Fractional Domination of the Cartesian Products in Graphs

Baogen XU
Department of Mathematics, East China Jiaotong University, Jiangxi 330013, P. R. China

Abstract Let $G = (V, E)$ be a simple graph. For any real function $g : V \rightarrow \mathbb{R}$ and a subset $S \subseteq V$, we write $g(S) = \sum_{v \in S} g(v)$. A function $f : V \rightarrow [0, 1]$ is said to be a fractional dominating function (FDF) of $G$ if $f(N[v]) \geq 1$ holds for every vertex $v \in V(G)$. The fractional domination number $\gamma_f(G)$ of $G$ is defined as $\gamma_f(G) = \min \{ f(V) | f \text{ is an FDF of } G \}$. The fractional total dominating function $f$ is defined just as the fractional dominating function, the difference being that $f(N[v]) \geq 1$ instead of $f(N[v]) \geq 1$. The fractional total domination number $\gamma_f^t(G)$ of $G$ is analogous. In this note we give the exact values of $\gamma_f(C_m \times P_n)$ and $\gamma_f^t(C_m \times P_n)$ for all integers $m \geq 3$ and $n \geq 2$.

Keywords Cartesian products; fractional domination number; fractional total domination number

MR(2010) Subject Classification 05C69

1. Introduction

We use Bondy and Murty [1] for terminology and notation not defined here and consider finite simple graph only.

Let $G = (V, E)$ be a graph. The open neighborhood of a vertex $v$ in $G$ is $N(v) = \{ u \in V | uv \in E(G) \}$, while $N[v] = N(v) \cup \{ v \}$ is the closed neighborhood of $v$. $C_n$ and $P_n$ denote the cycle and the path of order $n$, respectively. If $u, v \in V(G)$, then $u \sim v$ denotes $u$ is adjacent to $v$ in $G$.

For any two disjoint graphs $G$ and $H$, the Cartesian product $G \times H$ is defined as follows:

$$V(G \times H) = V(G) \times V(H),$$

$$E(G \times H) = \{(u_1, v_1)(u_2, v_2) | (u_1 = u_2 \text{ and } v_1 \sim v_2) \text{ or } (v_1 = v_2 \text{ and } u_1 \sim u_2) \}.$$ 

Let $G = (V, E)$ be a graph. For any real function $g : V \rightarrow \mathbb{R}$ and a subset $S \subseteq V$, we write $g(S) = \sum_{v \in S} g(v)$.

Hare [3] and Stewart [4] introduced the following concept of the fractional domination and the fractional total domination in graphs.
Let $G = (V, E)$ be a graph. A function $f : V \to [0, 1]$ is said to be a fractional dominating function (FDF) of $G$ if $f(N[v]) \geq 1$ holds for every vertex $v \in V(G)$. The fractional domination number $\gamma_f(G)$ of $G$ is defined as $\gamma_f(G) = \min \{ f(V) | f \text{ is an FDF of } G \}$.

A fractional total dominating function (FTDF) $f$ of $G$ is defined similarly, the difference being that $f(N(v)) \geq 1$ instead of $f(N[v]) \geq 1$. The fractional total domination number $\gamma^t_f(G)$ of $G$ is defined as $\gamma^t_f(G) = \min \{ f(V) | f \text{ is an FTDF of } G \}$.

Fractional packing numbers are defined analogously; a real function $f : V(G) \to [0, 1]$ is a fractional packing function of $G$ if $f(N[v]) \leq 1$ holds for every vertex $v \in V(G)$. A fractional packing function $f$ is maximal if for every $u \in V(G)$ with $f(u) < 1$, there exists a vertex $v \in N[u]$ such that $f(N[v]) = 1$. The upper fractional packing number $P_f(G)$ of $G$ is defined as $P_f(G) = \max \{ f(V) | f \text{ is a maximal packing function of } G \}$.

Lemma 1.1 ([2]) For any graph $G$, $P_f(G) = \gamma_f(G)$.

Lemma 1.2 ([2]) For any $r$-regular graph $G$ ($r \geq 1$), then

1. $\gamma_f(G) = \frac{n}{r+1}$;
2. $\gamma^t_f(G) = \frac{n}{r}$.

