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A Note on Almost Topological Groups

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Abstract In this paper, we mainly discuss some generalized metric properties and the cardinal invariants of almost topological groups. We give a characterization for an almost topological group to be a topological group and show that: (1) Each almost topological group that is of countable π-character is submetrizable; (2) Each left λ-narrow almost topological group is λ-narrow; (3) Each separable almost topological group is ω-narrow. Some questions are posed.

Keywords almost topological group; submetrizable; λ-narrow; separable

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1. Introduction

All spaces are T_2 unless stated otherwise. We denote by $\mathbb{N}$ the set of all natural numbers and $\omega = \mathbb{N} \cup \{0\}$. The letter e denotes the neutral element of a group. Readers may refer [3, 5, 7] for notations and terminology not explicitly given here.

A group G endowed with a topology τ is called a semitopological group if the left and right translations of G are continuous. We also say that G is a paratopological group if the multiplication in G is continuous as a mapping of $G \times G$ into G, where $G \times G$ is given product topology. A topological group is a paratopological group with continuous inversion. Obviously, each topological group is a paratopological group, and each paratopological group is a semitopological group. Paratopological groups were discussed and many results have been obtained in [1, 3, 8, 9, 11–14].

In the class of paratopological groups, it is well known that the closure of a subgroup of a paratopological group is not necessarily a subgroup. Therefore, Fernéandez in [6] introduced some class of paratopological groups (that is, almost topological groups) such that the closure of each subgroup of arbitrary such paratopological group must be a subgroup. In this paper, we shall discuss some generalized metric properties and the cardinal invariants of almost topological groups.

2. Preliminaries

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Definition 2.1 ([3]) *Let λ be a cardinal. A subset H of a semitopological group is left λ-narrow (resp., right λ-narrow) if for every open neighborhood U of the neutral element e in G, there exists a subset F of H such that $|F| \leq \lambda$ and $H \subset FU$ (resp., $H \subset UF$). A subset H of a semitopological group is λ-narrow if it is left λ-narrow and right λ-narrow.*

Definition 2.2 ([15]) *A semitopological group is left precompact (resp., right precompact) if for each open neighborhood U of the neutral element e in G, there exists a finite set $A \subset G$ such that $AU = G$ (resp., $UA = G$). A semitopological group is precompact if it is left precompact and right precompact.*

Remark 2.3 Recently, the following results have been obtained:

(1) Every left precompact paratopological group is right precompact [17];

(2) Every left ω-narrow Baire paratopological group is ω-narrow [15];

(3) A dense subgroup of a precompact paratopological group is precompact [18].

However, in the class of topological groups, the following results are well known:

(i) Every left ω-narrow topological group is ω-narrow;

(ii) The subgroup H of an ω-narrow topological group is ω-narrow [3].

Definition 2.4 ([6]) *An almost topological group is a paratopological group (G, τ) which satisfies the following conditions:*

(a) The group G admits a Hausdorff topological group topology γ weaker than τ, and

(b) There exists a local base $\mathscr{B}$ at the neutral element e of the paratopological group (G, τ) such that the set $V = U \setminus \{e\}$ is open in (G, γ) for each $U \in \mathscr{B}$.

We will say that G is an almost topological group with structure $(\tau, \gamma, \mathscr{B})$.

Remark 2.5 (1) It is easy to check that Sorgenfrey line is an almost topological group. However, Sorgenfrey line is not a topological group.

(2) The closure of any subgroup of the product of a family of almost topological groups is a subgroup [6].

(3) Any discrete subgroup of a product of a family of almost topological groups is closed [6].

Recall that a family $\mathscr{U}$ of non-empty open sets of a space X is called a π-base at a point x if for each non-empty open neighborhood V of x in X, there exists $U \in \mathscr{U}$ such that $U \subset V$. The π-character of x in X is defined by

$$\pi\chi(x, X) = \min\{|\mathscr{U}| : \mathscr{U} \text{ is a local } \pi\text{-base at } x \text{ in } X\}.$$

The π-character of X is defined by

$$\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}.$$

The character of x in X is defined by

$$\chi(x, X) = \min\{|\mathscr{U}| : \mathscr{U} \text{ is a neighborhood base at } x \text{ in } X\}.$$

The character of X is defined by

$$\chi(X) = \sup\{\chi(x, X) : x \in X\}.$$

The density of X is defined by

$$d(X) = \min\{|F| : F \subset X, \overline{F} = X\}.$$

The cellurarity of X is defined by

$$c(X) = \sup\{|\mathscr{U}| : \mathscr{U} \text{ is a disjoint family of open subsets of } X\}.$$

3. Generalized metric properties on almost topological groups

First, we shall give a condition under which an almost topological group is a topological group.

