A New Characterization of Simple $K_4$-Groups

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Abstract In this paper, we characterize some simple $K_4$-groups only by using the group order and largest element orders, where a simple $K_4$-group is a simple group of order containing exactly four distinct primes.

Keywords finite group; the largest element order; the second largest element order; simple $K_4$-group; characterization

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1. Introduction

Shi put forward the approach to characterizing a finite group by using the group order and the set of element-orders in the 1980’s. At present, this characterization for finite simple groups was finished (some results can be seen in [1–8]). To weaken the quantitative condition, He and Chen began to characterize a finite group only by using the group order and the largest element order in 2009, and proved that simple $K_3$-groups, sporadic simple groups, some alternating groups, and some simple groups of Lie Type can be uniquely determined by the group order and largest element orders [9–17]. To continue this work, in this paper, we characterize some simple $K_4$-groups via the group order and largest element orders.

The groups mentioned in this paper are all finite groups, the number in bracket “( )” behind a group is the order of the group, e.g., $L_2(7)(2^3 \cdot 3 \cdot 7)$ means that $L_2(7)$ is of order $2^3 \cdot 3 \cdot 7$. We use $\pi_e(G)$ to denote the set of orders of elements in $G$, $k_1(G)$ and $k_2(G)$ to denote the largest element order and second largest element order of $G$ respectively, and $\pi(G)$ is the set of all prime divisors of $|G|$. Let $\Gamma(G)$ denote the prime graph of $G$ and $t(G)$ is the number of connected components of $\Gamma(G)$. We denote by $\{\pi_i, i = 1, \ldots, t(G)\}$ the sets of vertex of the connected components of the prime graph, and if the order of $G$ is even, denote by $\pi_1$ the component containing 2 (see [18]).

2. Preliminary results

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Lemma 2.1 ([19]) Suppose that $G$ is a simple $K_4$-group. Then $G$ is isomorphic to one of the following groups:

(1) $A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, Sz(8), Sz(32), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4(4), S_4(7), S_4(9), S_6(2), O_q^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), 3D_4(2), 2F_4(2)’$;

(2) $L_2(q)$, where $q$ is a prime power satisfying $q(q^2-1) = (2, q-1)2^{n_1} \cdot 3^{n_2} \cdot p^{n_3} \cdot r^{n_4}$, where $n_i$ $(1 \leq i \leq 4)$ are positive integers and $p, q, r$ are distinct primes.

Lemma 2.2 Suppose that $G$ has more than one prime graph components. Then one of the following holds:

(1) $G$ is a Frobenius group or a 2-Frobenius group;

(2) $G$ has a normal series $1 \leq H \leq K \leq G$, such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ a non-abelian simple group, where $\pi_1$ is the prime graph component containing 2, $H$ is a nilpotent group, and $|G/K| = |\text{Out}(K/H)|$.

**Proof** The lemma follows from Theorem A and Lemma 3 in [18]. □

Lemma 2.3 $\pi_e(S_4(7)) = \{2, 3, 4, 5, 6, 7, 8, 12, 14, 21, 24, 25, 28, 42, 56\}$, $\pi_e(S_4(9)) = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 20, 24, 30, 40, 41\}$. And therefore $k_1(S_4(7)) = 56$, $k_2(S_4(7)) = 42$, $k_1(S_4(9)) = 41$, $k_2(S_4(9)) = 40$.

**Proof** The lemma follows from Corollary 2 in [20]. □

Lemma 2.4 $\pi_e(L_3(17)) = \{2, 3, 4, 6, 8, 9, 12, 16, 17, 18, 24, 32, 34, 36, 48, 68, 72, 96, 136, 144, 272, 288, 307\}$, and therefore, $k_1(L_3(17)) = 307$, $k_2(L_3(17)) = 288$.

