A Note on the Signless Laplacian and Distance Signless Laplacian Eigenvalues of Graphs

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Abstract Let $G$ be a simple graph. We first show that $\delta_i \geq d_i - \sqrt{\left\lfloor \frac{i}{2} \right\rfloor \left\lceil \frac{i}{2} \right\rceil}$, where $\delta_i$ and $d_i$ denote the $i$-th signless Laplacian eigenvalue and the $i$-th degree of vertex in $G$, respectively. Suppose $G$ is a simple and connected graph, then some inequalities on the distance signless Laplacian eigenvalues are obtained by deleting some vertices and some edges from $G$. In addition, for the distance signless Laplacian spectral radius $\rho_Q(G)$, we determine the extremal graphs with the minimum $\rho_Q(G)$ among the trees with given diameter, the unicyclic and bicyclic graphs with given girth, respectively.

Keywords signless Laplacian; distance signless Laplacian; spectral radius; eigenvalues.

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1. Introduction

Let $G = G(V, E)$ be a graph with vertex set $V$ and edge set $E$. The order and size of $G$ are defined as $|V|$ and $|E|$, respectively. Denote by $N_G(u)$ the set of vertices adjacent to $u$, called the neighbor set of $u$. Then the degree of $u$ is defined as $|N_G(u)|$. The signless Laplacian matrix of a simple graph $G$ is defined to be $Q = A + D$, where $A$ denotes the adjacency matrix and $D$ is the diagonal matrix of vertex degrees of $G$. We suppose graph $G$ to be connected when distance of vertices is considered in $G$. The distance between vertex $u$ and $v$, denoted by $d_G(u, v)$, is the length of a shortest path from $u$ to $v$. The transmission $Tr(u)$ of vertex $u$ is defined to be the sum of distances from $u$ to all other vertices, i.e., $Tr(u) = \sum_{v \in V(G)} d_G(v, u)$. The distance matrix of $G$, denoted by $D(G)$, is a symmetric real matrix with $(i, j)$-entry being $d_G(v_i, v_j)$. Obviously, $Tr(v_i)$ is the sum of $i$-th row of $D(G)$. Denote by diag$(Tr)$ the diagonal matrix of the vertex transmissions in $G$. Similar to the signless Laplacian matrix of a graph, the distance signless Laplaian matrix of graph $G$ is introduced by Aouchiche and Hansen [1], defined as $Q(G) = \text{diag}(Tr) + D(G)$. The eigenvalues of $Q(G)$, called distance signless Laplaian eigenvalues of $G$, are written as $\{q_1(G), q_2(G), \ldots, q_n(G)\}$. Without loss of generality, assume that $q_n(G) \leq \cdots \leq q_2(G) \leq q_1(G)$. Denote by $\rho_Q(G) = q_1(G)$ the distance signless Laplacian spectral radius. Let $P_Q(t)$ denote the distance signless Laplacian characteristic polynomial. As

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usual, we use $K_n$, $C_n$, $P_n$ and $S_n$ to denote the complete graph, the cycle, the path and the star with order $n$, respectively. $K_{a:b}$ means the complete bipartite graph with two colour classes of order $a$ and $b$. Identity matrix is denoted by $I$ with order following from the context. Let $J_n$ be the matrix of order $n$ with all entries one. The clique, regarded as an induced subgraph of $G$, is a complete graph. Denote by $G - e$ the graph obtained by removing edge $e \in E(G)$ from $G$. $G - u$ denotes the graph obtained by deleting the vertex $u$ and all the edges incident to it.

Generally, for $S \subseteq V(G)$, $G - S$ denotes the graph derived from deleting all the vertices of $S$ and edges incident to each vertex of $S$ from graph $G$.

2. Lower bound for the signless Laplacian eigenvalues of a graph

Before giving the main result, some well-known conclusions are necessary.

