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The Constructions for Large Sets and Overlarge Sets of Resolvable Hybrid Triple Systems

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Abstract An LRHTS(v) (or LARHTS(v)) is a collection of $\{(X,\mathcal{B}_i): 1 \le i \le 4(v-2)\}$, where X is a v-set, each $(X,\mathcal{B}_i)$ is a resolvable (or almost resolvable) HTS(v), and all $\mathcal{B}_i$s form a partition of all cycle triples and transitive triples on X. An OLRHTS(v) (or OLARHTS(v)) is a collection $\{(Y\backslash\{y\},\mathcal{A}_y^j): y \in Y, j = 0,1,2,3\}$, where Y is a $(v+1)$-set, each $(Y\backslash\{y\},\mathcal{A}_y^j)$ is a resolvable (or almost resolvable) HTS(v), and all $\mathcal{A}_y^j$s form a partition of all cycle and transitive triples on Y. In this paper, we establish some directed and recursive constructions for LRHTS(v), LARHTS(v), OLRHTS(v), OLARHTS(v) and give some new results.

Keywords Hybrid triple system; large set; overlarge set; parallel class; almost parallel classes

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1. Introduction

Let X be a finite set. In what follows, an ordered pair of X will always be an ordered pair (x,y), where $x \ne y \in X$. A cycle triple on X is a set of three ordered pairs $(x,y),(y,z)$ and (z,x) of X, which is denoted by $\langle x,y,z\rangle$ (or $\langle y,z,x\rangle$, or $\langle z,x,y\rangle$). A transitive triple on X is a set of three ordered pairs $(x,y),(y,z)$ and (x,z) of X, which is denoted by (x,y,z). Usually, a cycle triple or a transitive triple is called an oriented triple.

An oriented triple system of order v is a pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a collection of oriented triple, such that every ordered pair of X occurs in exactly one block of $\mathcal{B}$. If the triples in $\mathcal{B}$ are all cycle, then $(X,\mathcal{B})$ is called a Mendelsohn triple system and denoted by MTS(v). If the triples in $\mathcal{B}$ are all transitive, then $(X,\mathcal{B})$ is called a Directed triple system and denoted by DTS(v). If the triples in $\mathcal{B}$ are cycle and transitive, then $(X,\mathcal{B})$ is called a Hybrid triple system and denoted by HTS(v).

An oriented triple system is resolvable (or almost resolvable), if its block set $\mathcal{B}$ can be partitioned into parallel classes (or almost parallel classes), i.e., a partition of X (or a partition of $X\backslash\{x\}$, where $x \in X$). It is easy to see that a resolvable MTS(v) (or DTS(v), or HTS(v)), denoted by RMTS(v) (or RDTS(v), or RHTS(v)) contains $v-1$ parallel classes. An almost resolvable

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MTS(v) (or DTS(v), or HTS(v)), denoted by ARMTS(v) (or ARDTS(v), or ARHTS(v)) contains v almost parallel classes.

Two oriented triple systems $(X,\mathcal{A})$ and $(X,\mathcal{B})$ on the same set are called disjoint if $\mathcal{A}\cap\mathcal{B}=\emptyset$.

A large set of Mendelsohn (or Directed, or Hybrid) triple system of order v, denoted by LMTS(v) (or LDTS(v), or LHTS(v)), is a collection of $\{(X,\mathcal{B}_i)\}_i$, where every $(X,\mathcal{B}_i)$ is an MTS(v) (or DTS(v), or HTS(v)), and all $\mathcal{B}_i$s form a partition of all cycle triples (or transitive triples, or cycle triples and transitive triples) on X. It is easy to see that an LMTS(v) (or LDTS(v), or LHTS(v)) contains $v-2$ (or $3(v-2)$, or $4(v-2)$) MTS(v)s (or DTS(v)s, or HTS(v)s).

The existence results for the three types of large sets of oriented triple system have been finally completed by Kang, Lei and Chang [1–3].

Lemma 1.1 *(1) There exists an LMTS(v) if and only if* $v\equiv 0,1 \bmod 3$, $v\geq 3$ *and* $v\neq 6$;

(2) There exists an LDTS(v) if and only if $v\equiv 0,1 \bmod 3$, $v\geq 3$;

(3) There exists an LHTS(v) if and only if $v\equiv 0,1 \bmod 3$, $v\geq 4$.

An LRMTS(v) (or LRDTS(v), or LRHTS(v)), is an LMTS(v) (or LDTS(v), or LHTS(v)), where every MTS(v) (or DTS(v), or HTS(v)) is resolvable. Similarly, we can define LARMTS(v) (or LARDTS(v), or LARHTS(v)).

So far, the existence of large sets of resolvable (or almost resolvable) oriented triple system has not yet been completed. Many researchers have investigated the problem [4–11]. We can list the known results as follows:

Lemma 1.2 *(1) There exist an LRMTS(v) and an LRDTS(v), when*

$v=3^k m$, *where* $k\geq 1$ *and*

$m\in\{1,4,5,7,11,13,17,23,25,35,37,41,43,47,53,55,57,61,65,67,91,123\}$;

$v=7^k+2,\ 13^k+2,\ 25^k+2,\ 2^{4k}+2$ *and* $2^{6k}+2$, *where* $k\geq 0$;

$v=12(t+1)$, *where* $t\in\{0,1,2,3,4,6,7,8,9,14,16,18,20,22,24,28,32\}$;

$v=3(2t+1)$, *where* $t\in\{35,38,46,47,48,51,56,60\}$;

$v=3^k\prod_{q,r}(2q^r+1)\prod_{s}(4^s-1)$, *where* $k,r,s\geq 1, q=12t+7$ $(t\geq 0)$ *is a prime power.*

(2) There exists an LRHTS(v), when

$v=3^a5^bm\prod_{i=1}^{r}(2\cdot 13^{n_i}+1)\prod_{j=1}^{p}(2\cdot 7^{m_j}+1)$, *where* $m\in\{1,4,11,17,35,43,67,91,123\}\cup$ $\{2^{2l+1}25^s+1: l\geq 0, s\geq 0\}, a, n_i, m_j\geq 1$ $(1\leq i\leq r, 1\leq j\leq p), b, r, p\geq 2$, $b\geq 1$ *and* $m\neq 1$.

$v=3^k\prod_{q,r}(2q^r+1)\prod_{s}(4^s-1), k,r,s\geq 1, q\equiv 7 \bmod 12$ *is a prime power.*

(3) There exist an LARMTS(v) and an LARDTS(v), when

$$v = 4^n, 2(7^n + 1), 2(31^n + 1) \text{ and } 2(127^n + 1), n \geq 1.$$

An overlarge set of Mendelsohn (or Directed, or Hybrid) triple system of order v, denoted by OLMTS(v) (or OLDTS(v), or OLHTS(v)), is a collection $\{(Y\backslash\{y\}, \mathcal{A}_i)\}_i$, where Y is a $(v+1)$-set, each $y \in Y$, $(Y\backslash\{y\}, \mathcal{A}_i)$ is an MTS(v) (or DTS(v), or HTS(v)) and all $\mathcal{A}_i$s form a partition of all cycle triples (or transitive triples, or cycle triples and transitive triples) on Y. It is easy to see that an OLMTS(v) (or OLDTS(v), or OLHTS(v)) contains $v+1$ (or $3(v+1)$, or $4(v+1)$) MTS(v)s (or DTS(v)s, or HTS(v)s).

Recently the existence of spectrum for the three types of overlarge sets of oriented triple system has been finally completed by Ji, Tian and Cheng [12–14].

Lemma 1.3 *(1) There exists an OLMTS(v) if and only if $v \equiv 0, 1 \bmod 3$, $v \geq 3$ and $v \neq 6$;*

(2) There exists an OLDTS(v) if and only if $v \equiv 0, 1 \bmod 3$, $v \geq 3$;

(3) There exists an OLHTS(v) if and only if $v \equiv 0, 1 \bmod 3$, $v \geq 4$.

An OLRMTS(v) (or OLRDTS(v), or OLRHTS(v)), is an OLMTS(v) (or OLDTS(v), or OLHTS(v)), where every MTS(v) (or DTS(v), or HTS(v)) is resolvable. Similarly, we can define OLARMTS(v) (or OLARDTS(v), or OLARHTS(v)). So far, the existence of spectrum for resolvable (almost resolvable) overlarge sets of oriented triple system has not been completed. We have known that [4–7]:

Lemma 1.4 *(1) There exist an OLRMTS(v) and an OLRDTS(v), when*

$v = 9,\ 4^k - 1,\ 2 \cdot 13^n + 1, k \geq 0, n \geq 1;$

$v = 2 \cdot q^n + 1,$ *where* $q \equiv 7 \bmod 12, q$ *is a prime power;*

$v = 6u + 3,$ *where* $u = 4^n 25^m,\ m + n \geq 1;$

$v = 3^k \prod_{q,r}(2q^r + 1) \prod_s (4^s - 1),$ *where* $k, r, s \geq 1, q = 12t + 7\ (t \geq 0), q$ *is a prime power;*

(2) There exist an OLARMTS(v) and an OLARDTS(v), when

$$v = 10, 4^n, 7^n, 13^n, 25^n, 25 \cdot 4^n, n \geq 1.$$

(3) There exists an OLRHTS(v), where

$v = 9, 4^n - 1, 2 \cdot 7^n + 1, 2 \cdot 13^n + 1, \cdot 31^n + 1, 2 \cdot 127^n + 1, n \geq 1;$

$v = 6u + 3,$ *where* $u = 4^n 25^m,\ m + n \geq 1;$

$v = 3^k \prod_{q,r}(2q^r + 1) \prod_s (4^s - 1),$ *where* $k, r, s \geq 1, q = 12t + 7\ (t \geq 0), q$ *is a prime power.*

2. Some small constructions

Let $Z_v = \{0, 1, \ldots, v-1\}$ be an additive group of residues modulo v. For an oriented triple $\langle x, y, z\rangle$ (or (x, y, z)), $\langle z, y, x\rangle$ (or (z, y, x)) is called its reverse.

Theorem 2.1 *There exists an LARHTS*(4).

Proof The desired LARHTS(4) $= \{(Z_4, \mathcal{B}_i) : 0 \le i \le 7\}$, where

$\mathcal{B}_i = \{\langle i, 1+i, 3+i\rangle, (2+i, i, 3+i), (3+i, 2+i, 1+i), (1+i, i, 2+i)\}, 0 \le i \le 3.$

$\mathcal{B}_i$ is constituted of the reverse blocks of $\mathcal{B}_{i-4}, 4 \le i \le 7$.

It is easy to see that each $\mathcal{B}_i$ is almost resolvable, and every block is an almost parallel classes. □

Theorem 2.2 *There exists an OLRHTS*(9).