For the Cartesian product $P_m \times P_n$, Hare [3] and Stewart [4] gave an exact formula for $\gamma_f(P_2 \times P_n)$ and some bounds of $\gamma_f(P_m \times P_n)$ for $3 \leq m \leq n$.

Lemma 1.3 ([2]) For all integers $n \geq 1$, then

1. when $n \equiv 1 \pmod{2}$, $\gamma_f(P_2 \times P_n) = \frac{n+1}{2}$;
2. when $n \equiv 0 \pmod{2}$, $\gamma_f(P_2 \times P_n) = \frac{n^2+2n}{2(n+1)}$.

However, there is no known formula of $\gamma_f(P_m \times P_n)$ for $3 \leq m \leq n$. It is very difficult to give the exact value of $\gamma_f(P_m \times P_n)$. Fisher [5] has tried without success to find such a formula for $\gamma_f(P_3 \times P_n)$. Up to now, few exact value of $\gamma_f(P_m \times P_n)$ is known when $3 \leq m \leq n$.

We are interested in the Cartesian products $C_m \times P_n$. In this note we give exact formulas of $\gamma_f(C_m \times P_n)$ and $\gamma^t_f(C_m \times P_n)$ for all integers $m \geq 3$ and $n \geq 2$.

2. Fractional total domination number for $C_m \times P_n$

Theorem 2.1 For all integers $m \geq 3$ and $n \geq 2$, we have $\gamma^t_f(C_m \times P_n) = \frac{m}{4(n+1)} \left( n^2 + n + 2 \left\lceil \frac{n}{2} \right\rceil \right)$.

Proof Let $G = C_m \times P_n$, $V(G) = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n \}$, and

$E(G) = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n - 1 \} \cup \{(i, j) | i \leq m, 1 \leq j \leq n \}$,

where $(m + 1, j) = (1, j)$ for every integer $j$ $(1 \leq j \leq n)$.

Define an FTDF $f$ of $G$ as follows:

Let $f((i, j)) = x_j$ ($i = 1, 2, \ldots, m$) for every integer $j$ $(1 \leq j \leq n)$.

Case 1 $n = 2k + 1$; for some $k \in \mathbb{N}^+$.

Let $x_{2j} = 0$ ($1 \leq j \leq k$) and $x_{2j-1} = \frac{1}{2}$ for every integer $j$ ($1 \leq j \leq k + 1$);

It is easy to check that $f(N(i, j)) = 1$ holds for all vertices $(i, j) \in V(G)$, and hence $f$ is an
Fractional domination of the Cartesian products in graphs

**FTDF of G**, which means

\[ \gamma^0_f(G) \leq f(V(G)) = \frac{m(n + 1)}{4}. \]  

(1)

On the other hand, let \( g \) be an **FTDF of G** such that \( \gamma^0_f(G) = g(V(G)) \). By the definition, for every vertex \( (i, 2j - 1) \in V(G) \) \( (1 \leq i \leq m, 1 \leq j \leq k + 1) \), we have \( g(N(i, 2j - 1)) \geq 1 \), and hence \( 2g(V(G)) = \sum_{i=1}^{m} \sum_{j=1}^{k+1} g(N(i, 2j - 1)) \geq m(k + 1) \), i.e.,

\[ \gamma^0_f(G) = g(V(G)) \geq \frac{m(k + 1)}{2} = \frac{m(n + 1)}{4}. \]

Combining with (1), we have \( \gamma^0_f(G) = \frac{m(n + 1)}{4} \), and the theorem holds for all odd \( n \geq 3 \).

**Case 2** \( n = 2k; \) for some \( k \in N^+ \).

Let \( x_{2j} = \frac{j}{n + 1} \) and \( x_{2j-1} = \frac{n-2j+2}{2(n+1)} \) for every integer \( j \) \( (1 \leq j \leq k) \).