Lemma 2.1 (Interlacing theorem) Let $A$ be a symmetric real matrix and $B$ be a principal submatrix of $A$ with order $n$ and $s$ ($s \leq n$), respectively. For the eigenvalues of $A$ and $B$, then

$$\lambda_{i+n-s}(A) \leq \lambda_i(B) \leq \lambda_i(A), \quad 1 \leq i \leq s.$$ 

Lemma 2.2 (Courant-Weyl inequality) Let $H_1$ and $H_2$ be symmetric real matrices with order $n$. For $1 \leq i \leq n$, the eigenvalues of $H_1$ and $H_2$ satisfy:

$$\lambda_n(H_2) + \lambda_i(H_1) \leq \lambda_i(H_1 + H_2) \leq \lambda_i(H_1) + \lambda_1(H_2).$$

Lemma 2.3 ([11, Proposition 2]) Let $G$ be a simple graph of order $n$. Then the least eigenvalue $\lambda_n(A)$ of the adjacency matrix $A$ of $G$ satisfies:

$$\lambda_n(A) \geq -\sqrt{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil},$$

and the equality holds if and only if $G = K_{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$, where $K_{\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil}$ is the complete bipartite graph with two color classes of order $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil$.

The following theorem demonstrates a lower bound for each signless Laplacian eigenvalue.

Theorem 2.4 Let $G$ be a simple graph of order $n$, and let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ be the signless
Laplacian eigenvalues of $G$. For $1 \leq i \leq n$, then
\[
\delta_i \geq d_i - \sqrt{\frac{i^2}{2}}.
\]

**Proof** Without loss of generality let us take $d_1 \geq d_2 \geq \cdots \geq d_n$, where $d_i$ means the degree of $v_i$. Let $M$ and $A'$ be the left-top $i \times i$ principal submatrix of the signless Laplacian matrix $Q$ and the adjacency matrix $A$, respectively. Let $H = d_i I + A'$ where $I$ is the identity matrix of order $i$. Then let $P = \text{diag}(d_1 - d_i, d_2 - d_i, \ldots, d_{i-1} - d_i, 0)$ be a diagonal matrix with the least eigenvalue $0$. Obviously, $M = H + P$ where $M, H$ and $P$ are Hermitian matrices of order $i$.

By Lemmas 2.1 and 2.2, we have $\delta_i \geq \lambda_i(M) \geq \lambda_i(H) + \lambda_i(P) = \lambda_i(H)$. Moreover, the eigenvalues of $H$ are $\lambda_k(H) = d_i + \lambda_k(A')$, $k = 1, 2, \ldots, i$. Actually, the matrix $A'$ is the adjacency matrix of the subgraph indexed by $\{v_1, v_2, \ldots, v_i\}$ of $G$. Then $\lambda_k(A') \geq -\sqrt{\frac{i^2}{2}}$ ($k = 1, 2, \ldots, i$) follows from Lemma 2.3. Finally, we get
\[
\delta_i \geq \lambda_i(M) \geq \lambda_i(H) + \lambda_i(P) \geq d_i - \sqrt{\frac{i^2}{2}}. \quad \square
\]

3. Inequalities on the distance signless Laplacian eigenvalues

For a simple and connected graph $G$, obviously, $Q(G)$ is a symmetric real matrix. Then by Lemma 2.1, the following corollary is clear.

**Corollary 3.1** Let $G$ be a graph with order $n$. Let $M$ be the principal submatrix of $Q(G)$ with order $n - 1$. Then,
\[
q_1(G) \geq \lambda_1(M) \geq q_2(G) \geq \cdots \geq \lambda_{n-1}(M) \geq q_n(G).
\]

A pendant vertex in a graph is a vertex with degree one. The diameter of graph $G$, denoted by $d(G)$ ($d$, for brevity), is defined as the largest value of distances of any two vertices in $G$. For two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ with order $n$, if $a_{ij} \leq b_{ij}$ ($1 \leq i, j \leq n$), we say $A \leq B$ and $A < B$, if $a_{ij} < b_{ij}$ ($1 \leq i, j \leq n$).