Proof The desired OLRHTS(9) $= \{(Z_{10}\backslash\{x\}, \mathcal{A}_x^r) :\ x \in Z_{10},\ r = 0,1,2,3\}$, where

$\mathcal{A}_0^0$:	(1 2 4)	(4 7 5)	(5 8 4)	(3 6 7)	(7 4 3)	(5 2 6)	(8 2 3)	⟨2 8 7⟩
	(6 3 5)	(8 6 9)	⟨7 6 2⟩	(4 2 9)	(1 6 8)	(3 4 8)	(9 6 4)	(4 6 1)
	(9 7 8)	(3 2 1)	(1 9 3)	(8 5 1)	(9 2 5)	(7 9 1)	(1 5 7)	(5 3 9)
$\mathcal{A}_0^1$:	⟨2 4 1⟩	(5 4 7)	(8 4 5)	(6 7 3)	(3 7 4)	(6 5 2)	(2 3 8)	(8 7 2)
	(3 5 6)	(6 9 8)	(2 7 6)	(9 4 2)	(8 1 6)	(4 8 3)	(4 9 6)	(6 1 4)
	(7 8 9)	⟨1 3 2⟩	(3 1 9)	(5 1 8)	(2 5 9)	(9 1 7)	(7 1 5)	(9 5 3)
$\mathcal{A}_0^2$:	(4 1 2)	⟨7 5 4⟩	(4 5 8)	(7 3 6)	⟨4 3 7⟩	(2 6 5)	(3 8 2)	(7 2 8)
	(5 6 3)	(9 8 6)	(6 2 7)	(2 9 4)	(6 8 1)	(8 3 4)	(6 4 9)	(1 4 6)
	(8 9 7)	(2 1 3)	(9 3 1)	(1 8 5)	(5 9 2)	(1 7 9)	(5 7 1)	(3 9 5)
$\mathcal{A}_0^3$:	(2 4 1)	(7 5 4)	⟨8 4 5⟩	⟨6 7 3⟩	(4 3 7)	⟨6 5 2⟩	⟨2 3 8⟩	(2 8 7)
	⟨3 5 6⟩	⟨6 9 8⟩	(7 6 2)	⟨9 4 2⟩	⟨8 1 6⟩	⟨4 8 3⟩	⟨4 9 6⟩	⟨6 1 4⟩
	⟨7 8 9⟩	(1 3 2)	⟨3 1 9⟩	⟨5 1 8⟩	⟨2 5 9⟩	⟨9 1 7⟩	⟨7 1 5⟩	⟨9 5 3⟩

and $\mathcal{A}_x^r = \mathcal{A}_0^r + x,\ x \in Z_{10},\ r = 0,1,2,3$. □

Theorem 2.3 *There exists an OLARHTS*(v), *when* $v = 4, 7, 10$.

Proof (1) OLARDTS(4) $= \{(Z_5\backslash\{x\}, \mathcal{A}_x^r) :\ x \in Z_5, r = 0,1,2,3\}$, where

$$\mathcal{A}_0^0 = \{\langle 1,2,3\rangle, (1,4,3), (2,4,1), (3,4,2)\};$$
$$\mathcal{A}_0^1 = \{\langle 1,3,4\rangle, (3,2,1), (1,2,4), (4,2,3))\};$$
$$\mathcal{A}_0^2 = \{\langle 4,2,1\rangle, (2,3,4), (1,3,2), (4,3,1)\};$$
$$\mathcal{A}_0^3 = \{\langle 3,2,4\rangle, (2,1,3), (4,1,2), (3,1,4)\};$$
$$\mathcal{A}_x^r = \mathcal{A}_0^r + x, x \in Z_5, r = 0,1,2,3.$$

It is easy to see that every $\mathcal{A}_x^r$ is almost resolvable, and every block is an almost parallel classes.

(2) OLARHTS(7) $= \{(Z_8\backslash\{x\}, \mathcal{A}_x^r) :\ x \in Z_8, r = 0,1,2,3\}$, where

$\mathcal{A}_0^0$:	⟨2 5 6⟩	(4 6 1)	(7 1 6)	(2 3 1)	(1 7 2)	(5 1 3)	(1 5 4)
	(4 7 3)	(3 7 5)	(4 5 2)	(6 5 7)	(3 6 4)	(2 7 4)	⟨6 3 2⟩

$\mathcal{A}_0^1$:	(5 6 2)	⟨6 1 4⟩	(1 6 7)	(3 1 2)	(7 2 1)	(1 3 5)	(5 4 1)
	(7 3 4)	(5 3 7)	(2 4 5)	(7 6 5)	⟨4 3 6⟩	(4 2 7)	(2 6 3)
$\mathcal{A}_0^2$:	(6 2 5)	(1 4 6)	(6 7 1)	(1 2 3)	(2 1 7)	(3 5 1)	(4 1 5)
	⟨3 4 7⟩	⟨7 5 3⟩	(5 2 4)	(5 7 6)	(6 4 3)	(7 4 2)	(3 2 6)
$\mathcal{A}_0^3$:	(2 5 6)	(6 1 4)	⟨6 7 1⟩	⟨1 2 3⟩	⟨2 1 7⟩	⟨3 5 1⟩	⟨4 1 5⟩
	(3 4 7)	(7 5 3)	⟨5 2 4⟩	⟨5 7 6⟩	(4 3 6)	⟨7 4 2⟩	(6 3 2)

and $\mathcal{A}_x^r = \mathcal{A}_0^r + x, x \in Z_8, r = 0, 1, 2, 3$.

(3) OLARHTS(10) $= \{(Z_{11}\backslash\{x\}, \mathcal{A}_x^r) : \ x \in Z_{11}, \ r = 0, 1, 2, 3\}$, where

$\mathcal{A}_{10}^0$:	⟨7 1 9⟩	(0 4 5)	(5 7 0)	(6 0 9)	(8 1 0)	(4 0 6)	(0 7 8)	(0 1 3)	(9 0 2)	(3 2 0)
	(8 4 2)	(6 2 7)	(9 1 6)	(5 1 8)	(2 5 9)	(2 3 1)	(1 5 2)	(9 5 4)	(6 1 4)	(4 1 7)
	(5 6 3)	(8 9 3)	(3 4 8)	(7 2 4)	(7 3 6)	⟨9 8 7⟩	(4 3 9)	(2 6 8)	(3 7 5)	(8 6 5)
$\mathcal{A}_{10}^1$:	(1 9 7)	⟨4 5 0⟩	(7 0 5)	(9 6 0)	(0 8 1)	⟨0 6 4⟩	(8 0 7)	(1 3 0)	(0 2 9)	(2 0 3)
	(4 2 8)	(7 6 2)	(6 9 1)	(1 8 5)	(9 2 5)	(3 1 2)	(5 2 1)	(5 4 9)	(1 4 6)	(7 4 1)
	(3 5 6)	(3 8 9)	(8 3 4)	(2 4 7)	(6 7 3)	(7 9 8)	(9 4 3)	(8 2 6)	(5 3 7)	(6 5 8)
$\mathcal{A}_{10}^2$:	(9 7 1)	(5 0 4)	⟨0 5 7⟩	(0 9 6)	(1 0 8)	(6 4 0)	⟨7 8 0⟩	(3 0 1)	(2 9 0)	(0 3 2)
	(2 8 4)	(2 7 6)	(1 6 9)	(8 5 1)	(5 9 2)	(1 2 3)	(2 1 5)	(4 9 5)	(4 6 1)	(1 7 4)
	(6 3 5)	(9 3 8)	(4 8 3)	(4 7 2)	(3 6 7)	(8 7 9)	(3 9 4)	(6 8 2)	(7 5 3)	(5 8 6)
$\mathcal{A}_{10}^3$:	(7 1 9)	(4 5 0)	(0 5 7)	⟨0 9 6⟩	⟨1 0 8⟩	(0 6 4)	(7 8 0)	⟨3 0 1⟩	⟨2 9 0⟩	⟨0 3 2⟩
	⟨2 8 4⟩	⟨2 7 6⟩	⟨1 6 9⟩	⟨8 5 1⟩	⟨5 9 2⟩	⟨1 2 3⟩	⟨2 1 5⟩	4 9 5⟩	⟨4 6 1⟩	⟨1 7 4⟩
	⟨6 3 5⟩	⟨9 3 8⟩	⟨4 8 3⟩	⟨4 7 2⟩	⟨3 6 7⟩	(9 8 7)	⟨3 9 4⟩	⟨6 8 2⟩	⟨7 5 3⟩	⟨5 8 6⟩

and $\mathcal{A}_x^r = \mathcal{A}_{10}^r + x + 1, x \in Z_{11}, r = 0, 1, 2, 3$. □

From the above examples we can find that, if a transitive triple (a, b, c) was replaced by a cycle triple $\langle a, b, c\rangle$, then the $\mathcal{A}_x^r$ $(r = 0, 1, 2, 3)$ are the same. An MTS(v) can produce four HTS(v)s. The relation between an MTS(v) and an HTS(v) was discussed in [15].

For an MTS$(v) = (X, \mathcal{B})$, we denote the cycle triples which can form some subsystems MTS(3) as $\overline{\mathcal{B}}$. We define a block-incident graph $G(\mathcal{B})$, where the vertex set is $\mathcal{B}\backslash\overline{\mathcal{B}}$, and the vertices B and B' are joint if and only if there are two common elements in B and B'. Evidently, $G(\mathcal{B})$ is a 3-regular graph. Obviously, if a 2-factor of the block-incident graph $G(\mathcal{B})$ consists of some disjoint cycles with even length no less than 4, then $G(\mathcal{B})$ is 3-edge-chromatic. In this paper, a graph G is k-edge-chromatic, if the edges were coloured by k colours, such that the two adjacent edges have different colour. We summarize the above arguments in the following lemma.

Lemma 2.4 ([15]) *If the block-incident graph of MTS(v) is 3-edge-chromatic, then there exist four pairwise disjoint HTS(v)s, $v \geq 6$.*

Lemma 2.5 ([15]) *If there is a large set or overlarge set Ω of resolvable (or almost resolvable) MTS(v), and the block-incident graph of small set which constitutes Ω is 3-edge-chromatic, then there is a large set or overlarge set of resolvable (or almost resolvable) HTS(v).*

Lemma 2.6([15]) *If a large set or overlarge set Ω of resolvable (or almost resolvable) MTS(v) is generated by one or few MTS(v) under automorphism group, and the block-incident graph of every MTS(v) is 3-edge-chromatic, then there is a large set or overlarge set of resolvable (or almost resolvable) HTS(v).*

Next, we will use the relation between MTS(v) and HTS(v) to construct a large set or overlarge set of resolvable (or almost resolvable) HTS(v). For every $\{\langle x,y,z\rangle,\langle z,y,x\rangle\}\subseteq\overline{\mathcal{B}}$, define $\{\langle x,y,z\rangle,\ \langle z,y,x\rangle\}$, $\{(x,y,z),\ (z,y,x)\}$, $\{(y,z,x),(x,z,y)\}$, $\{(z,x,y),(y,x,z)\}$ belong to one of the four HTS(v)s, respectively.

Theorem 2.7 *There exists an OLARHTS*(13).

Proof From [6], there is an OLARMTS(13) on $Y=(Z_4\times Z_3)\cup\{\infty_1,\infty_2\}$, it contains 14 ARMTS(13)s, Ω_{∞_1}, Ω_{∞_2} and $\Omega_{x,t}$ ($x\in Z_4, t\in Z_3$). For every ARMTS(13), we give its 13 almost parallel classes which contains four blocks by row. For convenience, we denote $(u,v,w)=(3-x,2+x,1-x)$, $(p,q,r)=(1-x,2+x,3-x)$, where $x\in Z_4$.