**Proof** The lemma follows from Corollary 3 in [21]. □

Lemma 2.5 Let $G$ be a simple $K_4$-group, except that $L_2(q)$. Then $|G|$, $k_1(G)$ and $k_2(G)$ are as in Table 1:

| $G$    | $|G|$ | $k_1(G)$ | $k_2(G)$ |
|--------|-------|----------|----------|
| $A_7$  | $2^3 \cdot 3^2 \cdot 5 \cdot 7$ | 7        | 6        |
| $A_8$  | $2^6 \cdot 3^2 \cdot 5 \cdot 7$ | 15       | 7        |
| $A_9$  | $2^6 \cdot 3^4 \cdot 5 \cdot 7$ | 15       | 12       |
| $A_{10}$ | $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ | 21       | 15       |
| $M_{11}$ | $2^4 \cdot 3^2 \cdot 5 \cdot 11$ | 11       | 8        |
| $M_{12}$ | $2^6 \cdot 3^3 \cdot 5 \cdot 11$ | 11       | 10       |
| $J_2$  | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ | 15       | 12       |
| $Sz(8)$ | $2^6 \cdot 5 \cdot 7 \cdot 13$ | 24       | 21       |
| $Sz(32)$ | $2^{10} \cdot 5^2 \cdot 31 \cdot 41$ | 20       | 15       |
Let \( K \) be an element order, where \( p \) is a prime but not 2.

### 3. Main results

The lemma follows from [22], Lemmas 2.3 and 2.4.

**Proof** The lemma follows from [22], Lemmas 2.3 and 2.4. □

### 3. Main results

In [17], we discussed the simple \( K \)-groups of part (II) in Lemma 2.1, and proved that simple \( K \)-groups of type \( L_2(p) \) can be uniquely determined only by the group order and largest element order, where \( p \) is a prime but not \( 2^n - 1 \). In this paper, we will try to discuss the simple \( K \)-groups of part (I) in Lemma 2.1.

**Theorem 3.1** Let \( G \) be a group and \( M \) be one of the following simple \( K \)-groups: \( A_7, A_9, A_{10}, J_2, M_{11}, M_{12}, S_2(8), S_2(32), L_3(4), L_3(7), L_3(8), S_6(2), L_4(3), S_4(4), S_4(9), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), ^3D_4(2), L_3(17) \) and \(^2F_4(2)' \). Then \( G \cong M \) if and only if

(i) \( k_1(G) = k_1(M) \); 

(ii) \( |G| = |M| \).

| \( G \) | \(|G|\) | \( k_1(G)\) | \( k_2(G)\) |
|---|---|---|---|
| \( L_3(4) \) | \( 2^6 \cdot 3^2 \cdot 5 \cdot 7 \) | 7 | 5 |
| \( L_3(5) \) | \( 2^5 \cdot 3 \cdot 5^3 \cdot 31 \) | 31 | 24 |
| \( L_3(7) \) | \( 2^5 \cdot 3^2 \cdot 7^3 \cdot 19 \) | 19 | 16 |
| \( L_3(8) \) | \( 2^9 \cdot 3^2 \cdot 7^2 \cdot 73 \) | 73 | 63 |
| \( L_3(17) \) | \( 2^9 \cdot 3^2 \cdot 17^3 \cdot 307 \) | 307 | 288 |
| \( L_4(3) \) | \( 2^7 \cdot 3^6 \cdot 5 \cdot 13 \) | 20 | 13 |
| \( S_4(4) \) | \( 2^8 \cdot 3^2 \cdot 5^2 \cdot 17 \) | 17 | 15 |
| \( S_4(5) \) | \( 2^9 \cdot 3^2 \cdot 5^4 \cdot 13 \) | 30 | 20 |
| \( S_4(7) \) | \( 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4 \) | 56 | 42 |
| \( S_4(9) \) | \( 2^8 \cdot 3^3 \cdot 5^2 \cdot 41 \) | 41 | 40 |
| \( S_6(2) \) | \( 2^9 \cdot 3^4 \cdot 5 \cdot 7 \) | 15 | 12 |
| \( O_8^+(2) \) | \( 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7 \) | 15 | 12 |
| \( G_2(3) \) | \( 2^9 \cdot 3^3 \cdot 7^2 \cdot 13 \) | 13 | 12 |
| \( U_3(4) \) | \( 2^6 \cdot 3^2 \cdot 7^2 \cdot 13 \) | 15 | 13 |
| \( U_3(5) \) | \( 2^4 \cdot 3^2 \cdot 5^3 \cdot 7 \) | 10 | 8 |
| \( U_3(7) \) | \( 2^7 \cdot 3^3 \cdot 7^4 \cdot 43 \) | 56 | 48 |
| \( U_3(8) \) | \( 2^9 \cdot 3^4 \cdot 7^2 \) | 21 | 19 |
| \( U_3(9) \) | \( 2^5 \cdot 3^6 \cdot 5^2 \cdot 73 \) | 80 | 73 |
| \( U_4(3) \) | \( 2^7 \cdot 3^9 \cdot 5 \cdot 7 \) | 12 | 9 |
| \( U_5(2) \) | \( 2^{10} \cdot 3^5 \cdot 5 \cdot 11 \) | 18 | 15 |
| \(^3D_4(2)\) | \( 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13 \) | 28 | 21 |
| \(^2F_4(2)'\) | \( 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13 \) | 16 | 13 |