**Theorem 3.2** Let $u$ be a pendant vertex of $G$ and $d(G) = d$ be the diameter of $G$. For $i = 1, 2, \ldots, n - 1$,
\[
q_{i+1}(G) - d \leq q_i(G - u) \leq q_i(G) - 1.
\]

**Proof** Since $u$ is a pendant vertex, we can easily get $d_{G-u}(x, y) = d_G(x, y)$ for $x, y \in V(G - u)$, and $1 \leq d_G(u, w) \leq d$ for $w \in V(G - u)$. Therefore, $Tr_G(w) > Tr_{G-u}(w)$, $w \in V(G - u)$. Let $M$ be the principal submatrix of $Q(G)$ obtained by deleting the row and column corresponding to $u$. Then $M \geq Q(G - u)$ and $M \neq Q(G - u)$. Let $P = M - Q(G - u)$. Then $P$ is a diagonal matrix with the least diagonal entries not less than one and the largest diagonal entries not more than $d$ obviously, i.e., the eigenvalues of $P$ satisfy
\[
1 \leq \lambda_i(P) \leq d, \quad i = 1, 2, \ldots, n - 1. \quad (3.1)
\]
Thus by Lemma 2.2 and inequality (3.1), for $M, Q(G - u)$ and $P$, we can get
\[ q_i(G - u) + 1 \leq \lambda_i(M) \leq q_i(G - u) + d, \quad i = 1, 2, \ldots, n - 1. \] (3.2)

By Corollary 3.1, it is obtained that
\[ q_i(G) \geq \lambda_i(M) \geq q_{i+1}(G), \quad i = 1, 2, \ldots, n - 1. \] (3.3)

Combining the left inequalities of (3.2) and (3.3), we have
\[ q_i(G - u) + 1 \leq q_i(G), \quad i = 1, 2, \ldots, n - 1. \] (3.4)

Similarly, combining the right inequalities of (3.2) and (3.3) gives
\[ q_{i+1}(G) \leq q_i(G - u) + d, \quad i = 1, 2, \ldots, n - 1. \] (3.5)

The proof is completed by (3.4) and (3.5). □

**Corollary 3.3** Let $G$ be a graph on $n$ vertices with diameter $d(G) = 2$. Suppose vertex $v$ is adjacent to any other vertex of $G$ and $G - v$ is connected with $d(G - v) = d(G)$, then the eigenvalues of $Q(G - v)$ interlace those of $Q(G) - I$, i.e.,
\[ q_{i+1}(G) - 1 \leq q_i(G - v) \leq q_i(G) - 1, \quad i = 1, 2, \ldots, n - 1. \]

**Proof** Since $v$ is adjacent to any other vertex of $G$ and $d(G - v) = d(G) = 2$, we obtain $d_{G-v}(x, y) = d_G(x, y)$ for any $x, y \in V(G - v)$. Hence, $Tr_G(x) = Tr_{G-v}(x) + 1$ for each $x \in V(G - v)$. Let $M$ be the principal submatrix of $Q(G)$ derived from deleting the row and column corresponding to $v$ and $P = M - Q(G - v)$. Then $P$ is equal to the identity matrix $I$.

By Lemma 2.2, it is obtained that
\[ q_i(G - v) + 1 \leq \lambda_i(M) \leq q_i(G - v) + 1, \quad i = 1, 2, \ldots, n - 1, \]
where $\lambda_i(M)$ denotes the $i$-th largest eigenvalue of $M$.

From Corollary 3.1, we see $q_i(G) \geq \lambda_i(M) \geq q_{i+1}(G), \quad i = 1, 2, \ldots, n - 1$, where $\lambda_i(M)$ is defined as above. Thus similar to the method of Theorem 3.2, the conclusion is obtained. □

For graph $G$, $u, v \in V(G)$ are called multiplicate vertices, if $N_G(u) = N_G(v)$. Suppose $u$ is adjacent to $v$ and $N_{G-v}(u) = N_{G-v}(v)$, then $u, v$ are called quasi-multiplicate vertices. In general, $S \subset V(G)$ is a multiplicate vertex set, if $N_G(u) = N_G(v)$ for $u, v \in S$; $C \subset V(G)$ is a quasi-multiplicate vertex set, if the vertices of $C$ induce a clique and $N_G(u) - C = N_G(v) - C$ for $u, v \in C$. Obviously, if we add edges to any two vertices of a multiplicate vertex set, then we obtain a quasi-multiplicate vertex set.