$$\begin{aligned}
\Omega_{\infty_1}:&\{\langle(x,0),(x,2),(x,1)\rangle : x\in Z_4\};\\
&\{\langle\infty_2,(x,1),(x,2)\rangle\}\cup\{\langle(u,i),(v,i),(w,i+1)\rangle : i\in Z_3\}, x\in Z_4;\\
&\{\langle\infty_2,(x,2),(x,0)\rangle\}\cup\{\langle(u,i),(v,i+1),(w,i)\rangle : i\in Z_3\}, x\in Z_4;\\
&\{\langle\infty_2,(x,0),(x,1)\rangle\}\cup\{\langle(u,i+1),(v,i),(w,i)\rangle : i\in Z_3\}, x\in Z_4.\\
\Omega_{\infty_2}:&\{\langle(x,0),(x,1),(x,2)\rangle : x\in Z_4\};\\
&\{\langle\infty_1,(x,2),(x,1)\rangle\}\cup\{\langle(u,i),(v,i),(w,i-1)\rangle : i\in Z_3\}, x\in Z_4;\\
&\{\langle\infty_1,(x,0),(x,2)\rangle\}\cup\{\langle(u,i),(v,i-1),(w,i)\rangle : i\in Z_3\}, x\in Z_4;\\
&\{\langle\infty_1,(x,1),(x,0)\rangle\}\cup\{\langle(u,i-1),(v,i),(w,i)\rangle : i\in Z_3\}, x\in Z_4.\\
\Omega_{x,0}:&\{\langle\infty_2,(x,2),(x,1)\rangle\}\cup\{\langle(u,i),(v,i+2),(w,i+1)\rangle : i\in Z_3\};\\
&\{\langle\infty_1,(x,1),(x,2)\rangle\}\cup\{\langle(p,i),(q,i),(r,i)\rangle : i\in Z_3\};\\
&\{\langle\infty_2,\infty_1,(x,2)\rangle\}\cup\{\langle(p,i),(q,i+1),(r,i-1)\rangle : i\in Z_3\};\\
&\{\langle\infty_1,\infty_2,(x,1)\rangle\}\cup\{\langle(p,i),(q,i-1),(r,i+1)\rangle : i\in Z_3\};\\
&\{\langle\infty_1,(w,i),(u,i)\rangle,\langle\infty_2,(u,i-1),(v,i)\rangle,\\
&\qquad\langle(x,1),(w,i+1),(w,i-1)\rangle,\langle(x,2),(v,i-1),(v,i+1)\rangle\}, i\in Z_3;\\
&\{\langle\infty_1,(u,i),(v,i)\rangle,\langle\infty_2,(v,i-1),(w,i)\rangle,\\
&\qquad\langle(x,1),(u,i+1),(u,i-1)\rangle,\langle(x,2),(w,i-1),(w,i+1)\rangle\}, i\in Z_3;\\
&\{\langle\infty_1,(v,i),(w,i)\rangle,\langle\infty_2,(w,i-1),(u,i)\rangle,\\
&\qquad\langle(x,1),(v,i+1),(v,i-1)\rangle,\langle(x,2),(u,i-1),(u,i+1)\rangle\}, i\in Z_3.\\
\Omega_{x,1}:&\{\langle\infty_2,(x,0),(x,2)\rangle\}\cup\{\langle(u,i),(v,i+1),(w,i-1)\rangle : i\in Z_3\};\\
&\{\langle\infty_1,(x,2),(x,0)\rangle\}\cup\{\langle(p,i),(q,i),(r,i+1)\rangle : i\in Z_3\};\\
&\{\langle\infty_2,\infty_1,(x,0)\rangle\}\cup\{\langle(p,i),(q,i+1),(r,i)\rangle : i\in Z_3\};\\
&\{\langle\infty_1,\infty_2,(x,2)\rangle\}\cup\{\langle(p,i+1),(q,i),(r,i)\rangle : i\in Z_3\};\\
&\{\langle\infty_1,(w,i+1),(u,i)\rangle,\langle\infty_2,(u,i-1),(v,i-1)\rangle,(x,2),(w,i+2),(w,i)\rangle,\\
&\qquad\langle(x,0),(v,i-2),(v,i)\rangle\}, i\in Z_3;
\end{aligned}$$

$$\{\langle\infty_1,(u,i+1),(v,i)\rangle,\langle\infty_2,(v,i-1),(w,i-1)\rangle,$$
$$\langle(x,2),(u,i+2),(u,i)\rangle,\langle(x,0),(w,i-2),(w,i)\rangle\},i\in Z_3;$$
$$\{\langle\infty_1,(v,i+1),(w,i)\rangle,\langle\infty_2,(w,i-1),(u,i-1)\rangle,$$
$$\langle(x,2),(v,i+2),(v,i)\rangle,\langle(x,0),(u,i-2),(u,i)\rangle\},i\in Z_3.$$

$$\Omega_{x,2}:\{\langle\infty_2,(x,1),(x,0)\rangle\}\cup\{\langle(u,i),(v,i),(w,i)\rangle:i\in Z_3\};$$
$$\{\langle\infty_1,(x,0),(x,1)\rangle\}\cup\{\langle(p,i),(q,i),(r,i+2)\rangle:i\in Z_3\};$$
$$\{\langle\infty_2,\infty_1,(x,1)\rangle\}\cup\{\langle(p,i),(q,i+2),(r,i)\rangle:i\in Z_3\};$$
$$\{\langle\infty_1,\infty_2,(x,0)\rangle\}\cup\{\langle(p,i+2),(q,i),(r,i)\rangle:i\in Z_3\};$$
$$\{\langle\infty_1,(w,i-1),(u,i)\rangle,\langle\infty_2,(u,i-1),(v,i+1)\rangle,$$
$$\{\langle(x,0),(w,i),(w,i+1)\rangle,\langle(x,1),(v,i),(v,i-1)\rangle\},i\in Z_3;$$
$$\{\langle\infty_1,(u,i-1),(v,i)\rangle,\langle\infty_2,(v,i-1),(w,i+1)\rangle,$$
$$\langle(x,0),(u,i),(u,i+1)\rangle,\langle(x,1),(w,i),(w,i-1)\rangle\},i\in Z_3;$$
$$\{\langle\infty_1,(v,i-1),(w,i)\rangle,\langle\infty_2,(w,i-1),(u,i+1)\rangle,$$
$$\langle(x,0),(v,i),(v,i+1)\rangle,\langle(x,1),(u,i),(u,i-1)\rangle\},i\in Z_3.$$

To construct an OLARHTS(13), we only need to give a 2-factor of the block-incident graph Ω_{∞_1}, Ω_{∞_2} and $\Omega_{x,t}(t\in Z_3)$. For the cycle triples of the Mendelsohn triple systems, we omit "$\langle$", "$\rangle$" for short.

A 2-factor of Ω_{∞_1} (consisting of 4 cycles with 4-length and 3 cycles with 12-length):

$$((x,0),(x,2),(x,1)-\ \infty_2,(x,1),(x,2)-\ \infty_2,(x,2),(x,0)-\ \infty_2,(x,0),(x,1)),$$
$$x\in Z_4;$$

$$\begin{array}{lll}((0,i),(1,i+1),(2,i)- & (3,i),(2,i),(1,i+1)- & (2,i),(3,i),(0,i+1)-\\ \ (1,i),(0,i+1),(3,i)- & (0,i+1),(1,i),(2,i)- & (3,i+1),(2,i),(1,i)-\\ \ (2,i),(3,i+1),(0,i)- & (1,i),(0,i),(3,i+1)- & (0,i),(1,i),(2,i+1)-\\ \ (3,i),(2,i+1),(1,i)- & (2,i+1),(3,i),(0,i)- & (1,i+1),(0,i),(3,i)),\quad i\in Z_3.\end{array}$$

A 2-factor of Ω_{∞_2} (consisting of 4 cycles with 4-length and 3 cycles with 12-length):

$$((x,0),(x,1),(x,2)-\ \infty_1,(x,2),(x,1)-\ \infty_1,(x,0),(x,2)-\ \infty_1,(x,1),(x,0))$$
$$x\in Z_4;$$

$$\begin{array}{lll}((0,i),(1,i-1),(2,i)- & (3,i),(2,i),(1,i-1)- & (2,i),(3,i),(0,i-1)-\\ \ (1,i),(0,i-1),(3,i)- & (0,i-1),(1,i),(2,i)- & (3,i-1),(2,i),(1,i)-\\ \ (2,i),(3,i-1),(0,i)- & (1,i),(0,i),(3,i-1)- & (0,i),(1,i),(2,i-1)-\\ \ (3,i),(2,i-1),(1,i)- & (2,i-1),(3,i),(0,i)- & (1,i-1),(0,i),(3,i)),\quad i\in Z_3.\end{array}$$

A 2-factor of $\Omega_{x,0}$ (consisting of 3 cycles with 6-length and 7 cycles with 4-length):

$((x,1),(1-x,1),(1-x,2)-$ $(x,2),(1-x,2),(1-x,1)-$ $(x,2),(1-x,0),(1-x,2)-$
$(x,1),(1-x,2),(1-x,0)-$ $(x,1),(1-x,0),(1-x,1)-$ $(x,2),(1-x,1),(1-x,0));$

$((x,1),(3-x,1),(3-x,2)-$ $(x,2),(3-x,2),(3-x,1)-$ $(x,2),(3-x,0),(3-x,2)-$
$(x,1),(3-x,2),(3-x,0)-$ $(x,1),(3-x,0),(3-x,1)-$ $(x,2),(3-x,1),(3-x,0));$

$((x,1),(2+x,1),(2+x,2)-$ $(x,2),(2+x,2),(2+x,1)-$ $(x,2),(2+x,0),(2+x,2)-$
$(x,1),(2+x,2),(2+x,0)-$ $(x,1),(2+x,0),(2+x,1)-$ $(x,2),(2+x,1),(2+x,0));$

$(\infty_2,(x,2),(x,1)-\ \infty_1,(x,1),(x,2)-\ \infty_2,\infty_1,(x,2)-\ \infty_1,\infty_2,(x,1));$

$(\infty_2,(3-x,1),(2+x,2)-\ \infty_2,(2+x,2),(1-x,0)-\ \infty_2,(1-x,0),(3-x,1)-$
$(1-x,0),(2+x,2),(3-x,1));$

$(\infty_2,(3-x,2),(2+x,0)-\ \infty_2,(2+x,0),(1-x,1)-\ \infty_2,(1-x,1),(3-x,2)-$
$(1-x,1),(2+x,0),(3-x,2));$

$(\infty_2,(3-x,0),(2+x,1)\ -\ \infty_2,(2+x,1),(1-x,2)-\ \infty_2,(1-x,2),(3-x,0)-$
$(1-x,2),(2+x,1),(3-x,0));$

$(\infty_1,(1-x,i),(3-x,i)\ -\ \infty_1,(3-x,i),(2+x,i)-\ \infty_1,(2+x,i),(1-x,i)-$
$(1-x,i),(2+x,i),(3-x,i)),\quad i\in Z_3.$

A 2-factor of $\Omega_{x,1}$ (consisting of 3 cycles with 6-length, 1 cycle with 4-length and 1 cycle with 30-length):

$((x,2),(1-x,2),(1-x,0)-$ $(x,2),(1-x,0),(1-x,1)-$ $(x,0),(1-x,1),(1-x,0)-$
$(x,0),(1-x,0),(1-x,2)-$ $(x,0),(1-x,2),(1-x,1)-$ $(x,2),(1-x,1),(1-x,2));$

$((x,2),(3-x,2),(3-x,0)-$ $(x,2),(3-x,0),(3-x,1)-$ $(x,0),(3-x,1),(3-x,0)-$
$(x,0),(3-x,0),(3-x,2)-$ $(x,0),(3-x,2),(3-x,1)-$ $(x,2),(3-x,1),(3-x,2));$

$((x,2),(2+x,2),(2+x,0)-$ $(x,2),(2+x,0),(2+x,1)-$ $(x,0),(2+x,1),(2+x,0)-$
$(x,0),(2+x,0),(2+x,2)-$ $(x,0),(2+x,2),(2+x,1)-$ $(x,2),(2+x,1),(2+x,2));$

$(\infty_2,(x,0),(x,2)-\ \infty_1,(x,2),(x,0)-\ \infty_2,\infty_1,(x,0)-\ \infty_1,\infty_2,(x,2));$

$(\infty_1,(3-x,1),(2+x,0)$	$\infty_1,(1-x,2),(3-x,1)$	$\infty_1,(2+x,0),(1-x,2)$
$(1-x,2),(2+x,0),(3-x,2)$	$\infty_2,(1-x,2),(3-x,2)$	$\infty_2,(3-x,2),(2+x,2)$
$\infty_2,(2+x,2),(1-x,2)$	$(1-x,2),(2+x,2),(3-x,0)$	$\infty_1,(3-x,0),(2+x,2)$
$\infty_1,(1-x,1),(3-x,0)$	$\infty_2,(2+x,2),(1-x,1)$	$(1-x,1),(2+x,2),(3-x,1)$
$\infty_2,(1-x,1),(3-x,1)$	$\infty_2,(2+x,1),(1-x,1$	$\infty_2,(3-x,1),(2+x,1)$
$(1-x,2),(2+x,1),(3-x,1)$	$(3-x,0),(2+x,1),(1-x,2)$	$(1-x,0),(2+x,1),(3-x,0)$
$\infty_2,(1-x,0),(3-x,0)$	$\infty_2,(2+x,0),(1-x,0)$	$\infty_2,(3-x,0),(2+x,0)$
$(1-x,1),(2+x,0),(3-x,0)$	$(3-x,2),(2+x,0),(1-x,1)$	$(1-x,1),(2+x,1),(3-x,2)$
$\infty_1,(3-x,2),(2+x,1)$	$\infty_1,(2+x,1),)(1-x,0)$	$\infty_1,(1-x,0),(3-x,2)$
$(1-x,0),(2+x,2),(3-x,2)$	$(3-x,1),(2+x,2),(1-x,0)$	$(1-x,0),(2+x,0),(3-x,1)).$

