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A Characterization of Bicyclic Signed Graphs with Nullity $n-7$

Guojun LI, Aimei YU*
Department of Mathematics, Beijing Jiaotong University, Beijing 100044, *P. R. China*

Abstract Let Γ be a signed graph and $A(\Gamma)$ be the adjacency matrix of Γ. The nullity of Γ is the multiplicity of eigenvalue zero in the spectrum of $A(\Gamma)$. In this paper, the connected bicyclic signed graphs (including simple bicyclic graphs) of order n with nullity $n-7$ are completely characterized.

Keywords nullity; signed graph; bicyclic graph; adjacency matrix

MR(2010) Subject Classification 05C50

1. Introduction

A signed graph is a graph with a sign attached to each of its edges. So a signed graph $\Gamma=(G,\sigma)$ consists of a simple graph $G=(V(G),E(G))$, referred to as its underlying graph, and a mapping $\sigma: E(G)\to\{+,-\}$, the edge labelling. The underlying graph G of Γ is denoted by ${}^u\Gamma$. The adjacency matrix of Γ is $A(\Gamma)=(a_{ij}^{\sigma})$ with $a_{ij}^{\sigma}=\sigma(v_iv_j)a_{ij}$, where (a_{ij}) is the adjacency matrix of the underlying graph G. In the case of $\sigma=+$, which is an all-positive edge labelling, the adjacency matrix $A(G,+)$ is exactly the classical adjacency matrix of G. The nullity of a signed graph Γ is defined as the multiplicity of the eigenvalue zero in spectrum of $A(\Gamma)$, and is denoted by $\eta(\Gamma)$. The rank of Γ is referred to the rank of $A(\Gamma)$, and is denoted by $r(\Gamma)$. Clearly, $\eta(\Gamma)+r(\Gamma)=n$ if Γ has n vertices.

Let C be a cycle of Γ. The sign of C is defined as $\mathrm{sgn}(C)=\Pi_{e\in C}\sigma(e)$, and denoted by $\mathrm{sgn}(C)$. If $\mathrm{sgn}(C)=+$ or $\mathrm{sgn}(C)=-$, we call the cycle C is balanced or unbalanced. A signed graph is said to be balanced if all its cycles are balanced, or equivalently, all cycles have an even number of negative edges; otherwise it is called unbalanced.

Let $\theta: V(G)\to\{+,-\}$ be any sign function. Switching Γ by θ means forming a new signed graph $\Gamma^\theta=(G,\sigma^\theta)$ whose underlying graph is the same as G, but whose sign function is defined on an edge uv by $\sigma^\theta(uv)=\theta(u)\sigma(uv)\theta(v)$. Note that switching does not change the signs or balance of the cycles of Γ. If we define a (diagonal) signature matrix D^θ with $d_v=\theta(v)$ for each $v\in V(G)$, then $A(\Gamma^\theta)=D^\theta A(\Gamma)D^\theta$. Two graphs Γ_1,Γ_2 are called switching equivalent,

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* Corresponding author
E-mail address: 843007613@qq.com (Guojun LI); yuaimeimath@yeah.net (Aimei YU)

denoted by $\Gamma_1 \sim \Gamma_2$, if there exists a switching function θ such that $\Gamma_2 = \Gamma_1$, or equivalently $A(\Gamma_2) = D^{\theta}A(\Gamma_1)D^{\theta}$. Note that switching equivalence is a relation of equivalence, and two switching equivalent graphs have the same nullities. A signed graph Γ is balanced if and only if $\Gamma = (G, \sigma) \sim (G, +)$ (see [1]). So the results on the nullity for simple graphs still hold for balanced signed graphs.

Signed graphs were introduced by Harary [2] in connection with the study of the theory of social balance in social psychology. The matroids of graphs were extended to those of signed graphs by Zaslavsky [3], and the Matrix-Tree Theorem for signed graphs was obtained by Zaslavsky [3] and by Chaiken [4]. More recent results on signed graphs can be found in [1] and [5].

Recently, some papers have investigated the nullity of simple graphs. It is known that $0 \le \eta(G) \le n-2$ if G is a simple graph of order n containing at least one edge. Cheng and Liu [6] characterized the simple graphs of order n with nullity $n-2$ and $n-3$. Chang et al. [7, 8] characterized the simple graphs of order n with nullity $n-4$ and $n-5$. Zhu et al. [9] characterized the unicyclic graphs with nullity $n-6$ and $n-7$. Li and Chang [10] gave the nullity set of three kinds of bicyclic graphs, and characterized two kinds of bicyclic graphs with nullity $n-6$. More results on the nullity of special classes of simple graphs can be found in the papers [11–19].