Proposition 3.1 *A non-discrete almost topological group G is a topological group if and only if G satisfies the following ($\heartsuit$):*

($\heartsuit$) For each open neighborhood U of the neutral element e there exist a point $y \in U \setminus \{e\}$ and an open neighborhood V of e such that $e \in yV \subset U$.

Proof Let G be an almost topological group with structure $(\tau, \gamma, \mathscr{B})$.

Let (G, τ) be a topological group. For each open neighborhood U of e in τ, there exists a symmetric open neighborhood V of e in τ such that $e \in V^2 \subset U$. Since (G, τ) is non-discrete, there exists a point $y \in V \setminus \{e\}$. Obviously, we have $e \in yV \subset U$.

Conversely, it suffices to show that $e \in \mathrm{int}_\tau(U^{-1})$ for each open neighborhood U of e in τ. By ($\heartsuit$), there exist a point $y \in U \setminus \{e\}$ and an open neighborhood V of e in (G, τ) such that $e \in yV \subset U$. We can assume that $V \in \mathscr{B}$, then $e \in y(V \setminus \{e\}) \subset U$. Since $V \in \mathscr{B}$, we know that $y(V \setminus \{e\})$ is an open neighborhood of e in (G, γ). Therefore, $(y(V \setminus \{e\}))^{-1}$ is also a neighborhood of e in γ, thus $e \in \mathrm{int}_\gamma((y(V \setminus \{e\}))^{-1})$. Since $\gamma \subset \tau$, the set $\mathrm{int}_\gamma((y(V \setminus \{e\}))^{-1}) \subset U^{-1}$ is an open neighborhood of e in (G, τ). Therefore, $e \in \mathrm{int}_\tau(U^{-1})$. □

Example 3.2 *There exists a Hausdorff paratopological group G which satisfies ($\heartsuit$). However, it is not a topological group.*

Proof Consider the additive group $(\mathbb{R}, +)$. Fix a natural number k and put $U_n(k) = k(\mathbb{N} \cup \{0\}) + (-\frac{1}{n}, \frac{1}{n})$ for each $n \in \mathbb{N}$. Let $\mathscr{U} = \{U_n(k) : k, n \in \mathbb{N}\}$. Then there exists a topology σ on $\mathbb{R}$ such that $G = (\mathbb{R}, \sigma)$ is a Hausdorff paratopological group and the family $\mathscr{U}$ is a local base at 0 in G, see [10]. Obviously, G is not a topological group and satisfies ($\heartsuit$). □

The following question is still open in the class of paratopological groups.

Question 3.3 ([2, Problem 20]) *Is every regular first countable paratopological group submetrizable?*

However, in the class of almost topological groups, the following theorem gives a positive answer to Question 3.3.

Theorem 3.4 *Let G be an almost topological group that is of countable π-character. Then G is submetrizable.*

Proof Let G be an almost topological group with structure $(\tau, \gamma, \mathscr{B})$, and let $\{U_n : n \in w\}$ be a countable π-base at the neutral element e in (G, τ). If G is discrete, then it is obvious that G is submetrizable. Therefore, we may assume that G is non-discrete. For each $n \in \omega$, take $x_n \in U_n$. Then we can find $B_n \in \mathscr{B}$ such that $x_n B_n \subset U_n$ since (G, τ) is a paratopological group. Note that G is a non-discrete almost topological group, hence the set $B_n \setminus \{e\}$ is a non-empty open set in (G, γ) for each $n \in \omega$. So $x_n(B_n \setminus \{e\}) = x_n B_n \setminus \{x_n\}$ is also an open set in (G, γ). Then the family $\mathscr{V} = \{x_n B_n \setminus \{x_n\} : n \in \omega\}$ is countable. We claim that $\mathscr{V}$ is a π-base at the neutral element e of (G, γ). Indeed, let W be an arbitrary open neighbourhood of the neutral element e in (G, γ). Clearly, W is also an open neighbourhood of the neutral element e in (G, τ), hence there exists $n \in \omega$ such that $U_n \subset W$. Therefore, we have $x_n B_n \setminus \{x_n\} \subset x_n B_n \subset U_n \subset W$. Thus $\mathscr{V}$ is a π-base at the neutral element e of (G, γ). It is well known that a Hausdorff topological group with a countable π-character is metrizable, so G is submetrizable. □

By the proof of Theorem 3.4, we have the following.

Corollary 3.5 *If G is an almost topological group with structure $(\tau, \gamma, \mathscr{B})$, then $\pi\chi(G, \gamma) \leq \pi\chi(G, \tau)$.*

However, the following question is still open.

Question 3.6 Let G be an almost topological group with structure $(\tau, \gamma, \mathscr{B})$. Does the equation $\chi(G, \tau) = \chi(G, \gamma)$ hold?

The following question is posed by Liu and Lin.