A 2-factor of $\Omega_{x,2}$ (consisting of 3 cycles with 6-length, 1 cycle with 4-length and 1 cycle with 30-length):

$$((x,0),(1-x,0),(1-x,1)- \quad (x,0),(1-x,1),(1-x,2)- \quad (x,0),(1-x,2),(1-x,0)-$$
$$(x,1),(1-x,0),(1-x,2)- \quad (x,1),(1-x,2),(1-x,1)- \quad (x,1),(1-x,1),(1-x,0));$$

$$((x,0),(3-x,0),(3-x,1)- \quad (x,0),(3-x,1),(3-x,2)- \quad (x,0),(3-x,2),(3-x,0)-$$
$$(x,1),(3-x,0),(3-x,2)- \quad (x,1),(3-x,2),(3-x,1)- \quad (x,1),(3-x,1),(3-x,0));$$

$$((x,0),(2+x,0),(2+x,1)- \quad (x,0),(2+x,1),(2+x,2)- \quad (x,0),(2+x,2),(2+x,0)-$$
$$(x,1),(2+x,0),(2+x,2)- \quad (x,1),(2+x,2),(2+x,1)- \quad (x,1),(2+x,1),(2+x,0));$$

$$(\infty_2,(x,1),(x,0)- \ \infty_1,(x,0),(x,1)- \ \infty_2,\infty_1,(x,1)- \ \infty_1,\infty_2,(x,0));$$

$((3-x,0),(2+x,0),(1-x,0)$	$(1-x,0),(2+x,0),(3-x,2)$	$\infty_2,(1-x,0),(3-x,2)$
$\infty_2,(2+x,1),(1-x,0)$	$\infty_2,(3-x,2),(2+x,1)$	$(1-x,2),(2+x,1),(3-x,2)$
$(3-x,2),(2+x,2),(1-x,2)$	$(1-x,1),(2+x,2),(3-x,2)$	$\infty_1,(1-x,1),(3-x,2)$
$\infty_1,(3-x,2),(2+x,0)$	$\infty_1,(2+x,0),(1-x,1)$	$(1-x,1),(2+x,0),(3-x,1)$
$(3-x,1),(2+x,1),(1-x,1)$	$(1-x,0),(2+x,1),(3-x,1)$	$\infty_1,(1-x,0),(3-x,1)$
$\infty_1,(2+x,2),(1-x,0)$	$\infty_1,(3-x,1),(2+x,2)$	$(1-x,2),(2+x,2),(3-x,1)$
$\infty_2,(1-x,2),(3-x,1)$	$\infty_2,(3-x,1),(2+x,2)$	$\infty_2,(2+x,0),(1-x,2)$
$(1-x,2),(2+x,0),(3-x,0)$	$\infty_1,(1-x,2),(3-x,0)$	$\infty_1,(2+x,1),(1-x,2)$
$\infty_1,(3-x,0),(2+x,1)$	$(1-x,1),(2+x,1),(3-x,0)$	$\infty_2,(1-x,1),(3-x,0)$
$\infty_2,(2+x,2),(1-x,1)$	$\infty_1,(3-x,0),(2+x,2)$	$(1-x,0),(2+x,2),(3-x,0))$. □

Theorem 2.8 *There exists an LRHTS*(18).

Proof By [16], we have known that every $\mathcal{T}_x$ of LMTS(18) $= \{(F_{16}\cup\{\infty_1,\infty_2\},\mathcal{T}_x) : x \in F_{16}\}$ contains the blocks as follows, where g is a primitive element ($g^4+g=1$), and $y \in F_{16}\backslash\{x\}$.

(1) $\langle\infty_1,\infty_2,x\rangle,\langle\infty_2,\infty_1,x\rangle$;
(2) $\langle\infty_1,y,g^{14}x+g^3y\rangle,\langle\infty_2,y,g^{12}x+g^{11}y\rangle$;
(3) $\langle x,y,g^4x+gy\rangle$;
(4) $\langle g^{14}x+g^3y,y,gx+g^4y\rangle,\langle g^2x+g^8y,y,g^{13}x+g^6y\rangle$;
(5) $\langle g^8x+g^2y,y,g^2x+g^8y\rangle$;
(6) $\langle g^{10}x+g^5y,y,g^5x+g^{10}y\rangle,\langle g^5x+g^{10}y,y,g^{10}x+g^5y\rangle$.

Notice that the repeat blocks in (6) appeared once. It is easy to see that, for $x \in F_{16}, \mathcal{T}_x = \mathcal{T}_0+x$. So we only need to partition $\mathcal{T}_0$ into parallel classes.

For (1) and (6), we obtain two parallel classes:

$$\mathcal{A} = \{\langle\infty_1,\infty_2,0\rangle,\langle g^{5+i},g^i,g^{10+i}\rangle : 0 \le i \le 4\},$$
$$\overline{\mathcal{A}} = \{\langle\infty_2,\infty_1,0\rangle,\langle g^{10+i},g^i,g^{5+i}\rangle : 0 \le i \le 4\}.$$

Further, taking a block each from (2)–(5), we obtain parallel classes:

$$\mathcal{A}_0 = \{\langle\infty_1,g^{10},g^{13}\rangle,\langle\infty_2,g^5,g\rangle,\langle 0,g^7,g^8\rangle,\langle g^3,1,g^4\rangle,\langle g^{14},g^6,g^{12}\rangle,\langle g^{11},g^9,g^2\rangle\}.$$

Let $\mathcal{A}_k = g^k\mathcal{A}_0, k \in Z_{15}$. We obtain all parallel classes of $\mathcal{T}_0$.

Next we construct an LRHTS(18) $= \{(F_{16} \cup \{\infty_1, \infty_2\}, \mathcal{B}_x^r) : x \in F_{16}, r = 0, 1, 2, 3\}$, where $\mathcal{B}_x^r = \mathcal{B}_0^r + x, x \in F_{16}, r = 0, 1, 2, 3$.

First, it is easy to see that the blocks in (1) and (6) of $\mathcal{T}_x$ are reverse respectively. Denote by $\overline{\mathcal{T}}_x$ the sets which consist of the blocks in (1) and (6). So the block-incident graph of the MTS(18) $= (F_{16} \cup \{\infty_1, \infty_2\}, \mathcal{T}_x)$ contains $|\mathcal{T}_x \backslash \overline{\mathcal{T}}_x| = 90$ points. We partition the points into 3 cycles with 10-length, 20-length and 60-length. In the following we will concretely give $\mathcal{B}_0^0, \mathcal{B}_0^1, \mathcal{B}_0^2$ and $\mathcal{B}_0^3$. For short, we denote g^k by k and 0 by $*$. The blocks in (1) and (6) do not appear. In $\mathcal{B}_0^0$, the blocks which have been underlined are transitive triples, and the others are cycle triples. In $\mathcal{B}_0^1, \mathcal{B}_0^2$ and $\mathcal{B}_0^3$, the blocks which have been underlined are cycle triples, and the others are transitive triples.

$\mathcal{B}_0^0$:

$\underline{0}$	$\underline{3}$	$\underline{\infty_1}$	$\underline{0}$	$\underline{11}$	$\underline{\infty_2}$	$*$	0	1	4	3	0	0	8	2	$\underline{8}$	$\underline{0}$	$\underline{6}$
1	4	∞_1	12	∞_2	1	1	2	$*$	5	4	1	1	9	3	9	1	7
5	∞_1	2	13	∞_2	2	$*$	2	3	2	6	5	2	10	4	10	2	8
6	∞_1	3	3	14	∞_2	3	4	$*$	3	7	6	5	3	11	11	3	9
7	∞_1	4	$\underline{\infty_2}$	$\underline{4}$	$\underline{0}$	$*$	4	5	4	8	7	12	6	4	4	10	12
8	∞_1	5	1	∞_2	5	5	6	$*$	5	9	8	7	5	13	11	3	5
9	∞_1	6	6	2	∞_2	$*$	6	7	6	10	9	$\underline{6}$	$\underline{14}$	$\underline{8}$	6	12	4
∞_1	7	10	∞_2	7	3	7	8	$*$	10	7	11	7	0	9	13	0	7
∞_1	8	11	8	4	∞_2	$*$	8	9	11	8	12	8	1	10	1	8	14
12	∞_1	9	9	5	∞_2	9	10	$*$	9	13	12	9	2	11	2	9	0
10	13	∞_1	∞_2	10	6	11	$*$	10	14	13	10	12	10	3	3	10	1
11	14	∞_1	11	7	∞_2	12	$*$	11	14	11	0	13	11	4	4	11	2
$\underline{\infty_1}$	$\underline{12}$	$\underline{0}$	∞_2	12	8	$*$	12	13	1	0	12	14	12	5	3	5	12
∞_1	13	1	∞_2	13	9	13	14	$*$	2	1	13	0	13	6	4	6	13
2	∞_1	14	10	∞_2	14	0	$*$	14	14	3	2	14	7	1	5	7	14

$\mathcal{B}_0^1$:

$\underline{0}$	$\underline{3}$	$\underline{\infty_1}$	∞_2	0	11	$*$	0	1	4	3	0	0	8	2	6	8	0
1	4	∞_1	12	∞_2	1	1	2	$*$	5	4	1	1	9	3	9	1	7
5	∞_1	2	13	∞_2	2	$*$	2	3	2	6	5	2	10	4	10	2	8
6	∞_1	3	3	14	∞_2	3	4	$*$	3	7	6	5	3	11	11	3	9
7	∞_1	4	0	∞_2	4	$*$	4	5	4	8	7	12	6	4	4	10	12
8	∞_1	5	1	∞_2	5	5	6	$*$	5	9	8	7	5	13	11	3	5
9	∞_1	6	6	2	∞_2	$*$	6	7	6	10	9	14	8	6	6	12	4
∞_1	7	10	∞_2	7	3	7	8	$*$	10	7	11	7	0	9	13	0	7
∞_1	8	11	8	4	∞_2	$*$	8	9	11	8	12	8	1	10	1	8	14
12	∞_1	9	9	5	∞_2	9	10	$*$	9	13	12	9	2	11	2	9	0
10	13	∞_1	∞_2	10	6	11	$*$	10	14	13	10	12	10	3	3	10	1
11	14	∞_1	11	7	∞_2	12	$*$	11	14	11	0	13	11	4	4	11	2
$\underline{\infty_1}$	$\underline{12}$	$\underline{0}$	∞_2	12	8	$*$	12	13	1	0	12	14	12	5	3	5	12
∞_1	13	1	∞_2	13	9	13	14	$*$	2	1	13	0	13	6	4	6	13
2	∞_1	14	10	∞_2	14	0	$*$	14	14	3	2	14	7	1	5	7	14

$\mathcal{B}_0^2$:

3	∞_1	0	$\underline{0}$	$\underline{11}$	$\underline{\infty_2}$	1	*	0	0	4	3	8	2	0	0	6	8
4	∞_1	1	1	12	∞_2	*	1	2	1	5	4	9	3	1	1	7	9
2	5	∞_1	∞_2	2	13	2	3	*	6	5	2	10	4	2	2	8	10
∞_1	3	6	14	∞_2	3	*	3	4	7	6	3	11	5	3	3	9	11
∞_1	4	7	$\underline{\infty_2}$	$\underline{4}$	$\underline{0}$	4	5	*	8	7	4	6	4	12	12	4	10
∞_1	5	8	∞_2	5	1	*	5	6	9	8	5	13	7	5	5	11	3
6	9	∞_1	2	∞_2	6	6	7	*	9	6	10	8	6	14	12	14	6
7	10	∞_1	3	∞_2	7	*	7	8	11	10	7	0	9	7	7	13	0
8	11	∞_1	4	∞_2	8	8	9	*	12	11	8	10	8	1	14	1	8
∞_1	9	12	5	∞_2	9	10	*	9	12	9	13	2	11	9	9	0	2
13	∞_1	10	10	6	∞_2	*	10	11	10	14	13	10	3	12	1	3	10
14	∞_1	11	7	∞_2	11	11	12	*	11	0	14	4	13	11	11	2	4
12	0	∞_1	8	∞_2	12	13	*	12	0	12	1	5	14	12	12	3	5
1	∞_1	13	13	9	∞_2	*	13	14	13	2	1	6	0	13	13	4	6
∞_1	14	2	∞_2	14	10	14	0	*	3	2	14	7	1	14	14	5	7

$\mathcal{B}_0^3$:

∞_1	0	3	11	∞_2	0	0	1	*	3	0	4	2	0	8	$\underline{8}$	$\underline{0}$	$\underline{6}$
∞_1	1	4	∞_2	1	12	2	*	1	4	1	5	3	1	9	7	9	1
∞_1	2	5	2	13	∞_2	3	*	2	5	2	6	4	2	10	8	10	2
3	6	∞_1	∞_2	3	14	4	*	3	6	3	7	3	11	5	9	11	3
4	7	∞_1	4	0	∞_2	5	*	4	7	4	8	4	12	6	10	12	4
5	8	∞_1	5	1	∞_2	6	*	5	8	5	9	5	13	7	13	5	11
∞_1	6	9	∞_2	6	2	7	*	6	10	9	6	$\underline{6}$	$\underline{14}$	$\underline{8}$	14	6	12
10	∞_1	7	7	3	∞_2	8	*	7	7	11	10	9	7	0	0	7	13
11	∞_1	8	∞_2	8	4	9	*	8	8	12	11	1	10	8	8	14	1
9	12	∞_1	∞_2	9	5	*	9	10	13	12	9	11	9	2	0	2	9
∞_1	10	13	6	∞_2	10	10	11	*	13	10	14	3	12	10	10	1	3
∞_1	11	14	∞_2	11	7	*	11	12	0	14	11	11	4	13	2	4	11
0	∞_1	12	12	8	∞_2	12	13	*	12	1	0	12	5	14	5	12	3
13	1	∞_1	9	∞_2	13	14	*	13	1	13	2	13	6	0	6	13	4
14	2	∞_1	14	10	∞_2	*	14	0	2	14	3	1	14	7	7	14	5

□

Theorem 2.9 *There exists an LRHTS*(66).

Proof By [16], there exists an LMTS(66) $= \{(F_{64} \cup \{\infty_1, \infty_2\}, \mathcal{T}_x) : x \in F_{64}\}$. Every $\mathcal{T}_x$ contains the following blocks, where g is a primitive element ($g^6 + g = 1$), $y \in F_{64} \backslash \{x\}$.

(1) $\langle \infty_1, \infty_2, x\rangle, \langle \infty_2, \infty_1, x\rangle$;

(2) $\langle \infty_1, y, g^{27}x + g^{18}y\rangle, \langle \infty_2, y, g^{45}x + g^{9}y\rangle$;

(3) $\langle x, y, g^{54}x + g^{36}y\rangle$;

(4) $\langle g^{54}x + g^{36}y, y, g^{9}x + g^{45}y\rangle$;

(5) $\langle g^{26}x+g^7y,y,g^7x+g^{26}y\rangle$, $\langle g^7x+g^{26}y,y,g^{26}x+g^7y\rangle$,
$\langle g^{35}x+g^{13}y,y,g^{13}x+g^{35}y\rangle$, $\langle g^{13}x+g^{35}y,y,g^{35}x+g^{13}y\rangle$,
$\langle g^{11}x+g^{25}y,y,g^{25}x+g^{11}y\rangle$, $\langle g^{25}x+g^{11}y,y,g^{11}x+g^{25}y\rangle$,
$\langle g^{6}x+gy,y,gx+g^{6}y\rangle$, $\langle gx+g^{6}y,y,g^{6}x+gy\rangle$,
$\langle g^{24}x+g^{4}y,y,g^{4}x+g^{24}y\rangle$, $\langle g^{4}x+g^{24}y,y,g^{24}x+g^{4}y\rangle$,
$\langle g^{33}x+g^{16}y,y,g^{16}x+g^{33}y\rangle$, $\langle g^{16}x+g^{33}y,y,g^{33}x+g^{16}y\rangle$,
$\langle g^{12}x+g^{2}y,y,g^{2}x+g^{12}y\rangle$, $\langle g^{2}x+g^{12}y,y,g^{12}x+g^{2}y\rangle$,
$\langle g^{48}x+g^{8}y,y,g^{8}x+g^{48}y\rangle$, $\langle g^{8}x+g^{48}y,y,g^{48}x+g^{8}y\rangle$,
$\langle g^{3}x+g^{32}y,y,g^{32}x+g^{3}y\rangle$, $\langle g^{32}x+g^{3}y,y,g^{3}x+g^{32}y\rangle$

(6) $\langle g^{42}x+g^{21}y,y,g^{21}x+g^{42}y\rangle$, $\langle g^{21}x+g^{42}y,y,g^{42}x+g^{21}y\rangle$.

Notice that the repeat blocks in (6) appeared once, and $\mathcal{T}_x=\mathcal{T}_0+x$, $x\in F_{64}$. Similarly to the construction of an LRHTS(18), we only need to partition $\mathcal{T}_0$ into parallel classes.

For (1) and (6), we obtain two parallel classes:

$$\mathcal{A}=\{\langle\infty_1,\infty_2,0\rangle,\langle g^{21+i},g^i,g^{42+i}\rangle:0\le i\le 20\},$$
$$\overline{\mathcal{A}}=\{\langle\infty_2,\infty_1,0\rangle,\langle g^{42+i},g^i,g^{21+i}\rangle:0\le i\le 20\}.$$

For (2)–(5), we obtain the following parallel classes:

$$\begin{aligned}\mathcal{A}_0=\ &\{\langle\infty_1,g^{31},g^{49}\rangle,\ \langle\infty_2,g^{53},g^{62}\rangle,\ \langle 0,g^{51},g^{24}\rangle,\ \langle g^{36},1,g^{45}\rangle,\\ &\langle g^{28},g^{3},g^{14}\rangle,\ \langle g^{15},g^{2},g^{37}\rangle,\ \langle g^{16},g^{5},g^{30}\rangle,\ \langle g^{39},g^{4},g^{17}\rangle,\\ &\langle g^{21},g^{20},g^{26}\rangle,\ \langle g^{35},g^{23},g^{25}\rangle,\ \langle g^{54},g^{50},g^{11}\rangle,\ \langle g^{19},g^{34},g^{42}\rangle,\\ &\langle g^{43},g^{40},g^{9}\rangle,\ \langle g^{60},g^{58},g^{7}\rangle,\ \langle g^{18},g^{57},g^{61}\rangle,\ \langle g^{56},g^{48},g^{33}\rangle,\\ &\langle g^{10},g^{41},g^{44}\rangle,\ \langle g^{52},g^{46},g^{47}\rangle,\ \langle g^{29},g^{59},g^{12}\rangle,\ \langle g^{38},g^{22},g^{55}\rangle,\\ &\langle g^{8},g,g^{27}\rangle,\ \langle g^{32},g^{6},g^{13}\rangle\}\\ \mathcal{A}_k=\ &g^k\mathcal{A}_0, k\in Z_{63}.\end{aligned}$$

Next we construct an LRHTS(66) $=\{(F_{64}\cup\{\infty_1,\infty_2\},\mathcal{B}_x^r):x\in F_{64},r=0,1,2,3\}$, where $\mathcal{B}_x^r=\mathcal{B}_0^r+x,x\in F_{64},r=0,1,2,3$.

First, it is easy to see that the blocks in (1), (5) and (6) are reverse respectively. Denote by $\overline{\mathcal{T}}_x$ the sets which consist of the blocks in (1), (5) and (6). So the block-incident graph of MTS(66) $=(F_{64}\cup\{\infty_1,\infty_2\},\mathcal{T}_x)$ contains $|\mathcal{T}_x\backslash\overline{\mathcal{T}}_x|=28\times 9$ points. We partition the points into 18 cycles with 14-length. In the following we will concretely give $\mathcal{B}_0^0,\mathcal{B}_0^1,\mathcal{B}_0^2$ and $\mathcal{B}_0^3$. For short, let k denote g^k, and let $*$ denote 0. The blocks in (1), (5) and (6) do not appear. For every $\mathcal{B}_0^r$, we only list 28 blocks, and the other blocks can be given by $x+i$ $(1\le i\le 8)$ mod 63, where x is different from ∞_1,∞_2 and $*$. In $\mathcal{B}_0^0$, the blocks which have been underlined are transitive triples, and the others are cycle triples. In $\mathcal{B}_0^1,\mathcal{B}_0^2$ and $\mathcal{B}_0^3$, the blocks which have been underlined are cycle triples, and the others are transitive triples.

$\mathcal{B}_0^0$:	$\underline{\infty_1}$	$\underline{54}$	$\underline{9}$	∞_2	18	27	18	54	$*$	54	18	0
	$\underline{9}$	$\underline{27}$	$\underline{\infty_1}$	27	36	∞_2	$*$	45	18	9	54	45
	45	∞_1	27	∞_2	36	45	45	$*$	9	27	18	45
	∞_1	18	36	45	54	∞_2	0	$*$	27	27	9	0
	36	54	∞_1	∞_2	0	9	27	$*$	54	54	36	27
	∞_1	45	0	9	18	∞_2	36	$*$	0	0	45	36
	0	18	∞_1	0	∞_2	54	9	$*$	36	36	18	9

$\mathcal{B}_0^1:$	9	∞_1	54	27	∞_2	18	$\underline{54}$	$\underline{*}$	$\underline{18}$	$\underline{18}$	$\underline{0}$	$\underline{54}$
	∞_1	9	27	∞_2	27	36	45	18	$*$	54	45	9
	27	45	∞_1	36	45	∞_2	$*$	9	45	18	45	27
	18	36	∞_1	∞_2	45	54	27	0	$*$	0	27	9
	54	∞_1	36	9	∞_2	0	$*$	54	27	36	27	54
	0	∞_1	45	18	∞_2	9	$*$	0	36	45	36	0
	∞_1	0	18	54	0	∞_2	36	9	$*$	9	36	18

$\mathcal{B}_0^2:$	54	9	∞_1	18	27	∞_2	$*$	18	54	0	54	18
	27	∞_1	9	36	∞_2	27	18	$*$	45	45	9	54
	$\underline{\infty_1}$	$\underline{27}$	$\underline{45}$	45	∞_2	36	9	45	$*$	45	27	18
	36	∞_1	18	54	∞_2	45	$*$	27	0	9	0	27
	∞_1	36	54	0	9	∞_2	54	27	$*$	27	54	36
	$\underline{45}$	$\underline{0}$	$\underline{\infty_1}$	∞_2	9	18	0	36	$*$	36	0	45
	18	∞_1	0	∞_2	54	0	$*$	36	9	18	9	36

$\mathcal{B}_0^3:$	$\underline{\infty_1}$	$\underline{54}$	$\underline{9}$	18	27	∞_2	$\underline{54}$	$\underline{*}$	$\underline{18}$	$\underline{18}$	$\underline{0}$	$\underline{54}$
	$\underline{9}$	$\underline{27}$	$\underline{\infty_1}$	36	∞_2	27	18	$*$	45	45	9	54
	$\underline{\infty_1}$	$\underline{27}$	$\underline{45}$	45	∞_2	36	9	45	$*$	45	27	18
	36	∞_1	18	54	∞_2	45	$*$	27	0	9	0	27
	∞_1	36	54	0	9	∞_2	54	27	$*$	27	54	36
	$\underline{45}$	$\underline{0}$	$\underline{\infty_1}$	∞_2	9	18	0	36	$*$	36	0	45
	18	∞_1	0	∞_2	54	0	$*$	36	9	18	9	36

□

3. Recursive constructions

An $S(t,K,v)$, $t,v \in N, K \subseteq N$ is a pair $(X,\mathcal{B})$, where X is a v-set, and $\mathcal{B}$ is a collection of subsets on X, called blocks, such that every t-set on X exactly appears once, and $|B| \in K$ is satisfied for every block $B \in \mathcal{B}$. When $t=3$, $S(3,K,v)$ is called Steiner 3-design. Especially, when $K=\{k\}$, $S(3,K,v)$ is denoted $S(3,k,v)$ for short. An $S(3,4,v)$ is called Steiner quaternary system $SQS(v)$. $SQS(v)=(X,\mathcal{B})$. If the block set $\mathcal{B}$ can be partitioned into $\mathcal{B}_1,\mathcal{B}_2,\dots,\mathcal{B}_{\frac{v-2}{2}}$, and every $(X,\mathcal{B}_i)$ is an $S(2,4,v)$, $1 \le i \le \frac{v-2}{2}$, then the $SQS(v)$ is 2-resolvable.

