu\) is a fuzzy ideal of \(A\). \(\square\)

It is noted that the converse of the above theorem need not be true, which is shown in the following example.

**Example 4.3** Let \(A = \{1, a, b, c\}\) in which \(*\) is defined by

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Table 3 Definition of \(*\) in Example 4.3

Then \((A, *, 1)\) is a distributive implication groupoid. Let \(\mu\) be a fuzzy subset of \(A\) defined by \(\mu(1) = \mu(a) = \mu(b) = 0.7\) and \(\mu(c) = 0.2\). Then, it is clear that \(\mu\) is a fuzzy implicative ideal of \(A\).
If \( \nu \) is a fuzzy subset of \( A \) defined by \( \nu(1) = 0.7 \) and \( \nu(a) = \nu(b) = \nu(c) = 0.2 \), then it is clear that \( \nu \) is fuzzy ideal of \( A \) but not an implicative ideal of \( A \).

For fuzzy ideals, we have the following theorems.

**Theorem 4.4** Let \( \mu \) be a fuzzy ideal of \( A \). Then the following are equivalent:

(i) \( \mu \) is a fuzzy implicative ideal of \( A \);

(ii) \( \mu(x) \geq \mu((x \ast y) \ast x) \) for all \( x, y \in A \);

(iii) \( \mu(x) = \mu((x \ast y) \ast x) \) for all \( x, y \in A \).

**Proof** (i)\( \Rightarrow \) (ii). Assume that (i) and \( x, y \in A \). Then

\[
\mu(x) \geq \min \{ \mu(1 \ast ((x \ast y) \ast x)), \mu(1) \} = \mu((x \ast y) \ast x).
\]

(ii)\( \Rightarrow \) (iii). Observe that \( x \leq (x \ast y) \ast x \) for all \( x, y \in A \). Then \( \mu(x) \leq \mu((x \ast y) \ast x) \). It follows from (ii) that \( \mu(x) = \mu((x \ast y) \ast x) \).

(iii)\( \Rightarrow \) (i). Assume (iii) and \( x, y, z \in A \). Since \( \mu \) is a fuzzy ideal, we have

\[
\mu((x \ast y) \ast x) \geq \min \{ \mu(z \ast ((x \ast y) \ast x)), \mu(z) \}.
\]

Combining (iii), we obtain

\[
\mu(x) \geq \min \{ \mu(z \ast ((x \ast y) \ast x)), \mu(z) \} = \min \{ \mu(z \ast x), \mu(z) \}.
\]

Hence \( \mu \) is a fuzzy implicative ideal of \( A \). \( \Box \)

**Theorem 4.5** A fuzzy subset \( \mu \) of \( A \) is a fuzzy ideal of \( A \) if and only if it satisfies

\[
z \ast (x \ast y) = 1 \Rightarrow \mu(y) \geq \min \{ \mu(x), \mu(z) \}
\]

for all \( x, y, z \in A \).

**Proof** Assume that \( \mu \) is a fuzzy ideal of \( A \). Let \( x, y, z \in A \) be such that \( z \ast (x \ast y) = 1 \). Then \( z \leq x \ast y \) and so \( \mu(z) \leq \mu(x \ast y) \). Therefore

\[
\mu(y) \geq \min \{ \mu(x \ast y), \mu(x) \} \geq \min \{ \mu(x), \mu(z) \}.
\]

Conversely, assume that the condition holds. Since \( x \ast (x \ast 1) = 1 \) for all \( x \in A \), we have \( \mu(1) \geq \min \{ \mu(x), \mu(x) \} = \mu(x) \) for all \( x \in A \). Also, \( x \ast ((x \ast y) \ast y) = 1 \) for all \( x, y \in A \). Therefore, \( \mu(y) \geq \min \{ \mu(x \ast y), \mu(x) \} \). Hence \( \mu \) is a fuzzy ideal of \( A \). \( \Box \)

The following theorem can be easily proved.

**Theorem 4.6** A fuzzy set \( \mu \) in \( A \) is a fuzzy implicative ideal of \( A \) if and only if it satisfies, for all \( x, y, z \in A \),

\[
z \leq u \ast ((x \ast y) \ast x) \Rightarrow \mu(x) \geq \min \{ \mu(z), \mu(u) \}.
\]

The Theorem 4.7 is a simple consequence of the transfer principle as described in Kondo and Dudek [13].