In this paper, we discuss the nullity of the signed graphs. Fan et al. [20] characterized the unicyclic signed graphs of order n with nullity $n-2, n-3, n-4$ and $n-5$, respectively. All signed graphs of order n with nullity $n-2$ and $n-3$ were characterized in [21]. The authors in [21] also determined the unbalanced bicyclic signed graphs of order n with nullity $n-3$ or $n-4$, and signed bicyclic graphs of order n with $n-5$. In this paper, we characterize the bicyclic signed graphs (including simple bicyclic graphs) with nullity $n-7$.

2. Some lemmas

We first introduce some concepts and notations of signed graphs. However, these definitions are based only on the underlying graph of the signed graph. Let $\Gamma = (G, \sigma)$ be a signed graph with order n. Γ is called an acyclic graph if it contains no cycles; Γ is called a unicyclic (resp., bicyclic) graph if it is connected and it has n (resp., $n+1$) edges.

Denote by $N(v)$ the neighbourhood of v in Γ and $d(v)$ the degree of v. A vertex of Γ is called pendant if it has degree one. An edge subset $M \subseteq E(\Gamma)$ is called a matching of Γ if no two edges of M share a common vertex. A matching M is called maximum in Γ if it has maximum cardinality among all matchings of Γ, and the maximum cardinality is denoted by $\mu(\Gamma)$.

The disjoint union of two graphs Γ_1 and Γ_2 is denoted by $\Gamma_1 \cup \Gamma_2$. The union of k disjoint copies of Γ is denoted by $k\Gamma$. Denote by P_n, C_n and S_n a path, a cycle and a star, respectively, all of which are simple graphs of order n. An isolated vertex is denoted by K_1.

The following result on the nullity of a signed graph is obvious.

Lemma 2.1 *Let Γ be a signed graph. If $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_t$, where $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ are the connected components of Γ, then $\eta(\Gamma) = \sum_{i=1}^{t} \eta(\Gamma_i)$.*

Denote by $Pv_1v_2v_3$ a path on vertices v_1, v_2, v_3 with edges v_1v_2, v_2v_3. If a signed graph Γ contains a path $Pv_1v_2v_3$ which satisfies $d(v_2) = 2, v_1v_3 \notin E(\Gamma), N(v_1) \cap N(v_3) = \{v_2\}$, then $Pv_1v_2v_3$ is called a special path of Γ.

Lemma 2.2 ([21]) *Let Γ be a signed graph containing a special path $Pv_1v_2v_3$, where $\sigma(v_1v_2) = -1, \sigma(v_2v_3) = 1$. If Γ' is a signed graph obtained from Γ by contracting the special path $Pv_1v_2v_3$ into a single vertex, i.e., deleting the vertex v_2 and the edges v_1v_2 and v_2v_3 and identifying the vertices v_1 and v_3, then $\eta(\Gamma) = \eta(\Gamma')$.*

Lemma 2.3 ([20]) *Let Γ be a signed graph containing a pendant vertex, and Γ' be the induced subgraph of Γ obtained by deleting this pendant vertex together with the vertex adjacent to it. Then $\eta(\Gamma) = \eta(\Gamma')$.*

Lemma 2.4 ([20]) *(1) If T is an acyclic signed graph or a signed tree of order n, then $\eta(T) = n - 2\mu(T)$, where $\mu(T)$ is the matching number of T.*

(2) If C_n is a balanced signed cycle, then $\eta(C_n) = 2$ if $n \equiv 0\,(\mathrm{mod} 4)$, and $\eta(C_n) = 0$ otherwise.

(3) If C_n is a unbalanced signed cycle, then $\eta(C_n) = 2$ if $n \equiv 2\,(\mathrm{mod} 4)$, and $\eta(C_n) = 0$ otherwise.

Fan et al. [21] used Lemma 2.2 to characterize the signed graphs with nullity $n-2$ and $n-3$. For a signed connected graph Γ, $\eta(\Gamma) = n - 2$ if and only if Γ is a balanced complete bipartite graph.

Let Γ be a signed graph and v be a vertex of Γ. The positive neighbourhood $N_+(v)$ (resp., the negative neighbourhood $N_-(v)$) of v is the set of vertices joining v with positive edges (resp., negative edges).

Lemma 2.5 ([21]) *Let Γ be a signed graph of order $n \geq 3$. Then $\eta(\Gamma) = n - 3$ if and only if Γ is a complete tripartite graph with partition (V_1, V_2, V_3) together with some isolated vertices, which satisfies that for each $i = 1, 2, 3$, there exits a vertex $u_i \in V_i$ such that for every other vertices $v \in V_i$, either $N_+(v) = N_+(u_i)$, $N_-(v) = N_-(u_i)$ or $N_+(v) = N_-(u_i)$, $N_-(v) = N_+(u_i)$.*

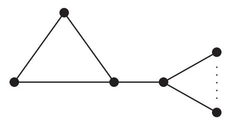

Figure 2.1 The graph U^*

Fan et al. [20] characterized the signed unicyclic graphs of order n with nullity $n-2$, $n-3$ and $n-5$, respectively. The results on the signed unicyclic graphs with nullity $n-2$ (resp., $n-3$) are the corollaries of the above two results.