Lemma 3.1 ([17]) *Let q be a prime power.*

(1) There is an $S(3,5,26)$;

(2) There is an $S(3,q+1,q^n+1)$, where $n \ge 1$;

(3) Suppose that there exist an $S(3,q+1,v+1)$ and an $S(3,q+1,w+1)$. Then there exists an $S(3,q+1,vw+1)$.

Lemma 3.2 ([18]) *There is a 2-resolvable $SQS(v+1)$, $v=4^n-1, 2\cdot 7^n+1, 2\cdot 31^n+1, 2\cdot 127^n+1$ and $n \ge 1$.*

Lemma 3.3 *Suppose that there exists an $S(3,K,v)$, and there exists an OLARHTS$(k-1)$ for any $k \in K$. Then there exists an OLARHTS$(v-1)$.*

Construction 3.4 *Let $S(3,K,v)=(X,\Omega)$, $|X|=v$. For any $B\in\Omega$, there is an OLARHTS$(|B|-1)=\{(B\backslash\{x\},\mathcal{B}_x(B,j)):x\in B,j=0,1,2,3\}$. The almost parallel classes of $\mathcal{B}_x(B,j)$ is $\mathcal{B}_x^y(B,j)$, $y\in B\backslash\{x\}$, and $\mathcal{B}_x^y(B,j)$ is the partition of $B\backslash\{x,y\}$. Define*

$$\mathcal{B}_x(j)=\bigcup_{x\in B\in\Omega}\mathcal{B}_x(B,j),\quad x\in X,\ \text{and}$$
$$\mathcal{B}_x^y(j)=\bigcup_{\{x,y\}\subset B\in\Omega}\mathcal{B}_x^y(B,j),\quad x\in X,y\in X\backslash\{x\}\ \text{and}\ j=0,1,2,3.$$

Then $\{(X\backslash\{x\},\mathcal{B}_x(j)):x\in X,j=0,1,2,3\}$ is an OLARHTS$(v-1)$, where $\mathcal{B}_x^y(j)$ is the almost parallel classes of $\mathcal{B}_x(j)$, $y\in X\backslash\{x\}$.

Proof First, for $x\in X,j=0,1,2,3$, every $(X\backslash\{x\},\mathcal{B}_x(j))$ is an ARHTS$(v-1)$.

For a given $x\in X$, let all the blocks which contain element x be $B_1,B_2,\ldots,B_s$. Then

$$\sum_{i=1}^{s}\binom{|B_i|-1}{2}=\binom{v-1}{2}.$$

Therefore

$$|\mathcal{B}_x(j)|=\sum_{i=1}^{s}\frac{(|B_i|-1)(|B_i|-2)}{3}=\frac{2}{3}\sum_{i=1}^{s}\binom{|B_i|-1}{2}=\frac{(v-1)(v-2)}{3}$$

is exactly the block number.

For a given ordered pair $P=(y,z),y\neq z\in X\backslash\{x\}$, there exists the unique $B\in\Omega$ such that $\{x,y,z\}\subset B$. So P is contained in a block of $\mathcal{B}_x(B,j)\subset\mathcal{B}_x(j)$. Since (X,Ω) is a 3-design, $\{B\backslash\{x,y\}:\{x,y\}\subset B\in\Omega\}$ is a partition of $X\backslash\{x,y\}$. And $\mathcal{B}_x^y(B,j)$ is a partition of $B\backslash\{x,y\}$. So $\mathcal{B}_x^y(j)$ is a partition of $X\backslash\{x,y\}$, $\mathcal{B}_x^y(j)$ is an almost parallel classes of $\mathcal{B}_x(j)\backslash\{y\}$.

Finally, for any triple $T=\langle a,b,c\rangle$ or (a,b,c) on X, there exists the unique $B\in\Omega$ such that $\{a,b,c\}\subset B$. Therefore, there exists an $x\in B\backslash\{a,b,c\}$ such that $T\in\mathcal{B}_x(B,j)\subset\mathcal{B}_x(j)$. So $\{(X\backslash\{x\},\mathcal{B}_x(j)):x\in X,j=0,1,2,3\}$ is an OLARHTS$(v-1)$. □

Corollary 3.5 *There exists an OLARHTS(v), when $v=25\cdot 4^k,4^n,7^n,13^n,25^n$, $k\geq 0,n\geq 1$.*

Proof From Lemma 3.1, we have $S(3,5,26)$ and recursive theorem:

"$S(3,q+1,v+1)\longrightarrow S(3,q+1,qv+1)$, where q is a prime power".

There exists an $S(3,4+1,4^k\cdot 25+1)$, $k\geq 0$. From Theorem 2.3, there exists an OLARHTS(4). From above lemma, there exists an OLARHTS$(25\cdot 4^k)$.

On the other side, for any prime power q, there exists an $S(3,q+1,q^n+1)$, particularly, for $q=4,7,13,5^2$. From Theorems 2.3, 2.7 and above conclusion, when $q=4,7,13,5^2$, there exists an OLARHTS(q). So from Lemma 3.3, there exists an OLARHTS(v), when $v=4^n,7^n,13^n,25^n$. □

An $S(3,K_0\bigcup K_1,v+1)$, we denote $S(3,(K_0,K_1),v+1)=(Z_v\cup\{\infty\},\Omega)$. If block sizes of $\mathcal{B}_0$ and $\mathcal{B}_1$ are from K_0 and K_1, where $\mathcal{B}_0$ does not contain ∞, $\mathcal{B}_1$ contains ∞.

An LHTS(v) $= \{(X, \mathcal{B}_i) : 1 \le i \le 4(v-2)\}$ is called quasi symmetric, denoted by LQHTS(v), if for every i,

$$\langle a,b,x\rangle \text{ (or } (a,b,x), \text{ or } (a,x,b), \text{ or } (x,a,b)) \in \mathcal{B}_i \Longleftrightarrow \langle b,a,x\rangle (\text{or } (x,b,a), \text{ or } (b,x,a), \text{ or } (b,a,x)) \in \mathcal{B}_i. \quad (*)$$

Lemma 3.6 *Suppose that there exists an $S(3,(K_0,K_1),v+1)$ on $Z_v \cup \{\infty\}$. Suppose that there exists an LRQHTS(k_1+1) for any $k_1 \in K_1$, an OLARHTS(k_0-1) for any $k_0 \in K_0$. Then there exists an LRHTS($v+2$).*

Construction 3.7 *Let $S(3,(K_0,K_1),v+1) = (Z_v \cup \{\infty\}, \Omega_0 \cup \Omega_1)$, where Ω_1 is the blocks set which contains ∞, Ω_0 is the blocks set which does not contain ∞. Denote $\overline{\Omega}_1 = \{B \backslash \{\infty\} : B \in \Omega_1\}$.*

For any block $A \in \Omega_0$, there exists an OLARHTS($|A|-1$) $= \{(A\backslash\{x\}, \mathcal{C}_A(x,j)) : x \in A, j = 0,1,2,3\}$. The almost parallel classes of $\mathcal{C}_A(x,j)$ is $\mathcal{C}_A^y(x,j)$, $y \in A\backslash\{x\}$. For any block $B \in \overline{\Omega}_1$, there exists an LRQHTS($|B|+2$) $= \{(\{\infty_0,\infty_1\} \cup B, \mathcal{A}_B(x,j))$: $x \in B, j = 0,1,2,3\}$, where the parallel classes of $\mathcal{A}_B(x,j)$ are $\mathcal{A}'_B(x,j)$, $\mathcal{A}''_B(x,j)$, and $\mathcal{A}^y_B(x,j)$, $y \in B\backslash\{x\}$. From the above condition (*), we have

$$\langle\infty_0,\infty_1,x\rangle \in \mathcal{A}'_B(x,0), \quad \langle\infty_1,\infty_0,x\rangle \in \mathcal{A}''_B(x,0),$$

$$(\infty_0,\infty_1,x) \in \mathcal{A}'_B(x,1), \quad (x,\infty_1,\infty_0) \in \mathcal{A}''_B(x,1),$$

$$(\infty_1,x,\infty_0) \in \mathcal{A}'_B(x,2), \quad (\infty_0,x,\infty_1) \in \mathcal{A}''_B(x,2),$$

$$(x,\infty_0,\infty_1) \in \mathcal{A}'_B(x,3), \quad (\infty_1,\infty_0,x) \in \mathcal{A}''_B(x,3).$$

For the $\mathcal{A}^y_B(x,j)$, y can be given in the following way: define a graph $G_B(x,j)$ on the vertex set $B\backslash\{x\}$, $\{y,z\}$ is its edge if and only if $\{\infty_0,y,z\}$ is an underlying triple of $\mathcal{A}_B(x,j)$ for $y \ne z \in B\backslash\{x\}$. It is easy to see that $G_B(x,j)$ is a 2-regular graph, so it can be partitioned into some disjoint cycles. Now, define an order on every cycle. Then we can get a directed graph $\overline{G}_B(x,j)$ from $G_B(x,j)$. Besides $\mathcal{A}'_B(x,j)$ and $\mathcal{A}''_B(x,j)$, there exists a unique block T which contains ∞_0 on every parallel class of $\mathcal{A}_B(x,j)$. Let y,z differ from ∞_0 as a directed edge of $\overline{G}_B(x,j)$. If the order of $\{y,z\}$ is from y to z, then the parallel class is $\mathcal{A}^y_B(x,j)$.

Let $X = \{\infty_0,\infty_1\} \cup Z_v$. Define:

$$\mathcal{B}^j_x = \Big(\bigcup_{x\in B\in\overline{\Omega}_1} \mathcal{A}_B(x,j)\Big) \bigcup \Big(\bigcup_{x\in A\in\Omega_0} \mathcal{C}_A(x,j)\Big), \quad x \in Z_v,\ j = 0,1,2,3,$$

then $\{(X, \mathcal{B}^j_x) : x \in Z_v, j = 0,1,2,3\}$ is an LRHTS($v+2$).