**Theorem 4.7** A fuzzy subset \( \mu \) of \( A \) is a fuzzy implicative ideal of \( A \) if and only if, for each \( t \in [0, 1], \mu_t \) is either empty or an implicative ideal of \( A \).
The following extension theorem for implicative ideals can be easily verified by Theorems 3.7 and 4.7.

**Theorem 4.8** (Extension property for fuzzy implicative ideals) Let \( \mu \) and \( \nu \) be fuzzy ideals of \( A \) with \( \mu \leq \nu \) and \( \mu(1) = \nu(1) \). If \( \mu \) is a fuzzy implicative ideal of \( A \), then so is \( \nu \).

For implicative ideals, we have the following theorem.

**Theorem 4.9** If \( I \) is an implicative ideal of \( A \), then there is a fuzzy implicative ideal \( \mu \) of \( A \) such that \( \mu_t = I \) for some \( t \in (0, 1) \).

**Proof** Let \( \mu \) be a fuzzy set in \( A \) defined by

\[
\mu(x) = \begin{cases} 
 t, & \text{if } x \in I; \\
 0, & \text{if } x \notin I,
\end{cases}
\]

where \( t \) is a fixed number \((0 < t < 1)\). Now we verify that \( \mu \) is a fuzzy implicative ideal of \( A \). Let \( x, y, z \in A \) be such that \( z * ((x * y) * x) \in I \) and \( z \in I \). Then \( x \in I \). Thus we have

\[
\mu(z * ((x * y) * x)) = \mu(z) = \mu(x) = t \quad \text{and so} \quad \mu(x) = \min\{\mu(z * ((x * y) * x)), \mu(z)\}.
\]

If at least one of \( z * ((x * y) * x) \) or \( z \) is not in \( I \), then at least one of \( \mu(z * ((x * y) * x)) \) and \( \mu(z) \) is 0. Hence we get \( \mu(x) = \min\{\mu(z * ((x * y) * x)), \mu(z)\} \). Therefore, we get \( \mu(x) = \min\{\mu(z * ((x * y) * x)), \mu(z)\} \) for all \( x, y, z \in A \). Since \( 1 \in I, \mu(1) = t \geq \mu(x) \) for all \( x \in A \). Therefore, \( \mu \) is a fuzzy implicative ideal of \( A \). It is clear that \( \mu_t = I \). \( \square \)

For any fixed element \( \omega \in A \), we consider the set

\[
\vec{\mu}_\omega = \{x \in A | \mu(\omega) \leq \mu(x)\}.
\]

Obviously, \( \omega \in \vec{\mu}_\omega \). If \( \mu \) is a fuzzy ideal of \( A \), then \( 1 \in \vec{\mu}_\omega \). Now, it is natural to ask the following question.

**Question** For a fuzzy set \( \mu \) of \( A \) satisfying \( \mu(1) \geq \mu(x) \) for all \( x \in A \), is \( \vec{\mu}_\omega \) an ideal of \( A \)?

The above question can be answered in the following:

In Example 4.3, if \( \mu \) is a fuzzy subset of \( A \) defined by \( \mu(1) = 0.9, \mu(a) = 0.8, \mu(b) = 0.5 \) and \( \mu(c) = 0.3 \), then \( \mu \) satisfies the condition \( \mu(1) \geq \mu(x) \) for all \( x \in A \), but it is not a fuzzy ideal of \( A \) since

\[
\mu(b) = 0.5 < 0.8 = \min\{\mu(a * b), \mu(a)\}.
\]

\( \vec{\mu}_b = \{a, b, 1\} \) is an ideal of \( A \).

We now present a condition under which \( \vec{\mu}_\omega \) is an ideal of \( A \).

**Theorem 4.10** Let \( \omega \in A \). If \( \mu \) is a fuzzy ideal of \( A \), then \( \vec{\mu}_\omega \) is an ideal of \( A \).

**Proof** Let \( \mu \) be a fuzzy ideal of \( A \). Then \( 1 \in \vec{\mu}_\omega \). Assume \( a * b \in \vec{\mu}_\omega \) and \( a \in \vec{\mu}_\omega \) for all \( a, b \in A \). Then \( \mu(a * b) \geq \mu(\omega) \) and \( \mu(a) \geq \mu(\omega) \). Since \( \mu \) is a fuzzy ideal of \( A \), we have \( \mu(b) \geq \min\{\mu(a * b), \mu(a)\} \geq \mu(\omega) \). So that \( b \in \vec{\mu}_\omega \). Therefore, \( \vec{\mu}_\omega \) is an ideal of \( A \). \( \square \)