Table 1 The \(|G|, k_1(G)\) and \( k_2(G)\) of simple \( K \)-groups, except that \( L_2(q)\)
**Proof** We only need to prove the sufficiency. If \( M = A_7, A_9, A_{10} \), then the proof can be seen in [14]. If \( M = M_{11}, M_{12} \), then the proof can be seen in [11]. If \( M = Sz(8), Sz(32) \), then the proof can be seen in [16]. If \( M = L_3(4), L_3(7), L_3(8), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9) \), then the proof can be seen in [12]. And if \( M = L_4(3), S_4(4), U_4(3), G_2(3), 2^5F_4(2)' \), then the proof can be seen in [13]. Therefore, we just need to consider the cases \( M = J_2, Sz(9), O_6^+(2), Sz(2), 3D_4(2), U_5(2), L_3(17) \). Now we will complete the proof through a case by case analysis.

**Case 1** If \( M = J_2 \), then \( G \cong M \).

In such case, \(|G| = 2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \) and \( k_1(G) = 15 \). Firstly, we can assert that the \( G \) has a normal series \( G \geq K \geq H \geq 1 \), such that \( \overline{K} = K/H \) is a non-abelian simple group, and \( \{5, 7\} \subseteq \pi(\overline{K}) \). In fact, let \( G = G_0 > G_1 > \cdots > G_{k-1} > G_k = 1 \) be a chief series of \( G \). Then there must exist an integer \( i \), such that \( \{5, 7\} \cap \pi(G_i) \neq \Phi \), and \( \{5, 7\} \cap \pi(G_{i+1}) = \Phi \). Let \( K = G_i \), \( H = G_{i+1} \). Then \( G \geq K \geq H \geq 1 \) is a normal series of \( G \), and \( \overline{K} = K/H \) is a minimal normal subgroup of \( G/H \). If \( 5 \in \pi(K) \), \( 7 \notin \pi(K) \), then \( 7 \in \pi(G/K) \). By Frattini’s argument, we have \( G = NG(S_5)K \), where \( S_5 \) is a Sylow 5-subgroup of \( K \). Therefore, we have \( 7 \in \pi(N_G(S_5)) \), from which we know that \( 35 \in \pi_e(G) \), a contradiction. So \( 7 \in \pi(K) \).

Similarly, we can prove that if \( 7 \in \pi(K) \), then \( 5 \in \pi(K) \). Thus we have \( \{5, 7\} \subseteq \pi(K) \), and therefore \( \{5, 7\} \subseteq \pi(\overline{K}) \). Since \( \overline{K} \) is the direct product of isomorphic simple groups, \( \overline{K} \) is a non-abelian simple group. From [22] we can assume that \( \overline{K} \) is isomorphic to one of the following simple groups: \( A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7), L_4(2)(2^5 \cdot 3^3 \cdot 5 \cdot 7), L_3(4)(2^6 \cdot 3^2 \cdot 5 \cdot 7), L_3(7), A_9(2^6 \cdot 3^2 \cdot 5 \cdot 7) \) and \( J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7) \). We first suppose that \( \overline{K} \) is isomorphic to \( A_7, L_4(2), A_9, L_3(4) \). Since \( G/C_G(\overline{K}) \cong \frac{\text{Aut}(\overline{K})}{|\text{Aut}(\overline{K})|} \cdot |\overline{K}| \), we have \( 5 \mid |C_G(\overline{K})| \), which means that \( 35 \in \pi_e(G) \), a contradiction. Therefore, \( \overline{K} \cong J_2 \). In such case, \( H = 1, K = G \), and thus \( G \cong J_2 \).