Proof First, let $\overline{\Omega}_1$ contain t blocks $B_1, B_2, \ldots, B_t$, Ω_0 contain s blocks $A_1, A_2, \ldots, A_s$. For a given $x \in Z_v$, we get:

$$\Big|\bigcup_{x\in B\in\overline{\Omega_1}} \mathcal{A}_B(x,j)\Big| = 2 + \sum_{i=1}^{t}\Big(\frac{(|B_i|+1)(|B_i|+2)}{3} - 2\Big) = 2 + \sum_{i=1}^{t}((|B_i|^2 + 3|B_i| - 4)/3,$$

$$\Big|\bigcup_{x\in A\in\Omega_0}\mathcal{C}_A(x,j)\Big| = \sum_{x\in A\in\Omega_0}|\mathcal{C}_A(x,j)| = \sum_{j=1}^{s}\frac{(|A_j|-1)(|A_j|-2)}{3},$$

$$\sum_{i=1}^{t}(|B_i|-1) = v-1,$$

$$\sum_{i=1}^{t}\binom{|B_i|-1}{2} + \sum_{j=1}\binom{|A_j|-1}{2} = \binom{v-1}{2}.$$

Therefore,

$$\begin{aligned}\Big|\bigcup_{B}\mathcal{A}_B(x,j)\Big| + \Big|\bigcup_{A}\mathcal{C}_A(x,j)\Big| &= 2+\frac{1}{3}(\sum_{i=1}^{t}(|B_i|^2+3|B_i|-4)+\sum_{j=1}^{s}(|A_j|-1)(|A_j|-2))\\ &= 2+\frac{1}{3}\Big(6\sum_{i=1}^{t}(|B_i|-1)+2\sum_{i=1}^{t}\binom{|B_i|-1}{2}+2\sum_{j=1}^{s}\binom{|A_j|-1}{2}\Big)\\ &= \frac{(v+1)(v+2)}{3},\end{aligned}$$

which is exactly the block number of $\mathcal{B}_x^j$.

In what follows, we will prove that for a given $x\in Z_v, j=0,1,2,3$, each ordered pair P of X appears in exactly one block of $\mathcal{B}_x^j$.

(i) $P=(\infty_k,\infty_{1-k})$, (∞_k,x) and (x,∞_k), $k=0,1$, belong to the blocks $\langle\infty_0,\infty_1,x\rangle$ and $\langle\infty_1,\infty_0,x\rangle$ $(j=0)$, or (∞_0,∞_1,x) and (x,∞_1,∞_0) $(j=1)$, or (∞_1,x,∞_0) and (∞_0,x,∞_1) $(j=2)$, or (x,∞_0,∞_1) and (∞_1,∞_0,x) $(j=3)$ of $\mathcal{A}_B(x,j)$, $x\in B\in\overline{\Omega}_1$. Notice, under "$\cup$", repeat triple block appears once in $\mathcal{A}_B(x,j)$.

(ii) $P=(\infty_k,y)$, (y,∞_k), (x,y) and (y,x), $k=0,1$, $y\in Z_v\backslash\{x\}$, there exists a unique block B containing $\{x,y\}$ on $\overline{\Omega}_1$, then P appears in one block of $\mathcal{A}_B(x,j)$.

(iii) $P=(y,z)$, $y\neq z\in Z_v\backslash\{x\}$, there exists a block $B\in\Omega_0\bigcup\overline{\Omega}_1$ such that $\{x,y,z\}\subset B$. If $B\in\overline{\Omega}_1$, then P appears in one block of $\mathcal{A}_B(x,j)$; If $B\in\Omega_0$, then P appears in one block of $\mathcal{C}_B(x,j)$.

More, every block T on X appears in a $\mathcal{B}_x^j$, $x\in Z_v, j=0,1,2,3$.

(i) For $T=\langle\infty_k,\infty_{1-k},x\rangle$, $k=0,1$, appear in $\mathcal{B}_x^0$; (∞_0,∞_1,x), (x,∞_1,∞_0) appear in $\mathcal{B}_x^1$; (∞_1,x,∞_0), (∞_0,x,∞_1) appear in $\mathcal{B}_x^2$; (x,∞_0,∞_1), (∞_1,∞_0,x) appear in $\mathcal{B}_x^3$.

(ii) For $T=\langle\infty_k,y,z\rangle$(or $(\infty_k,y,z),(y,\infty_k,z),(y,z,\infty_k)$), $k=0,1$, there is a block $B\in\overline{\Omega}_1$ such that $\{y,z\}\subset B$. So there is an $\mathcal{A}_B(x,j)$, $x\in B, j=0,1,2,3$, such that T appears in $\mathcal{A}_B(x,j)$.

(iii) For $T=\langle y,z,t\rangle$(or (y,z,t)), there is a block $B\in\Omega_0\bigcup\Omega_1$ such that $\{y,z,t\}\subset B$. If $\infty\in B$, then $\overline{B}=B\backslash\{\infty\}\in\overline{\Omega}_1$, there is an $x\in B, j=0,1,2,3$, such that T appears in $\mathcal{A}_B(x,j)\subset\mathcal{B}_x^j$. If $\infty\notin B$, then $B\in\Omega_0$, there is an element $x\in B, j=0,1,2,3$, such that T appears in $\mathcal{C}_B(x,j)\subset\mathcal{B}_x^j$.

Finally, every $\mathcal{B}_x^j$ can be partitioned into parallel classes as follows:

(i) $\mathcal{A}'(x,j)=\bigcup_{x\in B\in\overline{\Omega}_1}\mathcal{A}'_B(x,j)$. It contains

$$1+\sum_{i=1}^{t}(\frac{|B_i|+2}{3}-1)=1+\frac{1}{3}\sum_{i=1}^{t}(|B_i|-1)=1+\frac{v-1}{3}=\frac{v+2}{3}$$

triples, where every $\mathcal{A}'_B(x,j)$ is the partition of $B\bigcup\{\infty_0,\infty_1\}$. And $\{B\backslash\{x\}:x\in B\in\overline{\Omega}_1\}$ is the partition of $Z_v\backslash\{x\}$.

(ii) Similarly, $\mathcal{A}''(x,j)=\bigcup_{x\in B\in\overline{\Omega}_1}\mathcal{A}''_B(x,j)$ is another parallel class.

(iii) For $y\in Z_v\backslash\{x\}$, $\mathcal{A}^y(x,j)=\mathcal{A}^y_B(x,j)\bigcup(\bigcup_{\{x,y\}\in A\in\Omega_0}\mathcal{C}^y_A(x,j))$ is a parallel class, where B is the unique block in $\overline{\Omega}_1$ which contains $\{x,y\}$. It is easy to see that,

$$|\mathcal{A}^y(x,j)|=\frac{|B|+2}{3}+\sum_{i=1}^{w}\frac{|A_i|-2}{3}=\frac{v+2}{3},$$

$A_1,A_2,\ldots,A_w$ are all the blocks in Ω_0, each A_i $(1\le i\le x)$ contains $\{x,y\}$ and satisfies the equation $\sum_{i=1}^{w}(|A_i|-2)+|B|-2=v-2$. Note that B containing $\{x,y\}$ and $A_i\backslash\{x,y\}$, $1\le i\le w$, form the partition of Z_v, the parallel class is the partition of $\{\infty_1,\infty_2\}\cup B$, and the almost parallel class of $\mathcal{C}^y_{A_i}(x)$ is the partition of $A_i\backslash\{x,y\}$. □

Corollary 3.8 *There is an LRHTS*$(v+2)$, *when* $v=7^n,13^n,25^n,2^{4n}$, 2^{6n}, $n\ge 0$.

Proof Let $K_0=K_1=\{k\}$ and $k=7+1,13+1,5^2+1,2^4+1,2^6+1$, respectively. For $k-1=7,13,5^2,4^2$ and 4^3, from Theorems 2.3, 2.7 and Lemma 3.6, there is an OLARHTS$(k-1)$. For $k+1=9,15,27,18$ and 66, from Lemma 1.7, Theorems 2.8 and 2.9, there is an LRHTS$(k+1)$. These LRHTS$(k+1)$s are all LRQHTS$(k+1)$s. □

A quasigroup of order v is a pair $(X,\circ)$, where X is a $v-$set and $\circ$ is a binary operation on X such that equations $a\circ x=b$ and $y\circ a=b$ are uniquely solvable for every pair of element $a,b\in X$. A quasigroup $(X,\circ)$ is called idempotent if the identity $i\circ i=i$, holds for all $i\in X$. An idempotent quasigroup of order v is denoted by IQ(v). Furthermore, an idempotent quasigroup $(X,\circ)$ is called resolvable if all $v(v-1)$ pairs of distinct elements of X can be partitioned into subsets $T_i, 1\le i\le 3(v-1)$, such that every $\{(x,y,x\circ y):(x,y)\in T_i\}$ (called parallel class) is a partition of X. A resolvable idempotent quasigroup of order v is denoted by RIQ(v).

An IQ(v) is called first-transitive, if there exists a group G of order v acting transitively on X which forms an automorphism group of $(X,\circ)$. A first-transitive RIQ(v) is briefly denoted by TRIQ(v).

Take any fixed ordered pair $(i,j), i\ne j\in\{0,1,2\}$. For an IQ$(X,\circ)$ and the given ordered pair (i,j), define a set $T(i,j)$ of transitive triples of $X\times\{i,j\}$ as follows: for each ordered pair $(x,y),\ x\ne y\in X$, let $t(x,y,x\circ y)$ be the three transitive triples of $X\times\{i,j\}$ defined by

$$t(x,y,x\circ y)=\{((x,i),(y,i),(x\circ y,j)),((x,i),(x\circ y,j),(y,i)),((x\circ y,j),(x,i),(y,i))\},$$
$$T(i,j)=\bigcup_{x\ne y\in X}t(x,y,x\circ y).$$

The IQ$(X,\circ)$ is called second-transitive provided that $T(i,j)$ can be partitioned into three sets T_0, T_1 and T_2 such that

(i) The three transitive triples in $t(x,y,x\circ y)$ belong to different $T_k(i,j)$s $(k=0,1,2)$;

(ii) If $a\neq b\in X$, each of the ordered pairs $((a,i),(b,j))$, and $((b,j),(a,i))$ belongs to exactly one transitive triple in each of T_0, T_1 and T_2.

An IQ$(X,\circ)$ with both first- and second- transitivity is called doubly transitive. A doubly transitive RIQ(v) is denoted by DTRIQ(v). In [7], Chang, Zhou gave the results below:

Lemma 3.9 *A DTRIQ(v) exists if and only if v is a positive integer such that $3|v$ and $v\not\equiv 2$* mod 4.

In [19], Lei introduced the concept of LR-design. An LR-design of order v (briefly LR(v)) is a collection $\{(X,\mathcal{A}_k^j):1\leq k\leq \frac{u-1}{2}, j=0,1\}$ of $v-1$ KTS(v)s with the following properties:

(i) Let the resolution of $\mathcal{A}_k^j$ be $\mathcal{T}_k^j=\{\mathcal{A}_k^j(h):1\leq k\leq \frac{u-1}{2}\}$. There is an element in each $\mathcal{T}_k^j$, say, $\mathcal{A}_k^j(1)$, such that

$$\bigcup_{k=1}^{\frac{u-1}{2}} A_k^0(1)=\bigcup_{k=1}^{\frac{u-1}{2}} A_k^1(1)=\mathcal{A}$$

and $(X,\mathcal{A})$ is a KTS(v).

(ii) For any triple $T=\{x,y,z\}\subseteq X, x\neq y\neq z\neq x$, there exist k,j such that $T\in\mathcal{A}_k^j$.

Lemma 3.10 *There exists an $LR(3^a5^bm\prod_{i=1}^{r}(2\cdot 13^{n_i}+1)\prod_{j=1}^{p}(2\cdot 7^{m_j}+1))$, where $n_i,m_j\geq 1$ $(1\leq i\leq r, 1\leq j\leq p), a,b,r,p\geq 0$ with $a+r+p\leq 1$.*

Using the auxiliary design and results, Zhou [11] have given the following recently.

