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Reductions of Connected Simple r-Uniform Hypergraphs

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Abstract It is proved in this paper that if G is a simple connected r-uniform hypergraph with $\|G\| \geq 2$, then G has an edge e such that $G - e - V_1(e)$ is also a simple connected r-uniform hypergraph. This reduction is naturally called a combined Graham reduction. Under the simple reductions of single edge removals and single edge contractions, the minor minimal connected simple r-uniform hypergraphs are also determined.

Keywords graph families; reductions; uniform hypergraphs

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1. Graph theoretic background

Reductions preserving a graph theoretic property are interesting for at least two reasons: (1) they help obtain structural characterizations of families of graphs such as in the case of 3-connected graphs [22] and in the case of a simple proof of Kuratowski's theorem [21]; (2) they provide an order theoretic characterization of the family of graphs in question. If the reductions give rise to specifically defined minor inclusions, then they are particularly interesting. Work on reductions of hypergraphs in terms of specific minor inclusion is rare, while the graph minor theorem (the reader is referred to [16] for this theorem and a guide to the literature leading to it) holds for hypergraphs as well.

It is easy to see that the family of connected hypergraphs of orders at least 2 has an edge, contraction of which results in a connected hypergraph. Thus, the single edge contraction preserves connectivity if the hypergraph has an edge. It seems that the next step from this trivial case is the family of connected simple uniform hypergraphs where the degree of uniformity is fixed. This question will be addressed in this note.

As in [7], Tutte's inductive theorem on 3-connected graphs takes the form: Let G be a 3-connected graph. Then there exists a sequence

$$G_0 \leq G_1 \leq \cdots \leq G_n$$

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of 3-connected graphs, such that $G_0 = K_4$, $G_n = G$ and for each i, $1 \leq i \leq n$, there exists $e \in E(G_i)$ and $G_{i-1} = G_i/e$. With a slight over-use of notation, $\leq$ is the minor inclusion if it is placed between two graphs and denotes "not greater" if it is placed between two numbers. The family of 2-connected graphs also has a reduction theorem of this form [7, Proposition 3.1.2]. For the family of 4-connected graphs, there is also a reduction theorem of this form [2, 9, 10, 12–14, 18, 19].

The family of 3-connected triangle-free (i.e., girth at least 4) graphs has recently been shown to have a reduction structure theorem [11] where each reduction involved is a specific minor inclusion.

For triangulations of the sphere the reduction theorem was well known earlier in the twentieth century [20] where the reduction was a single edge contraction. Reductions for the family of quadrangulations of the sphere have been considered by several authors [1, 3, 5, 6, 15]. Among these, [3] meets both the requirements that the set of reductions preserve the property that the graphs are connected simple quadrangulations of the sphere, and that each reduction is one that provides a specific minor inclusion. As in [3], we adhere to these two essential requirements in this paper. For reductions and minors of hypergraphs the reader is referred to [17]. The notations and terminologies, except those explicitly declared otherwise, follow those of [4, 8].

For a hypergraphs $G = (V, E)$, the degree of a vertex $v \in V$ is defined to be

$$d(v) = |\{e \in E : v \in e\}|.$$

For an integer $k \geq 0$, denote

$$V_k = \{v \in V : d(v) = k\}.$$

Let G, H be hypergraphs. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a subhypergraph of G and this is denoted $H \subseteq G$. Here, again the symbol $\subseteq$ is used in two different contexts: first as subset inclusion and then as subhypergraph inclusion.

For a subhypergraph $H \subseteq G$, denote

$$d_H(v) = |\{e \in E(H) : v \in e\}|, \ V_k(H) = \{v \in V(H) : d_H(v) = k\}.$$

For an edge $e \in E(G)$, $\{e\}$ is a subhypergraph of G and hence we write $V_k(e) = \{v \in e : d_G(v) = k\}$. For a hypergraph G, the number of vertices $|G| = |V(G)|$ is usually called the order, and the number of edges $\|G\| = |E(G)|$ is usually called the size of G.