**Case 2** If \( M = O_6^+(2) \), then \( G \cong M \).

In such case, \(|G| = 2^{12} \cdot 3^5 \cdot 5^2 \cdot 7 \) and \( k_1(G) = 15 \). By the similar arguments in Case 1, we know that \( G \) has a normal series \( G \geq K \geq H \geq 1 \), such that \( \overline{K} = K/H \) is a non-abelian simple group, and \( \{5, 7\} \subseteq \pi(\overline{K}) \). From [22], we can assume that \( \overline{K} \) is isomorphic to one of the following simple groups: \( A_7(2^3 \cdot 3^2 \cdot 5 \cdot 7), A_8(2^6 \cdot 3^2 \cdot 5 \cdot 7), L_3(4)(2^3 \cdot 3^2 \cdot 5 \cdot 7), A_9(2^6 \cdot 3^4 \cdot 5 \cdot 7), J_2(2^7 \cdot 3^3 \cdot 5^2 \cdot 7), Sz(2)(2^5 \cdot 3^3 \cdot 5 \cdot 7), A_{10}(2^7 \cdot 3^4 \cdot 5^2 \cdot 7) \) and \( O_6^+(2)(2^{12} \cdot 3^5 \cdot 5^2 \cdot 7) \). Clearly, \( G/C_G(\overline{K}) \cong \frac{\text{Aut}(\overline{K})}{|\text{Aut}(\overline{K})|} \cdot |\overline{K}| = \frac{\text{Out}(\overline{K})}{|\text{Out}(\overline{K})|} \cdot |\overline{K}| \). If \( \overline{K} \) is isomorphic to \( A_7, A_8, L_3(4), A_9, Sz(2) \), then \( 3 \mid |C_G(\overline{K})| \), which means that \( 35 \in \pi_e(G) \), a contradiction. If \( \overline{K} \) is isomorphic to \( J_2, A_{10} \), then \( 3 \mid |C_G(\overline{K})| \). If \( 3 \mid |H| \), then \( \overline{G} = G/H \) has an element with order 21, a contradiction. Therefore, we assume that \( 3 \not| |H| \). Consider the action on \( H \) by an element of order 7. We get that there exists a Sylow 3-subgroup \( L \) of \( H \) fixed by this action. Since \(|L| \mid 3^2 \), we have \( 7 \mid |\text{Aut}(L)| \), which means that such action is trivial. So \( 21 \in \pi_e(G) \), a contradiction too. Therefore, \( \overline{K} \cong O_6^+(2) \). In such case, \( H = 1, K = G \), and thus \( G \cong O_6^+(2) \).