It is easy to see that \( f(N(i, j)) = 1 \) holds for all vertices \( (i, j) \in V(G) \), and hence \( f \) is an **FTDF of G**, which means

\[ \gamma^0_f(G) \leq f(V(G)) = m \sum_{j=1}^{k} \left( \frac{j}{n + 1} + \frac{n - 2j + 2}{2(n+1)} \right) = \frac{mk(n+2)}{2(n+1)} = \frac{m(n^2 + 2n)}{4(n+1)}. \]

Next we prove that \( \gamma^0_f(G) \geq \frac{m(n^2 + 2n)}{4(n+1)} \).

When \( n = 2, \) \( G \) is a 3-regular graph. By Lemma 1.2, Theorem 2.1 holds. Next suppose that \( n \geq 4 \) and \( n = 2k \) is even.

Assume, to the contrary, that

\[ \gamma^0_f(G) < \frac{m(n^2 + 2n)}{4(n+1)}. \]  

(2)

Let \( g \) be such an **FTDF of G** that \( \gamma^0_f(G) = g(V(G)) \), and for each \( j = 1, 2, \ldots, n \), let \( C(j) = \{(i, j)| 1 \leq i \leq m\} \subseteq V(G) \). Clearly, \( V(G) = \bigcup_{i=1}^{n} (C(2i - 1) \cup C(2i)) \), thus, there exists an odd integer \( r \) \( (1 \leq r \leq n) \), so that

\[ g(C(r)) + g(C(r + 1)) \leq \frac{2}{n} g(V(G)) = \frac{2}{n} \gamma^0_f(G) < \frac{m(n+2)}{2(n+1)}. \]

Let \( g(N(j)) = \sum_{i=1}^{m} g(N(i, j)) \) for every integer \( j \in \{1, 2, \ldots, n\} \). Since \( g(N(i, j)) \geq 1 \) holds for all vertices \( (i, j) \in V(G) \), we have \( g(N(j)) \geq m \) holds for all integers \( j \in \{1, 2, \ldots, n\} \). Note that \( r \) is odd and \( n = 2k \) is even. We have

\[ 2g(V(G)) + g(C(r)) + g(C(r + 1)) \]

\[ = g(N(1)) + g(N(3)) + \cdots + g(N(r)) + g(N(r + 1)) + g(N(r + 3)) + \cdots + g(N(n)) \]

\[ \geq \left( \frac{n}{2} + 1 \right)m. \]

And hence, we have

\[ 2g(V(G)) \geq \left( \frac{n}{2} + 1 \right)m - (g(C(r)) + g(C(r + 1)) \]

\[ \geq (\frac{n}{2} + 1)m - \frac{m(n+2)}{2(n+1)} = \frac{m(n^2 + 2n)}{2(n+1)}, \]
\[ \gamma_f^0(G) = g(V(G)) \geq \frac{m(n^2 + 2n)}{4(n + 1)}. \]

This contradicts (3). Combining with (2), we have proved that \( \gamma_f^0(G) = \frac{m(n^2 + 2n)}{4(n + 1)} \) holds for all even \( n \geq 2 \). The proof of Theorem 2.1 is completed. \( \square \)

3. Fractional domination number for \( C_m \times P_n \)

The following two lemmas are useful to obtain our main results.

**Lemma 3.1** Let \( A \) and \( B \) be both matrices of order \( n \geq 2 \), and

\[
A = \begin{pmatrix} 3 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 3 \\ \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 3 \\ \end{pmatrix}.
\]

Then
1. \( A_n = \det A = \frac{a^{n+1}-b^{n+1}}{a-b}, \) where \( a = \frac{3+\sqrt{5}}{2} \) and \( b = \frac{3-\sqrt{5}}{2} \);
2. \( B_n = \det B = \frac{1}{5}(A_n + A_{n-1} + (-1)^{n-1}), \) where let \( A_0 = 1. \)

**Proof** We use the induction on \( n \geq 1 \).

When \( n = 1 \), clearly, \( A_1 = 3 = a + b \), and \( B_1 = 1 \), and the result follows.