2. Contractions and minors in hypergraphs

In an axiomatic treatment of cycles and acyclic hypergraphs in [23], the concept of Graham reductions is used in an essential way. A Graham reduction of a hypergraph consists of repeatedly performing either (1) deletion of a vertex of degree 1; or (2) removal of an edge that is a proper subset of another. It will be seen in this paper that Graham reduction is also fundamental in our treatment of uniform hypergraphs. A Graham reduction is sometimes also called an ear reduction. If the two options of a Graham reduction are performed simultaneously, then the reduction is called a combined Graham reduction. More explicitly, for $e \in E(G)$, a combined Graham reduction of hypergraph G gives rise to hypergraph $H = G - e - V_1(e)$. The correspondence $\rho : G \to H$ is the combined Graham reduction.

A path in a hypergraph $G = (V, E)$ is defined to be a subhypergraph $P \subseteq G$ with

$$V(P) = \{v_1, v_2, \ldots, v_r\}, \quad E(P) = \{E_1, E_2, \ldots, E_{r-1}\}$$

such that for each $i = 1, 2, \ldots, r-1$, $v_i, v_{i+1} \in E_i$. Since $V(P)$ and $E(P)$ are given in set notations, their elements are distinct. The path P is said to connect vertices v_1 and v_r in G and the vertices v_1 and v_r are said to be end vertices of P and are connected by P in G. The edges E_1 and E_{r-1} are called the end edges of P. If each pair of vertices of G are connected by a path, then G is said to be connected. For $S \subseteq V$, the subhypergraph $G|_S$ induced by S is given by

$$G|_S = (S, 2^S \cap E).$$

A partition $\{V_1, \ldots, V_s\}$ of V is called a contraction, if for each $i = 1, \ldots, s$, $G|_{V_i}$ is connected. An automorphism is an extreme example of a contraction since it is a permutation of the trivial partition of $V(G)$ into single vertices. Another extreme example is the contraction of a connected hypergraph into the hypergraph with a single vertex, since if G is connected, then for the partition $\{V(G)\}$, $G|_{V(G)} = G$ is connected.

If a partition $\{V_1, \ldots, V_s\}$ is a contraction of a hypergraph G, then let

$$\begin{aligned} &V(H) = \{V_1, \ldots, V_s\}, \\ &E(H) = \{\{V_{i_1}, \ldots, V_{i_m}\} : \forall j \in \{1, \ldots, m\},\ \exists x_{i_j} \in V_{i_j}, \{x_{i_1}, \ldots, x_{i_m}\} \in E(G)\}. \end{aligned}$$

This defines a hypergraph H and a surjective mapping $f : G \to H$, specified by $f_V(x) = V_i$ if $x \in V_i$ and

$$f_E(e) = \begin{cases} V_i, & e \subseteq V_i \\ f_V(V(e)), & e \not\subseteq V_i,\ i = 1, \ldots, s. \end{cases}$$

Let χ_X denote the usual binary indicator variable for the event X. Then in a more compact expression,

$$f_E(e) = \chi_{e \subseteq V_i} V_i + [1 - \chi_{e \subseteq V_i}]\, f_V(V(e)).$$

Such a mapping may be called an egamorphism, as the image of an edge is an edge or a vertex. A contraction is therefore a preconnected egamorphism (an egamorphism for which the preimage of each vertex of H induces a connected subhypergraph in G).

A hypergraph H is a minor of G, denoted $H \leq G$, if there is a subhypergraph $K \subseteq G$ and a contraction $f : K \to H$. Clearly, $G \leq G$ for any hypergraph G (reflexivity) and $J \leq H$, $H \leq G \Rightarrow J \leq G$ (transitivity) both hold. In this paper, the main use of the symbol $\leq$ is that for minor inclusion. Where a partial order or a quasi order is denoted by $\leq$, it also refers exclusively to the minor inclusion of graphs or hypergraphs. Thus there is no great over-use of this symbol.

A reflexive and transitive binary relation is called a quasi ordering. A quasi ordering $\leq$ on a set S is a well quasi ordering if every descending chain is finite and every antichain is finite. Elements of a set with a well quasi ordering are called well quasi ordered. It may be proved easily that if a set S has a well quasi ordering, then the set of finite sequences of S is well quasi ordered [16].