Lemma 2.6 ([20]) *Let Γ be a unicyclic signed graph of order n. Then*

(1) $\eta(\Gamma) = n - 2$ if and only if Γ is the balanced cycle C_4.

(2) $\eta(\Gamma) = n - 3$ if and only if ${}^u\Gamma$ is C_3.

Lemma 2.7 ([20]) *Let Γ be a unicyclic signed graph of order $n \geq 5$. Then $\eta(\Gamma) = n - 5$ if and only if ${}^u\Gamma$ is C_5 or U^* (see Figure 2.1).*

3. The bicyclic signed graphs with nullity $n - 7$

Let C_k and C_l be two edge-disjoint cycles. Suppose that v_1 is a vertex of C_k and v_q is a vertex of C_l. Joining v_1 and v_q by a path $v_1 v_2 \cdots v_q$ of length $q - 1$, where $q \geq 1$ and $q = 1$ means identifying v_1 and v_q, the resultant graph, denoted by $\infty(k, q, l)$ shown in Figure 3.1, is called an ∞-graph. Let P_{l+1}, P_{p+1} and P_{q+1} be three vertex-disjoint paths, where $l, p, q \geq 1$, and at most one of them is 1. Identifying the three initial vertices and terminal vertices of them, respectively, the resultant graph, denoted by $\theta(l, p, q)$ shown in Figure 3.1, is called a θ-graph.

Let $\mathcal{B}_n$ be the set of all n-vertex bicyclic signed graphs. Obviously, the underlying graphs of the signed graphs in $\mathcal{B}_n$ consist of three types graphs: first type denoted by $\mathcal{B}_n^+$ is the set of those graphs each of which is an ∞-graph or an ∞-graph with trees attached when $q \geq 2$; second type denoted by $\mathcal{B}_n^{++}$ is the set of those graphs each of which is an ∞-graph or an ∞-graph with trees attached when $q = 1$; third type denoted by θ_n is the set of those graphs each of which is a θ-graph or a θ-graph with trees attached. Then $\mathcal{B}_n = (\mathcal{B}_n^+, \sigma) \cup (\mathcal{B}_n^{++}, \sigma) \cup (\theta_n, \sigma)$ with a mapping $\sigma : E \to \{+, -\}$, the edge labeling.

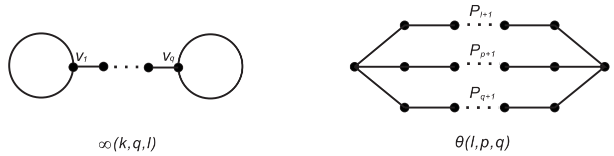

Figure 3.1 The graphs $\infty(k, q, l)$ and $\theta(l, p, q)$

Note that switching equivalence is a relation of equivalence, and two switching equivalent graphs have the same nullities. So, when we discuss the nullity of signed graphs, we can choose an arbitrary representative of each switching equivalent class.

Lemma 3.1 *Let Γ be a bicyclic signed graph of order n.*

(1) If ${}^u\Gamma \in \mathcal{B}_n^+ \cup \mathcal{B}_n^{++}$, then Γ is switching equivalent to a bicyclic signed graph of order n, which has exactly one (arbitrary) sign-uncertain edge on each cycles of Γ and the other edges are all positive.

(2) If ${}^u\Gamma \in \theta_n$, then Γ is switching equivalent to a bicyclic signed graph of order n, which has exactly one (arbitrary) sign-uncertain edge on two (arbitrary) paths of Γ and the other edges are all positive, where the paths are P_{l+1}, P_{p+1} and P_{q+1}.

Proof If ${}^u\Gamma \in \mathcal{B}_n^+ \cup \mathcal{B}_n^{++}$ (resp., ${}^u\Gamma \in \theta_n$), let e_1, e_2 be two arbitrary edges on each cycles of Γ (resp., two arbitrary paths of Γ) and $\Gamma - e_1 - e_2$ be the signed graph obtained from Γ by deleting the edges e_1 and e_2. Observe that $\Gamma - e_1 - e_2$ is balanced. So there exits a sign function $\theta : V(\Gamma - e_1 - e_2) \to \{+, -\}$ such that $(\Gamma - e_1 - e_2)^\theta$ consists of positive edges. Returning to the Γ^θ, the signs of e_1, e_2 are uncertain and the other signs are all positive. The result follows. $\square$

Lemma 3.2 *Let Γ be a bicyclic signed graph of order $n \geq 5$. Then $\eta(\Gamma) = n-5$ if and only if Γ is one of the following graphs (see Figure 3.2) with certain properties:*

(1) ${}^u\Gamma = B_1$;

(2) ${}^u\Gamma \in \{B_2, B_3, B_5\}$ and the two triangles of Γ have the same balance;

(3) ${}^u\Gamma \in \{B_4, B_8\}$ and the quadrangle of Γ is balanced;

(4) ${}^u\Gamma = B_6$ and the quadrangle of Γ is unbalanced;

(5) ${}^u\Gamma = B_7$ and the cycle C_6 of Γ is unbalanced.