**Theorem 4.11** Let \( \mu \) be a fuzzy set in \( A \) and \( \omega \in A \). Then we have the following statements.
Theorem 4.12

(i) If  is an ideal of  then  satisfies, for all  are such that , the following implication

\[ \mu(x) \leq \min\{\mu(y \ast z), \mu(y)\} \Rightarrow \mu(x) \leq \mu(z). \]  

\[ \text{(i) Assume that } \mu(x) \leq \min\{\mu(y \ast z), \mu(y)\}. \]  

\[ \text{Then } y \ast z \in \overrightarrow{m}_\omega^\ast \text{ and } y \in \overrightarrow{m}_\omega^\ast. \]  

\[ \text{Since } \overrightarrow{m}_\omega^\ast \text{ is an ideal of } A \text{ for each } \omega \in A, \text{ we have } z \in \overrightarrow{m}_\omega^\ast. \]  

Therefore \( \mu(x) \leq \mu(z) \).

(ii) Suppose that \( \mu \) satisfies \((*)\) and \( \mu(1) \geq \mu(x) \) for all \( x \in A \). For each \( \omega \in A \), let \( x, y \in A \) be such that \( x \ast y \in \overrightarrow{m}_\omega^\ast \) and \( x \in \overrightarrow{m}_\omega^\ast \). Then \( \mu(x \ast y) \geq \mu(\omega) \) and \( \mu(x) \geq \mu(\omega) \), which imply that \( \mu(\omega) \leq \min\{\mu(x \ast y), \mu(x)\} \). Using \((*)\), we have \( \mu(\omega) \leq \mu(y) \) and so \( y \in \overrightarrow{m}_\omega^\ast \). Since \( \mu \) satisfies \( \mu(1) \geq \mu(x) \) for all \( x \in A \), it follows that \( 1 \in \overrightarrow{m}_\omega^\ast \). Therefore, \( \overrightarrow{m}_\omega^\ast \) is an ideal of \( A \).

\[ \square \]

Theorem 4.12 If \( \mu \) is a fuzzy ideal of \( A \) satisfying \( \mu(x) \geq \mu(y \ast x) \) for all \( x, y \in A \), then \( \mu \) is a fuzzy implicative ideal of \( A \).

Proof Let \( \mu \) be a fuzzy ideal of \( A \) and \( x, y, z \in A \) such that \( \mu(x) \geq \mu(y \ast x) \). Then

\[ \mu(y \ast x) \geq \min\{\mu(z \ast (y \ast x)), \mu(z)\}. \]  

\[ \text{(I)} \]

We know that \( y \leq x \ast y \Rightarrow (x \ast y) \ast x \leq y \ast x \Rightarrow z \ast ((x \ast y) \ast x) \leq z \ast (y \ast x) \Rightarrow \mu(z \ast ((x \ast y) \ast x)) \leq \mu(z \ast (y \ast x)). \]  

Therefore from \( (I) \) and hypothesis, we have

\[ \mu(x) \geq \mu(y \ast x) \geq \min\{\mu(z \ast ((x \ast y) \ast x)), \mu(z)\}. \]

Hence \( \mu \) is a fuzzy implicative ideal of \( A \).

\[ \square \]

We conclude this paper with the following theorem for the distributive implication groupoid \( A \).

Theorem 4.13 Let \( \omega \in A \). If \( \mu \) is a fuzzy implicative ideal of \( A \), then \( \overrightarrow{m}_\omega^\ast \) is an implicative ideal of \( A \).

Proof Let \( \mu \) be a fuzzy implicative ideal of \( A \). Then \( \mu \) is fuzzy ideal of \( A \) and hence \( (FI_1) \) holds. Let \( \omega \in A \). Then \( 1 \in \overrightarrow{m}_\omega^\ast \). Let \( z \ast ((x \ast y) \ast x) \in \overrightarrow{m}_\omega^\ast \) and \( z \in \overrightarrow{m}_\omega^\ast \). Then \( \mu(\omega) \leq \mu(z \ast ((x \ast y) \ast x)) \) and \( \mu(\omega) \leq \mu(z) \). Therefore

\[ \mu(x) \geq \min\{\mu(z \ast ((x \ast y) \ast x)), \mu(z)\} \geq \mu(\omega). \]

So that \( x \in \overrightarrow{m}_\omega^\ast \). This shows that \( \overrightarrow{m}_\omega^\ast \) is indeed an implicative ideal of \( A \).

\[ \square \]

References

Implicative ideals and fuzzy implicative ideals of a distributive implication groupoid