A partially ordered set (or poset) $P = (S, \leq)$ is a set S with an antisymmetric and transitive binary relation $\leq$. Elements $x, y \in S$ are comparable if $x \leq y$ or $y \leq x$. If $x \leq y$ and $x \neq y$, then write $x < y$. If $x < y$ and there is no z with $x < z < y$, then y covers x, and it will be written $x \lhd y$. If there is a unique element $z \in S$ such that $z \leq x$ for all $x \in S$, then z is called the least element of P. An element $x \in S$ with no $y \in S$ such that $y < x$ is called a minimal element of P, and $x \in S$ with no $y \in S$ such that $x < y$ is called a maximal element. If every two elements of a subset $T \subseteq S$ are comparable, then T is said to form a chain. If no two elements of a subsets $T \subseteq S$ are comparable, then T is an antichain. A descending chain is:

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots$$

A partially ordered set is said to be well founded if every strictly descending chain is finite (this is the well known Jordan-Dedekind descending chain condition). A well founded partial order is also abbreviated as a well founded order.

In a partial order, if the antisymmetry is replaced by the weaker condition $a \leq b,\ b \leq a \Rightarrow a \sim b$, where $\sim$ is an equivalence relation on S, then the condition is called a weak antisymmetry and the order is called a weak partial order. If a weak partial order satisfies the Jordan-Dedekind condition that every strictly descending chain is finite, then it is called a weak well founded order.

For finite hypergraphs G and H, if $H \leq G$ and $G \leq H$, then $G \simeq H$. Hence for hypergraphs the equivalence $\sim$ will be isomorphism. Isomorphic hypergraphs are regarded as equal when the properties considered are invariant under isomorphisms.

3. Connected uniform hypergraphs

In this section a structure theorem for the family of simple connected uniform hypergraphs will be observed. After making this observation, critical hypergraphs will be determined under single edge removal or single edge contration.

Theorem 3.1 *Let $r \geq 2$ be an integer and let G be a connected simple r-uniform hypergraph. If $\|G\| \geq 2$, then there exists $e \in E(G)$ such that $G - e - V_1(e)$ is a connected simple r-uniform hypergraph.*

Proof Suppose that G is a connected simple r-uniform hypergraph with $\|G\| \geq 2$ where $r \geq 2$ is an integer. Let P be the longest path in G and let e be an end edge of P. Then we claim that $G - e - V_1(e)$ is connected. Suppose that $G - e - V_1(e)$ is not connected. Then there exists at least a component Q different from the component of $G - e - V_1(e)$ containing $P - e$. That is there is an edge $f \notin E(P)$ such that $e \cap f \neq \emptyset$. But then $P \cup f$ is a path with $\|P \cup f\| > \|P\|$, contradicting the choice of P. $\square$

For $r = 2$, this result reduces to the trivial observation that if G is a connected simple graph with $\|G\| \geq 1$, then there exists a vertex $x \in V(G)$ such that $G - x$ is connected and simple.

Corollary 3.2 *Let $r \geq 2$ be an integer and G be a connected simple r-uniform hypergraph. Then there exists a sequence*

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_s$$

of connected simple r-uniform hypergraphs such that $G_0 = K_1$, $G_s = G$ and for each $i \geq 1$, there exists $e \in E(G_i)$ with

$$G_{i-1} = G_i - e - V_1(e).$$

Theorem 3.1 implies the existence of a reasonably large proper minor in a connected simple r-uniform hypergraph with size at least 2, that is also a connected simple r-uniform hypergraph.

Corollary 3.3 *Let $r \geq 2$ be an integer and G be a connected simple r-uniform hypergraph with $\|G\| \geq 2$. Then there exists a connected simple r-uniform hypergraph H with $H \subseteq G$ and*

$$|G| - r + 1 \leq |H| \leq |G| - 1, \ \|H\| = \|G\| - 1.$$

Proof The proof follows since subgraph inclusion is a minor inclusion. The inequality is true since at least one and at most r vertices are being deleted and exactly one edge is being removed in obtaining H from G by the combined Graham reduction. □

4. Critical and minimal hypergraphs

Let S be a set and R be a binary relation on S, that is $R \subseteq S \times S$. Then this gives a graph G with

$$V(G) = S, \ E(G) = R.$$

If the binary relation R is a given transformation, the graph G is usually called a transformation graph. It may easily be seen from this definition that the concept of a transformation graph is as general as that of a graph. Examples abound of which only a few will be mentioned here: (1) if G is a graph, then the line graph $L(G)$ is a transformation graph, with $V(L(G)) = E(G)$ and for $x, y \in V(L(G))$, $xy \in E(L(G))$ if the edges x and y have a common end vertex in G; (2) if G is a hypergraph, then its corresponding bipartite incidence graph H is the one with $V(H) = V(G) \cup E(G)$ and

$$E(H) = \{xy : x \in V(G), \, y \in E(G), x \in y\}.$$

(3) Let S be a set of nonisomorphic finite groups and for groups x and y, $(x, y) \in R$ if there exists a homomorphism $f : x \to y$. This transformation graph (a complete graph) may as well be called the homomorphism graph of these groups. Of course if S is the set of all finite groups, then this graph is isomorphic to the category of all finite groups.