Proof Fan et al. [21] determined the bicyclic signed graphs of order $n \geq 5$ with nullity $n-5$. By Theorem 3.3 of [21], if Γ contains pendant vertices or Γ contains no pendant vertices and ${}^u\Gamma \in \mathcal{B}_n^+ \cup \mathcal{B}_n^{++}$, then Γ is one of the following graphs (see Figure 3.2) with certain properties:

(1) ${}^u\Gamma = B_1$;

(2) ${}^u\Gamma \in \{B_2, B_3, B_5\}$ and the two triangles of Γ have the same balance;

(3) ${}^u\Gamma \in \{B_4\}$ and the quadrangle of Γ is balanced.

But when Γ contains no pendant vertices and ${}^u\Gamma \in \theta_n$, they omitted the signed graphs with underlying graph B_8 and the quadrangles of them are balanced. So we only show the case when Γ contains no pendant vertices and ${}^u\Gamma \in \theta_n$.

If Γ contains no special paths, then ${}^u\Gamma$ is one of the graphs $\theta(1,2,3)$, $\theta(1,3,3)$, $\theta(2,2,2)$. By a direct calculation according to Lemma 3.1, ${}^u\Gamma = \theta(1,2,3) = B_6$ and the quadrangle of Γ is unbalanced.

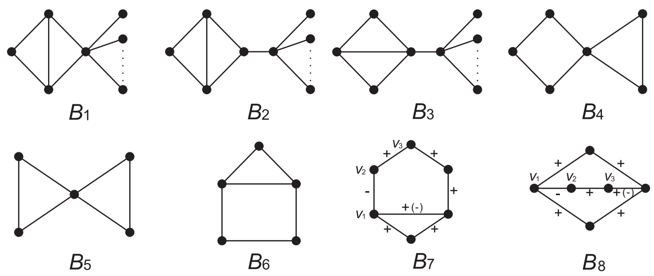

Figure 3.2 The graphs $B_1, B_2, \ldots, B_8$

If Γ contains a special path $Pv_1v_2v_3$, without loss of generality, we assume that the sign of v_1v_2 is negative, the sign of v_2v_3 is positive and v_1 is a vertex of degree 3. Now we contract the path $Pv_1v_2v_3$ into a single vertex, say u, then the resulting graph Γ' has rank 3 by Lemma 2.2. By Lemma 2.5, Γ' is a complete tripartite graph, i.e., ${}^u\Gamma' = \theta(1,2,2)$, and the two triangles of Γ' have the same balance. So there exists a sign function τ defined on $V(\Gamma')$ such that all edges of Γ' are positive or all edges of Γ' are positive except the common edge of the two triangles. Returning to the original graph, ${}^u\Gamma = \theta(1,2,4)$ or $\theta(2,2,3)$. Noting that $V(\Gamma) \setminus \{v_1, v_2, v_3\} = V(\Gamma') \setminus \{u\}$, we extend the sign function τ on $V(\Gamma')$ to a sign function θ on $V(\Gamma)$, where $\theta(v_1) = \theta(v_2) = \theta(v_3) = \tau(u)$ and $\theta(v) = \tau(v)$ for other vertex v of Γ. Then Γ^θ is the graph $B_i, i = 7, 8$ with edge labelling as in Figure 3.2. Therefore, ${}^u\Gamma = B_7$ and the cycle C_6 of Γ is unbalanced or ${}^u\Gamma = B_8$ and the quadrangle of Γ is balanced. This completes the proof of the necessity. The sufficiency is clear by simple calculation. □

Lemma 3.3 *Let Γ be a bicyclic signed graph of order $n \geq 7$ and ${}^u\Gamma$ be an ∞-graph or a θ-graph. Then $\eta(\Gamma) = n-7$ if and if Γ is one of the following graphs (see Figure 3.3) with certain properties:*

(1) ${}^u\Gamma \in \{\infty(4,2,3), \theta(1,3,4)\}$ and the quadrangle of Γ is unbalanced;

(2) ${}^u\Gamma \in \{\infty(4,3,3), \infty(4,1,5), \theta(2,2,5)\}$ and the quadrangle of Γ is balanced;

(3) ${}^u\Gamma \in \{\infty(6,1,3), \theta(2,3,4)\}$ and the cycle C_6 of Γ is unbalanced;

(4) ${}^u\Gamma \in \{\theta(2,3,3), \theta(1,2,5)\}$ and the cycle C_6 of Γ is balanced;

(5) ${}^u\Gamma \in \{\theta(1,2,6), \theta(1,4,4)\}$ and the cycle C_8 of Γ is balanced;

(6) ${}^u\Gamma = \infty(5,1,3)$ and the sign of C_5 is different from that of C_3;

(7) ${}^u\Gamma = \infty(3,3,3)$ and the two triangles of Γ have the same balance.