**Case 3** If \( M = Sz(2) \), then \( G \cong M \).

In such case, \(|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7 \) and \( k_1(G) = 15 \). By the similar arguments in Case 1, we know \( G \) has a normal series \( G \geq K \geq H \geq 1 \), such that \( \overline{K} = K/H \) is a non-abelian simple
group, and \(\{5, 7\} \subseteq \pi(K)\). From [22], we can assume that \(K\) is isomorphic to one of the following simple groups: \(A_7 (2^{2} \cdot 3^{2} \cdot 5 \cdot 7)\), \(A_8 (2^{6} \cdot 3^{2} \cdot 5 \cdot 7)\), \(L_3(4) (2^{2} \cdot 3^{2} \cdot 5 \cdot 7)\), \(A_9 (2^{11} \cdot 3^{4} \cdot 5 \cdot 7)\) and \(S_6(2) (2^{9} \cdot 3^{4} \cdot 5 \cdot 7)\). Clearly, \(G/C_{G}(K) \leq \text{Aut}(K)\) and \(|\text{Aut}(K)| = |\text{Out}(K)|\cdot |K|\). If \(K\) is isomorphic to \(A_7, A_8, L_3(4)\), then \(3 \mid |C_{G}(K)|\), which means \(21 \in \pi_{e}(G)\), a contradiction. If \(K\) is isomorphic to \(A_9\), then \(2 \mid |C_{G}(K)|\). As \(A_9\) has an element with 15, \(G\) has an element of order 30, a contradiction. Therefore, \(K \cong S_6(2)\). In such case, \(H = 1, K = G\), and thus \(G \cong S_6(2)\).

**Case 4** If \(M = S_4(9)\) or \(3D_4(2)\), then \(G \cong M\).

The proof is similar to the above cases.

**Case 5** If \(M = U_3(2)\), then \(G \cong M\).

In such case, \(|G| = 2^{10} \cdot 3^{5} \cdot 5 \cdot 11\) and \(k_1(G) = 18\). Since \(k_1(G) = 18, 11\) is an isolated point in \(\Gamma(G)\). If \(G\) is a Frobenius group with kernel \(K\) and complement \(H\), then \(H\) is of order 11 as \(|H|\) divides \(|K| - 1\). Now \(H\) acts trivially on a Sylow 5-subgroup of \(K\) and so \(55 \in \pi_{e}(G)\), which contradicts \(k_1(G) = 18\). Suppose that \(G\) is a 2-Frobenius group with normal series \(1 \leq H \leq K \leq G\), where \(|K/H| = 11\) and \(|G/K||10\). In such case, \(3||H|\). Consider the action on \(H\) by the element of order 11. One can see that \(K\) has a Sylow 3-subgroup \(L\) fixed by this action. Since \(G = 2^{10} \cdot 3^{5} \cdot 5 \cdot 11\), we have \(|L| = 3^5\). Clearly, \(\Omega_{1}(Z(L))\) is an elementary abelian 3-group. Because \(k_1(G) = 18, |\Omega_{1}(Z(L))| \mid 3^4\). Consider the action on \(\Omega_{1}(Z(L))\) by the element of order 11. We know such action is trivial for \(11 \mid |\text{Out}(\Omega_{1}(Z(L)))|\), which implies that \(33 \in \pi_{e}(G)\), a contradiction. Therefore, by Lemma 2.2, \(G\) has a normal series \(1 \leq H \leq K \leq G\), such that \(H \leq G/K\) are \(\pi_1\)-groups and \(K/H\) a non-abelian simple group, where \(\pi_1\) is the prime graph component containing \(2, H\) is a nilpotent group, and \(|G/K\mid |\text{Out}(K/H)|\). As \(|G| = 2^{10} \cdot 3^{5} \cdot 5 \cdot 11\), and \(11\) is an isolated point in \(\Gamma(G)\), we have \(\pi(H) \cup \pi(G/K) \subseteq \{2, 3, 5\}\) and \(11 \in \pi(K/H)\). From [22], we can suppose that \(K/H\) is isomorphic to one of the following simple groups: \(L_3(11) (2^2 \cdot 3 \cdot 5 \cdot 11), M_{11} (2^4 \cdot 3^2 \cdot 5 \cdot 11), M_{12} (2^6 \cdot 3^3 \cdot 5 \cdot 11)\) and \(U_3(2) (2^{10} \cdot 3^5 \cdot 5 \cdot 11)\).