Question 3.7([14, Question 2.2]) Let G be a first-countable paratopological group. If G is a p-space, is G developable?

A space X is a wΔ-space [4] if there exists a sequence $\{\mathscr{H}_n\}$ of open covers of X such that if $x_n \in \mathrm{st}(x, \mathscr{H}_n)$ for each $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point in X.

Definition 3.8 ([19]) *Let X be a space and $\{\mathscr{P}_n\}_n$ a sequence of collections of open subsets of X.*

(1) $\{\mathscr{P}_n\}_n$ is called a development for X if $\{st(x, \mathscr{P}_n)\}_n$ is a neighborhood base at x in X for each point $x \in X$.

(2) X is called developable, if X has a development.

(3) X is called Moore, if X is regular and developable.

Clearly, each developable space is a wΔ-space.

The following corollary gives a partial answer to Question 3.7.

Corollary 3.9 *Let G be an almost topological group that is of countable π-character. If G is a wΔ-space, then G is developable.*

Proof It follows from Theorem 3.4 that G is submetrizable. Then (G, τ) is developable since G

is a wΔ-space [7]. □

It is well known that a topological group with countable pseudocharacter is submetrizable. Moreover, the authors in [11] have given a paratopological group which is of countable pseudocharacter and non-submetrizable. Therefore, we have the following question.

Question 3.10 Let G be an almost topological group that is of countable pseudocharacter. Is G submetrizable?

The authors in [11] showed that a Moore paratopological group needs not be metrizable. Indeed, that paratopological group is an almost topological group. However, the following questions are still open in the class of paratopological groups.

Question 3.11 Let G be a regular paratopological group or an almost topological group. If G is regular and has a uniform base, is G metrizable?

Question 3.12 Let G be a paratopological group or an almost topological group. If G is regular and has a point-countable base, is G metrizable?

4. Some cardinal invariants of almost topological groups

The following question was posed by Guran.

Question 4.1 ([15, Question 2.2]) Is every left ω-narrow paratopological group right ω-narrow?

The following theorem gives a partial answer to Question 4.1.

Theorem 4.2 *Let G be an almost topological group with structure $(\tau, \gamma, \mathscr{B})$. If (G, τ) is left λ-narrow, then (G, τ) is λ-narrow.*

Proof It suffices to show that (G, τ) is right λ-narrow. If (G, τ) is discrete, then it is obvious that (G, τ) is right λ-narrow. Therefore, we may assume that (G, τ) is non-discrete. Let U be an arbitrary open neighborhood of the identity e in (G, τ). Since (G, τ) is left λ-narrow, we can find a subset F_0 of (G, τ) such that $|F_0| \leq \lambda$ and $F_0U = G$. Since $e \in U$, there exists $B_0 \in \mathcal{B}$ such that $B_0 \subset U$ and $B_0 \setminus \{e\}$ is open in (G, γ). Since G is non-discrete, we have $B_0 \setminus \{e\} \neq \emptyset$. Let $y \in B_0 \setminus \{e\}$. Then $(B_0 \setminus \{e\})y^{-1} \in \gamma$ and $e \in (B_0 \setminus \{e\})y^{-1}$. Since (G, τ) is left λ-narrow, topological group (G, γ) is λ-narrow. So there exists a subset F_1 of G such that $|F_1| \leq \lambda$ and $(B_0 \setminus \{e\})y^{-1}F_1 = G$. Thus $G = U(y^{-1}F_1)$ and $|y^{-1}F_1| \leq \lambda$. □

Theorem 4.3 *Let G be an almost topological group with structure $(\tau, \gamma, \mathscr{B})$. Then (G, γ) is λ-narrow if and only if (G, τ) is λ-narrow.*

Proof Obviously, if (G, τ) is discrete, then theorem holds. Therefore, we may assume that (G, τ) is non-discrete.

Sufficiency. Since (G, γ) is weaker than (G, τ), it is obvious that (G, γ) is λ-narrow if (G, τ) is λ-narrow.

Necessity. By Theorem 4.2, it suffices to show that (G, τ) is left λ-narrow. Let U be an arbitrary non-empty open neighborhood of e in (G,τ). Then there exists $B \in \mathscr{B}$ such that $B \subset U$ and $B \setminus \{e\}$ is a non-empty open set in (G,γ). Take $x \in B \setminus \{e\}$. Since (G,γ) is topological group, there exists an open neighborhood V of e in (G,γ) such that $xV \subset B\setminus\{e\}$. Since (G,γ) is λ-narrow, there exists a subset F of G such that $|F| \leq \lambda$ and $FV = G$. Let $H = Fx^{-1}$. Clearly, $|H| \leq \lambda$. Then

$$G = FV = Fx^{-1}xV \subset H(B \setminus \{e\}) \subset HU.$$

Therefore, (G,τ) is left λ-narrow. □

Theorem 4.4 *If G is an almost topological group with structure $(\tau,\gamma,\mathscr{B})$, then $d(G,\gamma) = d(G,\tau)$.*