Proof The sufficiency can be verified by a little calculation according to Lemma 3.1 or Lemma 2.2.

Now assume that $\eta(\Gamma) = n-7$. We divide the discussion into two cases.

Case 1 Γ contains no special paths.

If ${}^u\Gamma = \infty(k,q,l)$, then the length of any cycle in Γ is less than 5 and $q \leq 2$, otherwise Γ contains a special path. If ${}^u\Gamma = \theta(l,p,q)$, where $l \leq p \leq q$, then $l = 1$ and $q \leq 3$, otherwise Γ contains a special path. Therefore, ${}^u\Gamma$ can be one of the graphs $\infty(4,2,4)$, $\infty(4,2,3)$ or $\infty(4,1,4)$. By a direct calculation according to Lemma 3.1, ${}^u\Gamma = \infty(4,2,3)$ and the quadrangle of Γ is unbalanced.

Case 2 Γ contains a special path $Pv_1v_2v_3$.

Without loss of generality, we assume that the sign of v_1v_2 is negative and the sign of v_2v_3 is positive. Contract the path $Pv_1v_2v_3$ into a single vertex, say u, then the resulting graph Γ' will have rank 5 by Lemma 2.2.

If Γ' contains no special path, then ${}^u\Gamma' = B_i, i = 4,5,6$ (see Figure 3.1) by Lemma 3.2. Returning to the original graph Γ, we need to replace an arbitrary vertex in $B_i, i = 4,5,6$ with special path $Pv_1v_2v_3$. Then we get the following signed graphs by considering all situations.

(i) When ${}^u\Gamma' = B_4 = \infty(4,1,3)$ and the quadrangle of Γ' is balanced.

${}^u\Gamma = \infty(6,1,3)$ and the cycle C_6 of Γ is unbalanced, or ${}^u\Gamma = \infty(4,3,3)$ and the quadrangle of Γ is balanced, or ${}^u\Gamma = \infty(4,1,5)$ and the quadrangle of Γ is balanced.

(ii) When ${}^u\Gamma' = B_5 = \infty(3,1,3)$ and the two triangles of Γ' have the same balance.

${}^u\Gamma = \infty(5,1,3)$ and the sign of C_5 is different from that of C_3, or ${}^u\Gamma = \infty(3,3,3)$ and the two triangles of Γ have the same balance.

(iii) When ${}^u\Gamma' = B_6 = \theta(1,2,3)$ and the quadrangle of Γ' is unbalanced.

${}^u\Gamma = \theta(1,3,4)$ and the quadrangle of Γ is unbalanced, or ${}^u\Gamma = \theta(2,3,3)$ and the cycle C_6 of Γ is balanced, or ${}^u\Gamma = \theta(1,2,5)$ and the cycle C_6 of Γ is balanced.

If Γ' contains a special path, then ${}^u\Gamma' = B_i, i = 7,8$ (see Figure 3.1) by Lemma 3.2. Returning to the original graph Γ, we need to replace an arbitrary vertex in $B_i, i = 7,8$ with special path $Pv_1v_2v_3$. Therefore, Γ is one of the following graphs: ${}^u\Gamma = \theta(1,2,6)$ and the cycle C_8 of Γ is balanced, ${}^u\Gamma = \theta(2,3,4)$ and the cycle C_6 of Γ is unbalanced, ${}^u\Gamma = \theta(1,4,4)$ and the cycle C_8 of Γ is balanced or ${}^u\Gamma = \theta(2,2,5)$ and the cycle C_4 is balanced. This completes the proof. □