Suppose that \(K/H \cong L_3(11), M_{11}\) or \(M_{12}\). In such case, we can get that \(3 \mid |\text{Out}(K/H)|\) and thus \(3 \mid |H|\) by comparing the order of \(G\). Let \(L\) be a Sylow 3-subgroup of \(H\). We have \(L \leq G\) and \(|L| \mid 3^4\). Clearly, \(\Omega_{1}(Z(L))\) is an elementary abelian 3-group, and \(|\Omega_{1}(Z(L))| \mid 3^4\). Consider the action on \(\Omega_{1}(Z(L))\) by the element of order 11. Because \(11 \mid |\text{Out}(\Omega_{1}(Z(L)))|\), this action is trivial, which implies that \(33 \in \pi_{e}(G)\), a contradiction. Therefore, we have \(K/H \cong U_3(2)\). So \(H = 1, K = G\), and therefore, \(G \cong U_5(2)\).

**Case 6** If \(M = L_3(17)\), then \(G \cong M\).

The proof is similar to Case 5.

The proof of Theorem 3.1 is completed. □

**Theorem 3.2** Let \(G\) be a group and \(M\) be one of the following simple \(K_4\)-groups: \(A_8, L_3(5)\) and \(S_4(5)\). Then \(G \cong M\) if and only if

1. \(k_i(G) = k_i(M)\), where \(i = 1, 2\);
2. \(|G| = |M|\).
Remark 3.4

For simple groups, we characterize them in the way used in this paper. Connected only by their largest element order and second largest element order, so we cannot have $G \cong S_4(5)$. Now $H$ acts trivially on the Sylow $3$-subgroup of $K$ and so $93 \in \pi_e(G)$, which contradicts $k_3(G) = 31$. Suppose that $G$ is a $2$-Frobenius group with normal series $1 \leq H \leq K \leq G$, where $|K/H| = 31$ and $|G/K||30$. In such case, $2||H|$. Consider the action on $H$ by the element of order $11$. We can get that there exists a Sylow $2$-subgroup $L$ of $K$ with this action. Since $G = 2^5 \cdot 3 \cdot 5^3 \cdot 31$, we have $|L| = 2^5$. Clearly, $\Omega_1(Z(L))$ is an elementary abelian $2$-group. Because $k_2(G) = 24$, $G$ has an element with order $8$, and thus $|\Omega_1(Z(L))| = 2^3$. Consider the action on $\Omega_1(Z(L))$ by the element of order $31$. We know such action is trivial for $31 \mid |\text{Out}(\Omega_1(Z(L)))|$, which implies that $62 \in \pi_e(G)$, a contradiction. Therefore, by Lemma 2.2, we know that $G$ has a normal series $1 \leq H \leq K \leq G$, such that $H$ and $G/K$ are $\pi_1$-groups and $K/H$ a non-abelian simple group, where $\pi_1$ is the prime graph component containing $2$, $H$ is a nilpotent group, and $|G/K| / |\text{Out}(K/H)|$.Because $|G| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$, and $31$ is an isolated point of $\Gamma(G)$, we have $\pi(\Omega) \cup \pi(G/K) \subseteq \{2, 3, 5\}$ and $31 \in \pi(K/H)$. From [22] we know that $L = \text{L}_2(31)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$ or $L = \text{L}_3(5)(2^5 \cdot 3 \cdot 5^3 \cdot 31)$.

Suppose that $K/H$ is isomorphic to $L_2(31)$ or $L_3(5)$. If $5 \mid |\text{Out}(K/H)|$, and thus $5 \mid |H|$. Let $L$ be a Sylow $5$-subgroup of $H$. We know that $L \trianglelefteq G$ and $|L| = 5^2$. Consider the action on $L$ by the element of order $31$. Clearly, this action is trivial. It implies that $155 \in \pi_e(G)$, which is a contradiction. Therefore, we have $K/H \cong L_3(5)$. So $H = 1$, $K = G$, and therefore, $G \cong L_3(5)$.

This completes the proof. □

As a corollary of preceding theorems, we have

Theorem 3.3 Let $G$ be one of the simple $K_4$-groups mentioned in part (I) in Lemma 2.1, except that $S_4(7)$. Then $G$ can be uniquely determined by the order of $G$ and $k_4(G)$, where $i \leq 2$.

Remark 3.4 For simple $K_4$-group $S_4(7)$, we cannot judge whether their prime graphs are connected only by their largest element order and second largest element order, so we cannot characterize them in the way used in this paper.

References