Proof Obviously, we have $d(G,\tau) \geq d(G,\gamma)$. Next, we shall show that $d(G,\gamma) \geq d(G,\tau)$. Clearly, if (G,τ) is discrete, then theorem holds. Therefore, we may assume that (G,τ) is non-discrete. Let D be a dense subset of (G,γ). Next we shall show that D is a dense subset in (G,τ). Indeed, for an arbitrary non-empty open set U in (G,τ), take $x \in U$. There exists $B \in \mathscr{B}$ such that $xB \subset U$ and $xB \setminus \{x\}$ is open in (G,γ). Since (G,γ) is non-discrete, the interior of $xB \setminus \{x\}$ in (G,γ) is non-empty. So we have $(xB \setminus \{x\}) \cap D \neq \phi$, and therefore $(xB \setminus \{x\}) \cap D \subset xB \cap D \subset U \cap D \neq \phi$. Therefore, D is dense in (G,τ). □

Theorem 4.5 *Every separable almost topological group is ω-narrow.*

Proof Let G be an almost topological group with structure $(\tau,\gamma,\mathscr{B})$. Since (G,τ) is separable, (G,γ) is also separable since (G,γ) is weaker than (G,τ). So (G,γ) is w-narrow [3]. Therefore, Theorem 4.3 implies that (G,τ) is w-narrow. □

Theorem 4.6 *If G is an almost topological group with structure $(\tau,\gamma,\mathscr{B})$, then $c(G,\tau) = c(G,\gamma)$.*

Proof Obviously, $c(G,\gamma) \leq c(G,\tau)$. Next, it suffices to show that $c(G,\tau) \leq c(G,\gamma)$. Let $c(G,\tau) = \kappa$. If (G,τ) is discrete, then the theorem holds. Therefore, we may assume that (G,τ) is non-discrete. Let $\mathscr{U}$ be the maximum family of pairwise disjoint open sets in (G,τ). Since $c(G,\tau) = \kappa$, we may assume that $\mathscr{U} = \{U_\alpha : \alpha \in \kappa\}$. For each $\alpha \leq \kappa$, take $x_\alpha \subset U_\alpha$, then there exists $B_\alpha \in \mathscr{B}$ such that $x_\alpha B_\alpha \subset U_\alpha$. By Definition 2.4, we know that $x_\alpha B_\alpha \setminus \{x_\alpha\}$ is a non-empty open set in (G,γ). Let

$$\mathscr{V} = \{x_\alpha B_\alpha \setminus x_\alpha : \alpha \leq \kappa\}.$$

Clearly, $\mathscr{V}$ is the family of pairwise disjoint open sets in (G,γ), hence $|\mathscr{V}| \leq c(G,\gamma)$. Moreover, it is obvious that $|\mathscr{V}| = |\mathscr{U}|$. Thus $\kappa \leq c(G,\gamma)$. Therefore, we have $c(G,\tau) \leq c(G,\gamma)$. □

Theorem 4.7 *Suppose that G is an almost topological group with structure $(\tau,\gamma,\mathscr{B})$. If $c(G,\tau) \leq \omega$, then (G,τ) is ω-narrow.*

Proof Suppose $c(G,\tau) \leq \omega$. Then it follows from Theorem 4.6 that $c(G,\gamma) \leq \omega$. Since (G,γ) is topological group, (G,γ) is ω-narrow [3]. By Theorem 4.3, (G,τ) is ω-narrow. □

In [16], the author showed that a paratopological group G is λ-narrow if it contains a dense λ-narrow subgroup. The following theorem is complementary to I. Sánchez's result.

Theorem 4.8 *Every subgroup H of an ω-narrow almost topological group G is ω-narrow.*

Proof Let G be an almost topological group with structure $(\tau, \gamma, \mathscr{B})$. If (G, τ) is discrete, then the theorem holds. Therefore, we may assume that (G, τ) is non-discrete. Suppose H is an arbitrary subgroup of G. By Theorem 4.3, (G, γ) is ω-narrow, so H is w-narrow in (G, γ). For an arbitrary open neighborhood U of the neutral element e in (G, τ), take $x \in U$. Then there exists $B \in \mathscr{B}$ such that $xB \subset U$ and $xB \setminus \{x\}$ is a non-empty open subset of (G, γ). So there exists a countable subset A of H such that $(xB \setminus \{x\}) \cdot A \supseteq H$. Then

$$H \subset (xB \setminus \{x\}) \cdot A \subset xB \cdot A \subset UA.$$

Therefore, H is w-narrow in (G, τ). $\square$

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