Lemma 3.11 *If there exist a DTRIQ(v) and an LRHTS(v), then there exists an LRHTS$(3v)$.*

Lemma 3.12 *If there exist an LRHTS(u), a DTRIQ(u), and an LR(v), then there exists an LRHTS(uv).*

Corollary 3.13 *There exists an LRHTS(v), when $v=3^a5^bm\prod_{i=1}^{r}(2\cdot 13^{n_i}+1)\prod_{j=1}^{p}(2\cdot 7^{m_j}+1)$, where $m\in\{1,4,11,17,35,43,67,91,123,7^n+2,13^n+2,25^n+2\}\cup\{2^{2l+1}25^s+1:l\geq 0,s\geq 0\}$, $a,n_i,m_j\geq 1$ $(1\leq i\leq r,\ 1\leq j\leq p)$, $b,r,p\geq 2$. $b\geq 1$ and $m\neq 1$.*

Proof From Lemmas 3.10, 3.11, 3.12 and Theorem 1.2, we have the results. □

In [20], Teirlinck introduced the concept of overlarge set.

An $LS(\lambda,1;t,(k,K),v)$, $k\geq t$, $\lambda\in N\backslash\{0\}$ and $k\leq\min\{j;\ j\in K\}$ is a set $\{(X,\mathcal{B}_r):\ r\in R\}$ which is constituted by some $S(t,K,v)=(X,\mathcal{B}_r)$s, such that $(X,\bigcup_{r\in R}\mathcal{B}_r)$ is an $S(k,K,v)$, and for every $B\in\bigcup_{r\in R}\mathcal{B}_r$, there exist $\lambda\binom{|B|-t}{k-t}$ elements r on R, satisfying $B\in\mathcal{B}_r$ (Note that $\bigcup_{r\in R}\mathcal{B}_r$ does not contain repeated set). Let $LS(1,1;t,(k,K),v)$ denote $LS(t,(k,K),v)$. We have known that $LS(\lambda,1;t,(k,K),v)$, $v\geq k$ contains $\lambda\binom{v-t}{k-t}S(t,K,v)$s. Repeating $LS(t,(k,K),v)$ λ times, we get an $LS(\lambda,1;t,(k,K),v)$.

Lemma 3.14 *Suppose there exists an $LS(4,1;2,(3,K),v)$, and there exists an LARHTS(k) for any $k\in K$. Then there exists an LARHTS(v).*

Construction 3.15 *Let $\{(X,\mathcal{B}_r):\ r\in R\}$ be an $LS(4,1;2,(3,K),v)$. For any $B\in\bigcup_{r\in R}\mathcal{B}_r$,* *there is an LARHTS*$(|B|) = \{(B, \mathcal{C}_r(B)) :\ r \in R_B\}$, *where* $R_B = \{r :\ B \in \mathcal{B}_r\}$. *Define*

$$\mathcal{A}_r = \bigcup_{B \in \mathcal{B}_r} \mathcal{C}_r(B), \quad r \in R.$$

Then $\{(X, \mathcal{A}_r) :\ r \in R\}$ *is an LARHTS*(v).

Proof First, $|X| = v$, $|R| = 4(v-2)$, $|R_B| = 4(|B|-2)$.

For a given $r \in R$, we consider the ordered pair $P = (x, y)$ on X. Since $(X, \mathcal{B}_r)$ is an $S(2, K, v)$, the unordered pair $\{x, y\}$ appears in the unique block $B \in \mathcal{B}_r$. Since $(B, \mathcal{C}_r(B))$ is an HTS$(|B|)$, P appears in the unique block of $\mathcal{C}_r(B) \subset \mathcal{A}_r$. Then, $(X, \mathcal{A}_r)$ is an HTS(v). $\mathcal{C}_r(B)$ can be partitioned into $|B|$ almost parallel classes $\mathcal{C}_r(B, x)$, $x \in B$, where $\mathcal{C}_r(B, x)$ is a partition of $B \backslash \{x\}$. Denote

$$\mathcal{A}_r(x) = \bigcup_{x \in B \in \mathcal{B}_r} \mathcal{C}_r(B, x), \quad x \in X.$$

For a given $r \in R$, $x \in X$, $\mathcal{A}_r(x)$ is a partition of $X \backslash \{x\}$. In fact, for a given $y \in X \backslash \{x\}$, ordered pair $\{x, y\}$ appears in the unique block $B \in \mathcal{B}_r$, and y appears in the unique triple of $\mathcal{C}_r(B, x) \subset \mathcal{A}_r(x)$. And

$$\mathcal{A}_r = \bigcup_{B \in \mathcal{B}_r} \mathcal{C}_r(B) = \bigcup_{B \in \mathcal{B}_r} \bigcup_{x \in B} \mathcal{C}_r(B, x) = \bigcup_{x \in X} \bigcup_{x \in B \in \mathcal{B}_r} \mathcal{C}_r(B, x) = \bigcup_{x \in X} \mathcal{A}_r(x),$$

so, $(X, \mathcal{A}_r)$ is an ARHTS(v).

Finally, for any block $T = \langle x, y, z \rangle$ or (x, y, z) on X. Since $(X, \bigcup_{r \in R} \mathcal{B}_r)$ is an $S(3, K, v)$, there exists block $B \in \bigcup_{r \in R} \mathcal{B}_r$ such that $\{x, y, z\} \subset B$. Since $\{(B, C_r(B)) :\ r \in R_B\}$ is an LHTS$(|B|)$, $\bigcup_{r \in R_B} C_r(B)$ contains all the cycle and transitive triples on B, So T appears in $\mathcal{C}_r(B) \subset \mathcal{A}_r$.

Lemma 3.16 *If there exists a 2- resolvable* $S(3, 4, v)$*, then there exists an LS*$(2, (3, \{4\}), v)$. *Therefore there exists an LS*$(\lambda, 2, (3, \{4\}), v)$.

Corollary 3.17 *For* $v = 4^n, 2(7^n+1), 2(31^n+1), 2(127^n+1)$, $n \geq 1$, *there is an LARHTS*(v).

Proof By Lemma 3.2, there is a 2- resolvable $S(3, 4, v)$ for $v = 4^n, 2(7^n+1), 2(31^n+1), 2(127^n+1)$, $n \geq 1$. So there is an LS$(2, (3, \{4\}), v)$. There is an LARHTS(4) by Theorem 2.1, and we have the results from Lemma 3.14. □

References

[1] Qingde KANG, Yanxun CHANG. *A completion of the spectrum for large sets of disjoint transitive triple systems*. J. Combin. Theory Ser. A, 1992, **60**(2): 287–294.

[2] Qingde KANG, Jianguo LEI. *A completion of the spectrum for large sets of disjoint Mendelsohn triple systems*. Bull. Inst. Combin. Appl., 1993, **9**: 14–26.

[3] Qinde KANG, Jianguo LEI. *On large sets of hybrid triple systems*. J. Statist. Plann. Inference, 1996, **51**(2): 181–188.

[4] Qingde KANG, Jianguo LEI. *On large sets of resolvable and almost resolvable oriented triple systems*. J. Combin. Des., 1996, **4**(2): 95–104.

[5] Qinde KANG. *Large sets of resolvable MTS and DTS of order* p^n+2. J. Combin. Des., 1996, **4**(5): 301–321.

[6] Qinde KANG, Zihong TIAN. *Large sets of oriented triple systems with resolvability*. Discrete Math., 2000, **212**(2): 199–221.

[7] Junling ZHOU, Yanxun CHANG. *Product constructions for large sets of resolvable MTSs and DTSs*. Australas. J. Combin., 2005, **33**: 47–56.

[8] Junling ZHOU, Yanxun CHANG. *Tripling construction for large sets of resolvable directed triple systems*. Acta Math. Sin. (Engl. Ser.), 2006, **22**(1): 311–318.

[9] Qinde KANG, Hongtao ZHAO. *More large sets of resolvable MTS and resolvable DTS with odd orders*. Discrete Math., 2008, **308**(5-6): 886–895.

[10] Yanxun Chang. *Transitively resolvable idempotent quasigroup and large sets of resolvable Mendelsohn triple systems*. Discrete Math., 2009, **309**: 5926–5931.

[11] Junling ZHOU. *Recursive constructions for large sets of resolvable hybrid triple systems*. Australas. J. Combin., 2006, **36**: 113–122.

[12] Lijun JI, Ruizhong WEI. *The spectrum of 2-idempotent 3-quasigroups with conjugate invariant subgroups*. J. Combin. Des., 2010, **18**(4): 292–304.

[13] Zihong TIAN, Lijun JI. *The spectrum for overlarge sets of directed triple systems*. Sci. China Ser. A, 2007, **50**(10): 1369–1381.

[14] Meihui CHENG, Zihong TIAN. *The spectrum for overlarge sets of hybrid triple systems*. Acta Math. Appl. Sin. Engl. Ser., 2014, **30**(3): 635–646.

[15] Zihong TIAN, Qinde KANG. *A conjecture on the relation between three types of oriented triple systems*. J. Math. Res. Exposition, 2008, **28**(4): 769–778.

[16] Qingde KANG. *The construction of large sets of disjoint Mendelsohn triple systems of order $2n+2$*. Discrete Math., 1991, **90**(2): 199–205.

[17] C. J. COLBOURN, J. H. DINITZ. *Handbook of Combinatorial Designs* (Second Edition). CRC Press, Boca Raton, 2006.

[18] L. TEIRLINCK. *Some new 2-resolvable $S(3,4,v)$s*. Des. Codes Cryptogr., 1994, **4**(1): 5–10.

[19] Jianguo LEI. *On large sets of disjoint Kirkman triple systems*. Discrete Math., 2002, **257**(1): 63–81.

[20] L. TEIRLINCK. *Large sets with holes*. J. Combin. Des., 1993, **1**(1): 69–94.

[21] TH. BETH, D. JUNGNICKEL, H. LENZ. *Design Theory*. Bibliographisches Institut, Mannheim-Wien-Zurich, 1985.

[22] Qinde KANG, Yanxun CHANG. *Symmetric Mendelsohn triple system and large sets of disjoint Mendelsohn triple systems*. Dekker, New York, 1990.

[23] T. P. KIRKMAN. *On a problem in combinations*. Cambridge and Dublin Math. J., 1847, **2**: 191–204.

[24] D. K. RAY-CHAUDHURI, R. M. WILSON. *Solution of Kirkman's school-girl problem*. Proc. Symp. Pure Math., 1971, **19**: 187–203.

[25] Jiaxi LU. *On large sets of disjoint Steiner triple systems* (I, II, III). J. Combin. Theory Ser. A, 1983, **34**(2): 140–146, 147–155, 156–182.

[26] Jiaxi LU. *On large sets of disjoint Steiner triple systems* (IV, V, VI). J. Combin. Theory Ser. A, 1984, **37**(2): 136–163, 164–188, 189–192.

[27] L. TEIRLINCK. *A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems*. J. Combin. Theory Ser. A, 1991, **57**(2): 302–305.

[28] M. J. SHARRY, A. P. STREET. *A doubling construction for overlarge sets of Steiner triple systems*. Ars Combin., 1991, **32**: 143–151.

[29] Qingde KANG, Zihong TIAN. *Large sets of oriented triple systems with resolvability*. Discrete Math., 2000, **212**(3): 199–221.

[30] L. TEIRLINCK. *On the maximum number of disjoint Steiner triple systems*. Discrete Math., 1973, **6**: 299–300.

[31] N. S. MENDELSOHN. *A Natural Generalization of Steiner Triple Systems*. Academic Press, London, 1971.

[32] C. J. COLBOURN, A. ROSA. *Directed and Mendelsohn Triple Systems*. Wiley, New York, 1992.

[33] J. C. BERMOND, A. GERMA, D. SOTTEAU. *Resolvable decomposition of K_n^**. J. Combin. Theory Ser. A, 1979, **26**(2): 179–185.

[34] F. E. BENNETT, D. SOTTEAU. *Almost resolvable decomposition of K_n^**. J. Combin. Theory Ser. B, 1981, **30**(2): 228–232.