**Corollary 3.4** For graph $G$ of order $n$ and $u, v \in V(G)$, if $u, v$ are multiplicate (or quasi-multiplicate) vertices, then
\[ q_{i+1}(G) - d \leq q_i(G - v) \leq q_i(G) - 1. \]

In fact, in Corollary 3.4, since $u, v$ are multiplicate (or quasi-multiplicate) vertices, then $d_G(u, w) = d_G(v, w)$, for $w \in V(G)$ and $w \neq u, v$. Moreover, for $x, y \in V(G - v)$, $d_{G-v}(x, y) =$
dc(x, y). Then by Lemma 2.2 and Corollary 3.1, the conclusion can be proved in the similar way as Theorem 3.2.

In [1], the authors demonstrate that the eigenvalues of \(Q(G)\) are non-decreasing when some edges are removed with the resultant graph also connected. The following lemma is on the behavior of distance signless Laplacian eigenvalues when the edge between quasi-multiplicate vertices is removed. And by it, we have a theorem in general.

**Lemma 3.5** Let \(x\) and \(y\) be quasi-multiplicate vertices of \(G\) and \(|V(G)| = n \geq 3\). Denote the edge between \(x\) and \(y\) by \(e\). Let \(q_i\) be the eigenvalues of \(Q(G)\) and \(q'_i\) be the eigenvalues of \(Q(G-e)\). For \(i = 1, 2, \ldots, n\), then \(q_i \leq q'_i \leq q_i + 2\).

**Proof** As \(x\) and \(y\) are quasi-multiplicate vertices, apart from the change of distance between \(x\) and \(y\) from one to two, the distances of other vertices are invariable. So \(Q(G-e) \geq Q(G)\) and let \(P = Q(G-e) - Q(G)\). Then \(P\) can be partitioned into \(\begin{pmatrix} J_2 & 0 \\ 0 & 0 \end{pmatrix}\) and the eigenvalues of \(P\) are 2 and 0 with multiplicity 1 and \(n-1\), respectively. Thus, the conclusion follows by Lemma 2.2. □

**Theorem 3.6** Let \(C \subset V(G)\) be a quasi-multiplicate set of graph \(G\) and \(2 \leq m = |C| < |V(G)| = n\). Suppose \(G'\) is the graph obtained by removing all the edges between vertices of \(C\). Let \(q_i\) be the eigenvalues of \(Q(G)\) and \(q'_i\) be those of \(Q(G')\), \(i = 1, 2, \ldots, n\). Then,

\[ q_i \leq q'_i \leq q_i + 2m - 2, \quad i = 1, 2, \ldots, n. \]

**Proof** Obviously, \(C\) becomes a multiplicate set in \(G'\). Similarly to Lemma 3.5, in the process of deleting edges, only the distances of vertices in \(C\) change from one to two. Let \(P = Q(G') - Q(G)\). Then \(P\) can be partitioned into \(\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}\), where \(M = (m-2)I + J_m\) with order \(m\). It is easy to know the eigenvalues of \(M\) are \(2m - 2\) and \(m - 2\) with multiplicity 1 and \(m - 1\), respectively. Hence, the eigenvalues of \(P\) are \(2m - 2\), \(m - 2\) and 0 with multiplicity 1, \(m - 1\) and \(n - m\), respectively. Thus the theorem follows from Lemma 2.2. □

**4. Extremal graphs with minimum \(\rho_Q(G)\)**

For trees with given diameter \(d\), the following theorem shows that \(P_{d+1}\) is the extremal graph with the minimum \(\rho_Q(G)\).

**Theorem 4.1** Let \(T_d\) be the set of all trees with given diameter \(d \geq 1\). Then for any tree \(T \in T_d\), the distance signless Laplacian spectral radius \(\rho_Q(T) \geq \rho_Q(P_{d+1})\) with equality if and only if \(T = P_{d+1}\), where \(P_{d+1}\) denotes the path of order \(d + 1\).