The aim of this section is to determine critical and minimal connected simple r-uniform hypergraphs. A connected simple r-uniform hypergraph G is called contraction critical if for each $e \in E(G)$, G/e is not a connected, simple and r-uniform hypergraph. Some illustrative examples of simple, connected and 3-uniform contraction critical hypergraphs are shown in Figure 1 using their bipartite incidence graphs. In this figure, the black vertices denote edges.

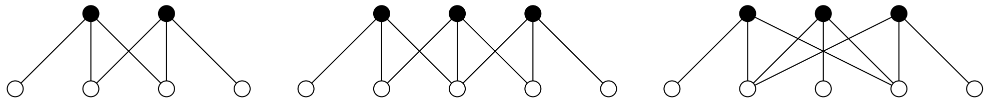

Figure 1 Examples of 3-uniform contraction critical hypergraphs

Let G be a simple, connected and r-uniform hypergraph with $\|G\| \geq 2$. If for $e \in E(G)$ the hypergraph $H = G - e - V_1(e)$ is connected and has the property that $|V(e) \cap V(H)| \geq 2$, then G/e is not simple and r-uniform. It is clear that G/e is connected. We observe that the converse is also true.

Lemma 4.1 *Let $r \geq 2$ be an integer and G be a connected simple r-uniform hypergraph with $\|G\| \geq 2$. Then G is contraction critical if and only if for each $e \in E(G)$ with $H = G - e - V_1(e)$, $|V(e) \cap V(H)| \geq 2$.*

Proof The sufficiency was observed before the statement of the theorem.

For the necessity let G be a simple, connected and r-uniform hypergraph and let $e \in E(G)$ such that G/e is not a simple, connected and r-uniform hypergraph.

It is clear that G/e is connected. Suppose that G/e is not simple. Let $f, h \in E(G/e)$ with $f \subseteq h$ and let $f : G \to G/e$ be the contraction. Since f, h are edges of G/e, $f^{-1}(f), f^{-1}(h) \in E(G)$. But then $f^{-1}(f) \subseteq f^{-1}(h)$ since f is a contraction mapping. This contradicts the condition that G is simple.

Hence G/e is not r-uniform, and hence $|V(e) \cap V(H)| \geq 2$. $\square$

Suppose that for each i with $1 \leq i \leq s$, H_i is a connected simple r-uniform hypergraph with $V_1(e) \neq \emptyset$ for each $e \in E(H_i)$. Let $|e_j| = r$ $(j = 1, 2, \ldots, t)$ be distinct sets satisfying $V_1(e) - e_j \neq \emptyset$ for each $e \in E(H_i)$. Let $\mathbf{T}$ be the graph with

$$V(\mathbf{T}) = \{H_i : 1 \leq i \leq s\} \cup \{e_j : 1 \leq j \leq t\}.$$

Suppose that $\mathbf{T}$ is connected and each $\{e_j\}$ is a separator. Then clearly the hypergraph

$$G = H_1 \cup \cdots \cup H_s \cup e_1 \cup \cdots \cup e_t$$

is a connected simple r-uniform hypergraph such that for each $e \in E(G)$, $G - e$ is not connected. That is, each of these hypergraph is minimal under edge removal. Each edge f of every H_i is not removable since it satisfies $V_1(f) - e_j \neq \emptyset$ for each e_j, and no e_j is removable since it is a separating edge by the definition of $\mathbf{T}$.

The next aim of this section is to establish the converse of this statement. A connected simple r-uniform hypergraph G is minimal under edge removal if for each $e \in E(G)$, $G - e$ is not a connected simple r-uniform hypergraph.

Theorem 4.2 *A connected simple r-uniform hypergraph G with $\|G\| \geq 2$ is minimal under edge removal if and only if*

$$G = H_1 \cup \cdots \cup H_s \cup e_1 \cup \cdots \cup e_t$$

such that for each $i \in \{1, 2, \ldots, s\}$, $j \in \{1, 2, \ldots, t\}$ and each $f \in E(H_i)$, $V_1(f) - e_j \neq \emptyset$ and each e_j is separating.