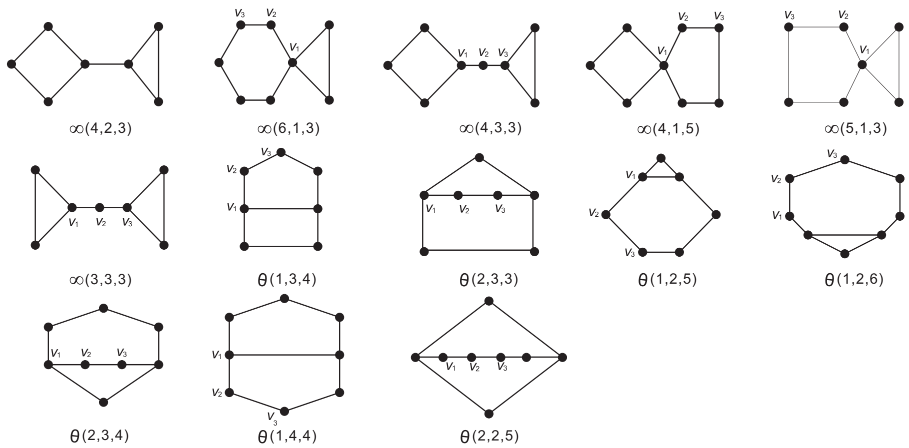

Figure 3.3 The underlying graphs of the ∞-graphs and θ-graphs with order n and nullity $n-7$

If Γ is a bicyclic signed graph with pendent vertices, Γ can be obtained from a signed $\infty(k,q,l)$ (or $\theta(p,l,q)$) by attaching some signed trees. We call the signed $\infty(k,q,l)$ (or $\theta(p,l,q)$) the base of the corresponding Γ, denoted by χ_Γ.

Lemma 3.4 *Let Γ be a bicyclic signed graph of order n with pendent vertices. Then $\eta(\Gamma) = n-7$ if and only if Γ is one of the following graphs (see Figure 3.4) with certain properties:*

(1) ${}^u\Gamma = G_i^*, i = 1,2,\ldots,5,30,31,\ldots,37;$

(2) ${}^u\Gamma = G_i^*, i = 6,7,\ldots,13,18,19$ *and the two triangles of Γ have the same balance;*

(3) ${}^u\Gamma = G_i^*, i = 14,15,\ldots,17,27,28,29,38$ *and the quadrangle of Γ is balanced;*

(4) ${}^u\Gamma = G_i^*, i = 20,21,22$ *and the quadrangle of Γ is unbalanced;*

(5) ${}^u\Gamma = G_i^*, i = 23,24,25,26$ *and the cycle C_6 of Γ is unbalanced.*

Proof The sufficiency can be verified by Lemmas 2.1–2.7, Lemmas 3.2 and 3.3.

Now we show the necessity. Assume that Γ is a bicyclic signed graph with pendent vertices with $\eta(\Gamma) = n-7$. Let $T = V(\Gamma) \setminus V(\chi_\Gamma)$ and S be the set of vertices in χ_Γ which are adjacent to a vertex of T. Let $v_0v_1\cdots v_h$ be a longest signed path P_{h+1} in Γ such that $v_0 \in S$ and $v_1,\ldots,v_h \in T$.

In the rest of proof, we distinguish two cases.

Case 1 $h \geq 2$.

Let Γ_1 be the signed graph obtained from Γ by deleting v_h and v_{h-1}. Since P_{h+1} is the longest signed path of Γ such that $v_0 \in S$ and $v_1,\ldots,v_h \in T$ and $h \geq 2$, Γ_1 has a unique nontrivial connected component Γ_{11}. Let $|\Gamma_{11}| = n_1$. Then $\Gamma_1 = \Gamma_{11} \cup (n-n_1-2)K_1$. Note that

Γ_{11} is a bicyclic signed graph with order n_1. By Lemmas 2.1 and 2.3, we have $\eta(\Gamma) = \eta(\Gamma_1) = \eta(\Gamma_{11}) + (n - n_1 - 2) = n - 7$, which implies that $\eta(\Gamma_{11}) = n_1 - 5$.