**Proof** Let tree \(T \in T_d\) with order \(n \geq d+1\). From Theorem 3.2, we see that \(q_1(G-u) \leq q_1(G)-1\), i.e., \(q_1(G-u) < q_1(G)\) where \(u\) is a pendent vertex. In other words, the distance signless Laplacian spectral radius \(\rho_Q(G)\) strictly decreases when the pendent vertices are removed from
Thus, the conclusion follows by continuously deleting the pendent vertices which are not on the diametrical line. □

Lemma 4.2 ([1]) The distance signless Laplacian characteristic polynomial of cycle \( C_n \) is as follows.

\[
P_Q(t) = \begin{cases} 
(t - \frac{n^2}{4})^{k-1} \cdot (t - \frac{n^2}{2}) \cdot \prod_{j=1}^{k} (t - \frac{n^2}{4} + \csc^2 \left( \frac{\pi}{n} \right)), & \text{if } n = 2k; \\
(t - \frac{n^2 - 1}{2}) \cdot \prod_{j=1}^{k} (t - \frac{n^2 - 1}{4} + \frac{1}{4} \sec^2 \left( \frac{\pi}{n} \right)) (t - \frac{n^2 - 1}{4} + \frac{1}{4} csc^2 \left( \frac{\pi}{2n} \right)), & \text{if } n = 2k + 1.
\end{cases}
\]

Then by calculating, for the distance signless Laplacian spectral radius of \( C_n \), we have

\[
\rho_Q(C_n) = \begin{cases} 
n^2, & \text{if } n = 2k \text{ (i.e., even);} 
n^2 - 1, & \text{if } n = 2k + 1 \text{ (i.e., odd).}
\end{cases}
\]

A simple connected graph \( G \) is called unicyclic if \( |V(G)| = |E(G)| \), bicyclic if \( |V(G)| + 1 = |E(G)| \). The girth of graph \( G \) is the length of the shortest cycle (if exists).

Theorem 4.3 Let \( U_g \) be the set of all unicyclic graphs with given girth \( g \geq 3 \). For any unicyclic graph \( G \in U_g \),

(i) \( \rho_Q(G) \geq \frac{g^2}{2} \), if \( g \) is even;

(ii) \( \rho_Q(G) \geq \frac{g^2 - 1}{2} \), if \( g \) is odd.

Equalities hold if and only if \( G = C_g \).

Proof Let \( G \in U_g \) and \( V(G) = V_1 \cup V_2 \). Without loss of generality, let the vertices of the cycle be \( V_1 = \{v_1, v_2, \ldots, v_g\} \). Then the components of subgraph induced by \( V_2 = \{v_{g+1}, v_{g+2}, \ldots, v_n\} \) are isolated vertices or trees. Assume that \( G \) has the minimum distance signless Laplacian spectral radius with order \( n > g \), then \( V_2 \neq \emptyset \). By Theorem 3.2, we obtain another graph \( G - v_i \), where \( v_i \) is a pendent vertex, a contradiction. Thus \( G = C_g \) has the minimum distance signless Laplacian spectral radius and the conclusion follows from Lemma 4.2. □

For graph \( G \), let \( e \in E(G) \) and the two incident vertices be \( u \) and \( v \). Replace \( e \) with a new vertex, say \( h \notin V(G) \), and make \( h \) adjacent to \( u \) and \( v \). This operation of graph is known as edge subdivision. Remove \( e \) from graph \( G \) and identify the two vertices incident to \( e \). We call this operation edge contraction. A cut-edge of connected graph \( G \) is an edge \( e \in E(G) \) such that \( G - e \) is disconnected.

Recall that the spectral radius of a nonnegative irreducible matrix increases if an entry increases [6, p.38]. Then before demonstrating the conclusion on bicyclic graphs, we first give the following important and useful lemmas.

Lemma 4.4 Let \( G_s \) be the graph derived from subdividing an edge, say \( e \), of graph \( G \). Then
\( \rho_Q(G_s) > \rho_Q(G) \).