We suppose that Lemma 3.1 is true for all matrices with determinants of order \( k \leq n - 1 \). Now we consider the two \( n \times n \) matrices \( A \) and \( B \). Note that \( a + b = 3 \) and \( ab = 1 \). By the induction hypothesis, we have

\[
A_n = 3A_{n-1} - A_{n-2} = (a + b)\frac{a^n - b^n}{a-b} - ab\frac{a^{n-1} - b^{n-1}}{a-b} = \frac{a^{n+1} - b^{n+1}}{a-b}
\]

\[
B_n = A_{n-1} - B_{n-1} = A_{n-1} - \frac{1}{5}(A_{n-1} + A_{n-2} + (-1)^{n-2})
\]

\[
= \frac{1}{5}(4A_{n-1} - A_{n-2} + (-1)^{n-1}) = \frac{1}{5}(A_n + A_{n-1} + (-1)^{n-1}).
\]

So, Lemma 3.1 is true for all determinants of order \( n \), this proof is completed. \( \square \)

**Lemma 3.2** Let \( X^T = (x_1, x_2, \ldots, x_n) \), and \( C^T = (1, 1, 1, \ldots, 1) \) be an \( n \)-dimensional vector \((n \geq 2)\). Then the linear equation

\[ AX = C \]

has the unique solution \((x_1, x_2, \ldots, x_n)\) which satisfies the following two conditions:
1. \( x_1 = x_n = \frac{B_n}{A_n}, \) and \( x_i = x_{n+1-i} \) \((1 \leq i \leq \lceil \frac{n}{2} \rceil)\);
2. \( 0 \leq x_i \leq 1 \) \((1 \leq i \leq n)\),

where \( A, A_n \) and \( B_n \) are defined as in Lemma 3.1.

**Proof** (1) Since \( A_n \neq 0 \), the linear equation \((*)\) has the unique solution \((x_1, x_2, \ldots, x_n)\), from the uniqueness of the solution and the symmetry of \( A \), and by Cramer’ Rule, we have \( x_1 = x_n = \frac{B_n}{A_n}, \) and \( x_i = x_{n+1-i} \) \((1 \leq i \leq \lceil \frac{n}{2} \rceil)\).
(2) When $2 \leq n \leq 6$, it is easy to check that $0 \leq x_i \leq 1$ ($1 \leq i \leq n$). The solution $(x_1, x_2, \ldots, x_n)$ is listed in the proof of Theorem 3.3 (1) for every $n \in \{2, 3, 4, 5, 6\}$.

Next we suppose $n \geq 7$.

Now we prove that $x_i \geq 0$ holds for every integer $i$ ($1 \leq i \leq n$).

Assume, to the contrary, that there exists an integer $i$ such that $x_i < 0$.

Let $r = x_j = \min\{x_i | 1 \leq i \leq n\}$. Note that $a = \frac{3 + \sqrt{5}}{2}$, $b = \frac{3 - \sqrt{5}}{2}$, $ab = 1$, we have $A_n = aA_{n-1} + b^n$, $A_{n-1} = bA_n - b^{n+1}$. By Lemma 3.1, we have

$$x_1 = x_n = \frac{B_n}{A_n} = \frac{1}{5}A_n + A_{n-1} + (-1)^{n-1} = \frac{1 + b}{5} + \frac{(-1)^{n-1} - b^{n+1}}{5A_n}.$$ 

Note that $\frac{1}{3} b \leq b = \frac{3 + \sqrt{5}}{2} \leq \frac{3}{2}$ and $A_n \geq 6$, we have $0 \leq x_1 \leq \frac{1}{3}$. It is easy to see from the linear equation $AX = C$ that $x_2 = x_{n-1} = 1 - 3x_1 \geq 0$, and hence $3 \leq j \leq n - 2$. Since $x_{j-1} + 3x_j + x_{j+1} = 1$, we have $x_{j-1} \geq \frac{1 - 3r}{2}$ or $x_{j+1} \geq \frac{1 - 3r}{2}$.

If $x_{j-1} \geq \frac{1 - 3r}{2}$, since $x_{j-2} + 3x_{j-1} + x_j = 1$, and note that $r \leq 0$, we have $x_{j-2} = 1 - 3x_{j-1} - r \leq 1 - \frac{3}{3}(1 - 3r) - r = \frac{1}{3} - r < 1 - r$, this contradicts the choice of $r$.