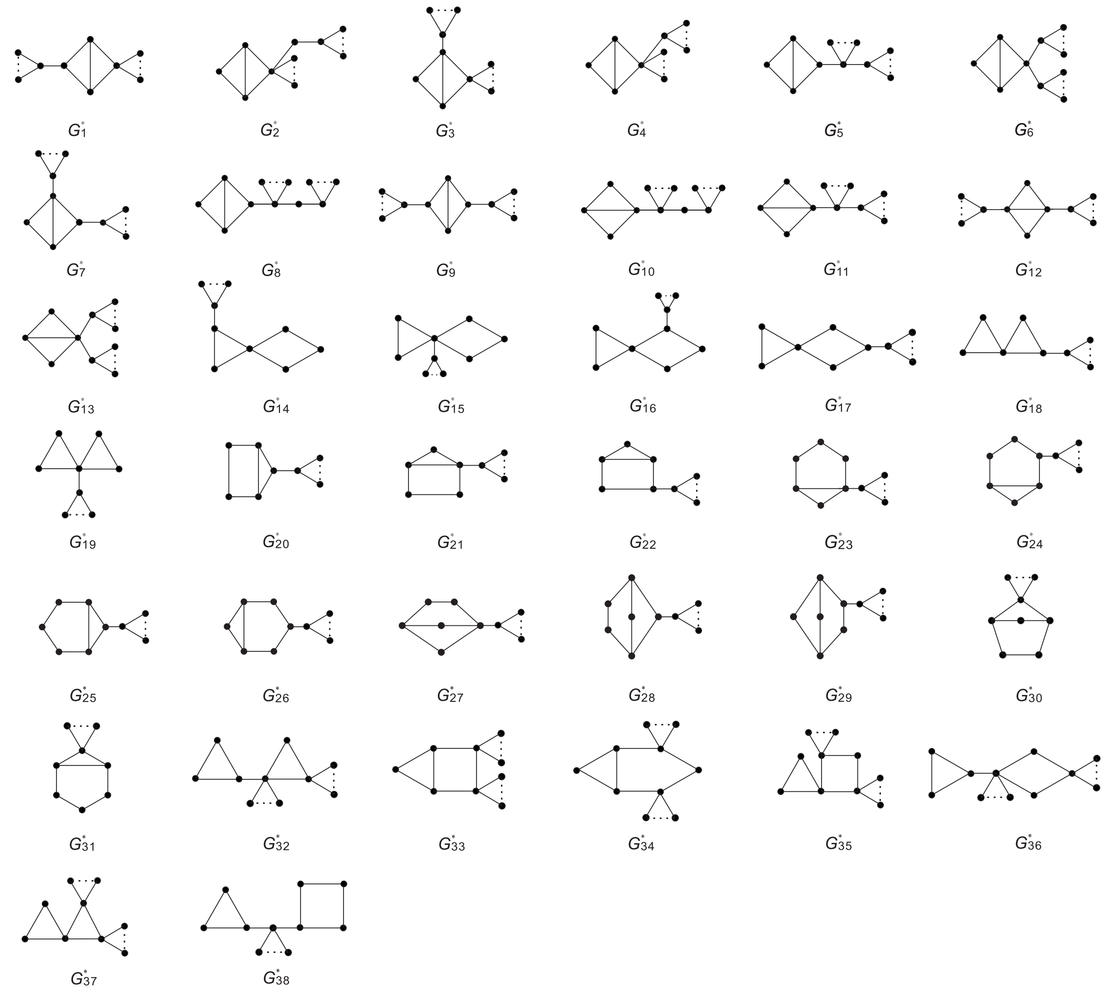

Figure 3.4 The underlying graphs of the bicyclic signed graphs containing pendent vertices with order n and nullity $n - 7$

Subcase 1.1 There are pendent vertices in Γ_{11}.

By Lemma 3.2, we have ${}^u\Gamma_{11} = B_1$, B_2 or B_3 and the two triangles of B_2 and B_3 have the same balance, where B_1, B_2 and B_3 are shown in Figure 3.2. So ${}^u\Gamma_1 = B_1 \cup (n - n_1 - 2)K_1$ or ${}^u\Gamma_1 = B_2 \cup (n - n_1 - 2)K_1$ or ${}^u\Gamma_1 = B_3 \cup (n - n_1 - 2)K_1$. Now recover v_h and v_{h-1} to Γ_1, we need to insert edges attached arbitrary signs from v_{h-1} to each $n - n_1 - 1$ isolated vertices. This gives a signed star S_{n-n_1}. Since Γ_{11} already contains two cycles, in order to recover Γ, we only need to connect the center of S_{n-n_1} to a vertex of $B_i, i = 1, 2, 3$ with arbitrary signed edge. Hence ${}^u\Gamma = G_i^*, i = 1, 2, \ldots, 4$, or ${}^u\Gamma = G_i^*, i = 5, 6, \ldots, 13$ and the two triangles of Γ have the same balance (see Figure 3.4).

Subcase 1.2 There are no pendent vertices in Γ_{11}, by Lemma 3.2, then the underlying graphs of ${}^u\Gamma_{11}$ are $B_i, i=4,5,\ldots,10$ (see Figure 3.2). In order to recover Γ, we only need to connect the center of S_{n-n_1} to a vertex of $B_i, i=4,5,\ldots,8$ with an arbitrary signed edge. Then the ${}^u\Gamma = G_i^*, i=14,15,\ldots,17$ and the quadrangle of Γ is balanced, or ${}^u\Gamma = G_i^*, i=18,19$ and the two triangles of Γ have the same balance, or ${}^u\Gamma = G_i^*, i=20,21,22$ and the quadrangle of Γ is unbalanced, or ${}^u\Gamma = G_i^*, i=23,24,25,26$ and the cycle C_6 of Γ is unbalanced, or ${}^u\Gamma = G_i^*, i=27,28,29$ and the quadrangle of Γ is balanced.

Case 2 $h=1$.