**Proof** Let the new vertex be \( h \). Then \( V(G_s) = V(G) \cup \{ h \} \). For \( \forall x, y \in V(G) \), by the definition of distance of vertex, we easily obtain \( d_G(x, y) \leq d_{G_s}(x, y) \) and \( Tr_{G_s}(x) > Tr_G(x) \). Suppose \( M \) is the principal submatrix of \( Q(G_s) \) derived from deleting the row and column corresponding to \( h \). So \( 0 < M \) is an irreducible matrix and \( M \geq Q(G_s) (M \neq Q(G)) \). Thus we get \( \rho(M) > \rho_Q(G) \), where \( \rho(M) \) denotes the spectral radius of \( M \). Therefore, \( \rho_Q(G_s) \geq \rho(M) > \rho_Q(G) \) from Lemma 2.1. \( \square \)

**Lemma 4.5** Let \( e \in E(G) \) be a cut-edge of graph \( G \). Let \( G_c \) be the graph obtained by contracting \( e \). Then \( \rho_Q(G_c) < \rho_Q(G) \).

**Proof** Let the vertices incident to edge \( e \) be \( u \) and \( v \). By contracting \( e \), without loss of generality, let \( v \) be identified with \( u \). Thus \( V(G) = V(G_c) \cup \{ v \} \). Moreover, in fact, \( d_G(x, y) \geq d_{G_s}(x, y) \) and \( Tr_{G_s}(x) > Tr_G(x) \) for any \( x, y \in V(G_c) \). The remaining proof is similar to that of Lemma 4.4, and is omitted. \( \square \)

For bicyclic graph \( G \), we call it type of \( \infty \), if it has an induced subgraph isomorphic to \( G_1 \) (see Figure 1), and type of \( \theta \), if it has an induced subgraph isomorphic to \( G_2 \) (see Figure 1).

![Graphs G1, G2, G3, G4](image)

Figure 1 The graphs \( G_1, G_2 \) and \( G_3, G_4 \) (\( g \) denotes the length of cycle)

**Theorem 4.6** Let \( G \) be a bicyclic graph with given girth \( g \geq 3 \). Then,

(i) If \( G \) is type of \( \infty \), \( \rho_Q(G) \geq \rho_Q(G_3) \);

(ii) If \( G \) is type of \( \theta \), \( \rho_Q(G) \geq \rho_Q(G_4) \).

For (i) and (ii), equalities hold if and only if \( G \) is isomorphic to \( G_3 \) and \( G_4 \) (see Figure 1), respectively.

**Proof** (i) Assume bicyclic graph \( G \) with girth \( g \) having the minimum distance signless Laplacian spectral radius is not isomorphic to \( G_3 \). Then through the following steps we get contradictions.

Step 1. Let \( G^{(1)} \) be the induced subgraph of \( G \) isomorphic to \( G_1 \). If \( G^{(1)} \) is the proper induced subgraph of \( G \), i.e., the order of \( G \) is more than that of \( G^{(1)} \). By the method of deleting pendent vertices and Theorem 3.2, we obtain \( \rho_Q(G) > \rho_Q(G^{(1)}) \), a contradiction.

Step 2. From Step 1, if \( G \) has the minimum \( \rho_Q(G) \), \( G \) is necessarily isomorphic to \( G_1 \). Then we let \( G \) be isomorphic to \( G_1 \). Furthermore assume the length of the other cycle in \( G \) is larger than \( g \). Then by the inverse of Lemma 4.4, we can get a graph, say \( G^{(2)} \), possessing less distance signless Laplacian spectral radius, which has two cycles with the same length \( g \), a contradiction.
Step 3. After the above steps, let $G$ be isomorphic to $G_1$ and have same length $g$ of cycles. Suppose the length of the path $P_m$ (see $G_1$ in Figure 1) between the two cycles of $G$ is more than zero (i.e., $m \geq 2$). If $m = 2$, by Lemma 4.5, we derive a new graph, say $G^{(3)}$, with less distance signless Laplacian spectral radius than $G$. If $m > 2$, we also obtain a contradiction from Lemmas 4.4 and 4.5. Thus the length of $P_m$ in $G$ is zero.

By the three steps, if $G$ is type of $\infty$ and has the minimum $\rho_Q(G)$, $G$ is isomorphic to $G_3$. Then the proof of (i) is done.

The proof of (ii) can be testified in the similar way, omitted. $\square$

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