If $x_{j+1} \geq \frac{1 - 3r}{2}$, similarly, since $x_j + 3x_{j+1} + x_{j+2} = 1$, we have $x_{j+2} = 1 - 3x_{j+1} - r \leq 1 - \frac{3}{3}(1 - 3r) - r = \frac{1}{3}r < 1 - r$, this contradicts the choice of $r$ as well.

Thus, $x_i \geq 0$ holds for every integer $i$ ($1 \leq i \leq n$), implying that $x_i \leq 1$ holds for every integer $i$ ($1 \leq i \leq n$). We have completed the proof of Lemma 3.2. □

**Theorem 3.3** For all integers $m \geq 3$ and $n \geq 2$, then

1. $\gamma_f(C_m \times P_2) = \frac{5}{2} m$, $\gamma_f(C_m \times P_3) = \frac{5}{2} m$, $\gamma_f(C_m \times P_4) = \frac{10}{3} m$, $\gamma_f(C_m \times P_5) = \frac{10}{3} m$.

2. When $n \geq 7$, $\gamma_f(C_m \times P_n) = \frac{(5n + 2)A_n + 2A_{n-1} + 2(1)^n - 1}{25A_n}$, where $A_n = \frac{(3 + \sqrt{5})^{n+1} - (3 - \sqrt{5})^{n+1}}{2n+1 + \sqrt{5}}$ for each integer $n \geq 1$.

**Proof** Let $G = C_m \times P_n$, and $V(G)$ and $E(G)$ be the same as in the proof of Theorem 2.1.

Next we define a maximum packing function $f$ of $G$ such that $f(N[v]) = 1$ holds for every vertex $v \in V(G)$.

For every vertex $(i, j) \in V(G)$, define $f((i, j)) = x_i$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, m$). $S(n) = \sum_{i=1}^{n} x_i$, clearly, $f(V(G)) = mS(n)$.

1. When $n = 2$; let $(x_1, x_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$, $S(2) = \frac{1}{2}$;
   when $n = 3$; let $(x_1, x_2, x_3) = \left(\frac{5}{6}, \frac{1}{2}, \frac{1}{2}\right)$, $S(3) = \frac{5}{6}$;
   when $n = 4$; let $(x_1, x_2, x_3, x_4) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, $S(4) = \frac{1}{2}$;
   when $n = 5$; let $(x_1, x_2, x_3, x_4, x_5) = \left(\frac{5}{6}, \frac{3}{6}, \frac{1}{2}, \frac{3}{6}, \frac{5}{6}\right)$, $S(5) = \frac{10}{6}$;
   when $n = 6$; let $(x_1, x_2, x_3, x_4, x_5, x_6) = \left(\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}\right)$, $S(6) = \frac{30}{6}$.

   It is easy to see that $f(N[v]) = 1$ holds for every vertex $v \in V(G)$, and hence $f$ is a maximum packing function. By Lemma 1.1, these five equalities in Theorem 3.3 hold.

2. When $n \geq 7$, let $(x_1, x_2, \ldots, x_n)$ be the unique solution of the linear equation (*). It is easy to see from Lemma 3.2 that $f$ is a maximum packing function of $G$. And

$$4(x_1 + x_n) + 5(x_2 + x_3 + \cdots + x_{n-1}) = C^T A X = C^T C = n.$$
By Lemmas 3.1 and 3.2, we have
\[ S(n) = \sum_{i=1}^{n} x_i = \frac{n + x_1 + x_n}{5} = \frac{n + 2B_n}{5A_n} = \frac{(5n + 2)A_n + 2A_{n-1} + 2(-1)^{n-1}}{25A_n}. \]

By Lemma 1.1,
\[ \gamma_f(G) = P_f(G) = f(V(G)) = mS(n) = \frac{(5n + 2)A_n + 2A_{n-1} + 2(-1)^{n-1}}{25A_n} m, \]
where \( A_n = \frac{a^{n+1} - b^{n+1}}{a-b} = \frac{(3+\sqrt{5})^{n+1}-(3-\sqrt{5})^{n+1}}{2n+1+\sqrt{5}}. \)

We have completed the proof of Theorem 3.3. \( \square \)

Acknowledgements I am very grateful to the referees for their careful reading with corrections.

References