In this case, Γ is obtained from a signed ∞-graph or a signed θ-graph attached some pendent signed edges. Let v be a pendent vertex of Γ, and u be the vertex in $V(\chi_\Gamma)$ adjacent to v. Delete v, u, and then we get the graph $\Gamma_1 = \Gamma_{11} \cup \Gamma_{12} \cup \cdots \cup \Gamma_{1t}$, where $\Gamma_{11}, \Gamma_{12}, \ldots, \Gamma_{1t}$ are connected components of Γ_1. Some of these components may be trivial, i.e., K_1. But not all components are trivial, otherwise, ${}^u\Gamma = S_n$, a contradiction. Moreover, it is not difficult to see that Γ_1 has at most two nontrivial components. Now we consider the following two subcases:

Subcase 2.1 There is a unique nontrivial component in Γ_1.

Let Γ_{11} be the unique nontrivial component of Γ_1 and $|V(\Gamma_{11})| = n_1$. Then $\Gamma_1 = \Gamma_{11} \cup (n-n_1-2)K_1$ and $\eta(\Gamma) = \eta(\Gamma_1) = \eta(\Gamma_{11}) + n - n_1 - 2$, by Lemmas 2.1 and 2.3. It follows that $\eta(\Gamma_{11}) = n_1 - 5$. Note that Γ_{11} is a unicyclic signed graph or signed tree. By Lemma 2.4, there is no signed tree with nullity $n_1 - 5$. Hence Γ_{11} is a unicyclic signed graph with $\eta(\Gamma_{11}) = n_1 - 5$. By Lemma 2.7, Γ_{11} is a signed C_5 or a signed graph with U^* (see Figure 2.1) as underlying graph. Then we get that Γ are signed graphs with $G_i^*, i=30,31,\ldots,37$ as underlying graph.

Subcase 2.2 There are two nontrivial components in Γ_1.

Without loss of generality, we assume that Γ_{11} and Γ_{12} are nontrivial. Let $|V(\Gamma_{11})| = n_1$ and $|V(\Gamma_{12})| = n_2$. Then $\Gamma_1 = \Gamma_{11} \cup \Gamma_{12} \cup (n-n_1-n_2-2)K_1$ and $\eta(\Gamma_1) = \eta(\Gamma_{11}) + \eta(\Gamma_{12}) + n - n_1 - n_2 - 2 = n - 7$. Hence $\eta(\Gamma_{11}) + \eta(\Gamma_{12}) = n_1 + n_2 - 5$. Without loss of generality, we assume $\eta(\Gamma_{11}) = n_1 - 2$ and $\eta(\Gamma_{12}) = n_2 - 3$. By Lemma 2.4, Γ_{11} and Γ_{12} are not both signed trees.

Then Γ_{11} is a signed tree and Γ_{12} is a unicyclic signed graph or Γ_{11} and Γ_{12} are both unicyclic signed graphs. If Γ_{11} is a signed tree and Γ_{12} is a unicyclic signed graph, then Γ_{11} is a signed S_{n_1} and Γ_{12} is a signed C_3 by Lemma 2.6. Then ${}^u\Gamma = G_i^*, i=32,36$ as underlying graph. If Γ_{11} and Γ_{12} are both unicyclic signed graphs, then Γ_{11} is a balanced C_4 and Γ_{12} is a signed C_3 by Lemma 2.6. Then ${}^u\Gamma = G_{38}^*$ and the C_4 of Γ is balanced. This completes the proof. □

Theorem 3.5 *Let Γ be a connected bicyclic signed graph of order $n \geq 7$. Then $\eta(\Gamma) = n-7$ if and only if ${}^u\Gamma = \infty(4,2,3), \infty(6,1,3), \infty(4,3,3), \infty(4,1,5), \infty(5,1,3), \infty(3,3,3), \theta(1,3,4), \theta(2,3,3), \theta(1,2,5), \theta(1,2,6), \theta(1,4,4), \theta(2,3,4), \theta(2,2,5), G_i^*, i=1,2,\ldots,38$ with certain properties (see Lemmas 3.3 and 3.4).*

Proof By Lemmas 3.3 and 3.4, Theorem 3.5 holds immediately. □

From Theorem 3.5, we easily get a characterization of simple bicyclic graphs (or balanced bicyclic signed graphs) of order n with nullity $n-7$.

Corollary 3.6 *Let G be a connected bicyclic simple graph of order $n \geq 7$. Then $\eta(G) = n-7$ if and only if $G = \infty(4,3,3), \infty(4,1,5), \infty(3,3,3), \theta(2,3,3), \theta(1,2,6), \theta(1,4,4), \theta(1,2,5), \theta(2,2,5)$, $G_i^*, i = 1, 2, \ldots, 19, 27, 28, \ldots, 38$ (see Figure 3.4).*

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