Proof The sufficiency was proved before the statement of the theorem.

The necessity will be proved by induction on the size of the hypergraphs. The conclusion is clearly true for $\|G\| = 1, 2$. Suppose $\|G\| \geq 3$ and the conclusion of the theorem holds for each connected simple r-uniform hypergraph with size less than that of G.

Suppose that G is a connected simple r-uniform hypergraph minimal under edge removal. Let $e \in E(G)$. Then by the assumption $G - e$ is not a connected simple r-uniform hypergraph. Note that $G - e$ is simple since $G - e \subseteq G$.

Suppose that $G - e$ is not r-uniform. Then $G - e$ has isolated vertices.

Hence suppose that $G - e$ is r-uniform. But then $G - e$ is not connected. Suppose that $H_1, \ldots, H_s$ are the components of $G - e$. Each H_i is a connected simple r-uniform hypergraph. Now for each i, $1 \leq i \leq s$, $\|H_i\| < \|G\|$. By the inductive hypothesis, each

$$H_i = J_{i,1} \cup \cdots \cup J_{i,s_i} \cup e_{i,1} \cup \cdots \cup e_{i,t_i}$$

where $J_{i,k}$ is a connected simple r-uniform hypergraph, $e_{i,l}$ is separating for each l, $1 \leq l \leq t_i$ and for each $f \in E(J_{i,k})$, $V_1(f) - e_{i,l} \neq \emptyset$. Hence G satisfies that condition of the theorem, that is

$$G = H_1 \cup \cdots \cup H_s \cup e_1 \cup \cdots \cup e_t$$

where each H_i is a connected simple r-uniform hypergraph, each e_j is separating and for each $f \in \cup E(H_i)$, $V_1(f) - e_j \neq \emptyset$. □

Note that the union in this theorem is edge disjoint union. More precisely, for $i \neq j$, $E(H_i) \cap E(H_j) = \emptyset$.

5. Contraction critical hypergraphs minimal under edge removal

It is clear that the first hypergraph of Figure 1 is both minimal under edge removal and contraction critical. It is minimal since removal of either of the edges results in isolated vertices and the hypergraph is not connected; it is contraction critical by Lemma 4.1.

Let

$$H_3^1 = \{e_1, e_2, e_3 : |e_1 \cap e_2| \geq 2, |e_1 \cap e_3| \geq 2, |e_2 \cap e_3| \geq 1, V_1(e_i) \neq \emptyset \ (i = 1, 2, 3)\}.$$

Then it may also be shown that H_3^1 is the only contraction critical connected simple r-uniform hypergraph of order 3 minimal under edge removal.

A contraction critical connected simple r-uniform hypergraph minimal under edge removal are characterized by the condition of Lemma 4.1 and the condition of Theorem 4.2. This observation will not be separately stated as a theorem.

As an illustrative example, consider the family of connected simple 3-uniform hypergraphs. Let

$$G_0 = \{\{a, b, c\}\}$$

and, in general, let $a_0, a_1, \ldots, a_n, b, c$ be distinct and let

$$G_n = \{\{a_i, b, c\} : 0 \leq i \leq n\}.$$

Let H be any connected multigraph without loops. A connected simple 3-uniform hypergraph may be constructed by replacing each edge with a G_n for some $n \geq 0$.

Suppose that $h = uv \in E(H)$ is separating. Denote by H_u and H_v the components of $H - h$ containing u and v, respectively. Replace h with G_i, $i \geq 0$ by identifying a vertex of G_i with u and another vertex of G_i with v.

Suppose that $h = uv$ is not a separating edge. Replace h with a G_i, $i \geq 0$ by identifying a vertex of G_i of degree at least 2 with u and another vertex of G_i of degree at least 2 with v.

Let $\mathscr{G}$ be the family of hypergraphs constructed in this manner, in addition to the member G_0. Then each $G \in \mathscr{G}$ is a connected simple 3-uniform hypergraph that is contraction critical and minimal under edge removal. The theorem guarantees that if G is any connected simple 3-uniform hypergraph that is contraction critical and minimal under edge removal, then $G \in \mathscr{G}$.

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