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On Friendly Index Sets of Cyclic Silicates

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Abstract Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A labeling $f: V(G) \to Z_2$ induces an edge labeling $f^*: E(G) \to Z_2$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in Z_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f of a graph G is said to be friendly if $|v_f(0) - v_f(1)| \le 1$. The friendly index set of the graph G, denoted $\mathrm{FI}(G)$, is defined as $\{|e_f(0) - e_f(1)|$: the vertex labeling f is friendly$\}$. This is a generalization of graph cordiality. We investigate the friendly index sets of cyclic silicates $\mathrm{CS}(n, m)$.

Keywords vertex labeling; friendly labeling; cordiality; friendly index set; cycle; $\mathrm{CS}(n, m)$; arithmetic progression

MR(2010) Subject Classification 05C78; 05C25

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let A be an abelian group. A labeling $f: V(G) \to A$ induces an edge labeling $f^*: E(G) \to A$ defined by $f^*(xy) = f(x) + f(y)$, for each edge $xy \in E(G)$. For $i \in A$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. Let $c(f) = \{|e_f(i) - e_f(j)| : (i,j) \in A \times A\}$. A labeling f of a graph G is said to be A-friendly if $|v_f(i) - v_f(j)| \le 1$ for all $(i,j) \in A \times A$. If $c(f)$ is a set for some A-friendly labeling f, then f is said to be A-cordial.

The notion of A-cordial labelings was first introduced by Hovey [1], who generalized the concept of cordial graphs of Cahit [2]. Cahit considered $A = Z_2$. For more details of known results and open problems on cordial graphs, the reader can see relevant papers.

In this paper, we will exclusively focus on $A = Z_2$, and drop the reference to the group. A vertex v is called a k-vertex if $f(v) = k$, $k \in \{0, 1\}$, an edge e is called a k-edge if $f^*(e) = k$, $k \in \{0, 1\}$. When the context is clear, we will drop the subscript f.

In [3] the following concept was introduced.

Definition 1.1 *The friendly index set* $\mathrm{FI}(G)$ *of a graph* G *is defined as* $\{|e_f(0) - e_f(1)|$: *the vertex labeling* f *is friendly*$\}$.

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Note that if 0 or 1 is in FI(G), then G is cordial. Thus the concept of friendly index sets could be viewed as a generalization of cordiality. Cairnie and Edwards [4] have determined the computational complexity of cordial labeling and Z_k-cordial labeling. They proved that deciding whether a graph admits a cordial labeling is NP-complete. Even the restricted problem of deciding whether a connected graph of diameter 2 has a cordial labeling is NP-complete. Thus, in general, it is difficult to determine the friendly index sets of graphs.

Example 1.2 Figure 1 illustrates the friendly index set of the cycle C_8 with two parallel chords.

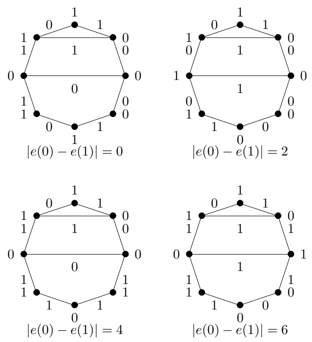

Figure 1 FI(PC(8, 2)) = {0, 2, 4, 6}

Example 1.3 Figure 2 illustrates the friendly index set of $K_{3,3}$ and $C_3 \times K_2$.

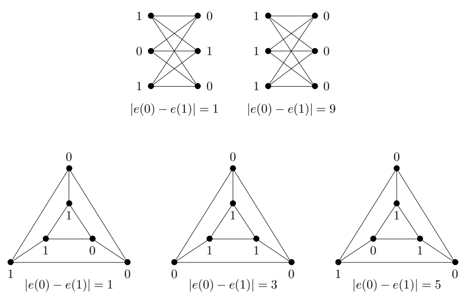

Figure 2 FI($K_{3,3}$) = {1, 9}, FI($C_3 \times K_2$) = {1, 3, 5}

In [5–7], the friendly index sets of a few classes of graphs, including complete bipartite graphs and cycles, are determined. In [8], the following results were established:

Theorem 1.4 *For any graph G with q edges, the friendly index set* $\mathrm{FI}(G) \subseteq \{0, 2, 4, \ldots, q\}$ *if q is even, and* $\mathrm{FI}(G) \subseteq \{1, 3, \ldots, q\}$ *if q is odd.*

Theorem 1.5 *The friendly indices of a cycle form an arithmetic sequence:*

(i) $\mathrm{FI}(C_{2n}) = \{0, 4, 8, \ldots, 2n\}$ *if n is even;* $\mathrm{FI}(C_{2n}) = \{2, 6, 10, \ldots, 2n\}$ *if n is odd.*

(ii) $\mathrm{FI}(C_{2n+1}) = \{1, 3, 5, \ldots, 2n-1\}$.

For more details of known results and open problems on cordial graphs, the reader can see [8–15].

In this paper, we consider the friendly index sets of cyclic silicates, denoted $\mathrm{CS}(n,m)$ $(n,m \geq 3)$, obtained from an n-cycle and n copies of K_m by gluing to each edge of C_n an edge from one copy of K_m. The graph labeling f of $\mathrm{CS}(n,m)$ by $G(a)$ in which $|e_f(1)-e_f(0)|=a$.

2. The friendly index sets of $\mathrm{CS}(n,3)$

When $m=3$, $\mathrm{CS}(n,3)$ is shown in Figure 3 with the K_3 subgraphs given by vertices in $\{u_1,u_n,w_n\}$ and in $\{u_i,u_{i+1},w_i\}$ for $1\leq i\leq n-1$.

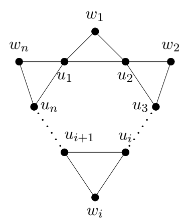

Figure 3 Graph $\mathrm{CS}(n,3)$

Since each vertex of a K_3 can be labeled with 0 or 1, it is easy to verify that $e(0)$ is either 3 or 1. The following lemma then follows.

Lemma 2.1 *In all possible* $(0,1)$*-labelings of the vertices of a* K_3*, we have* $e(0)-e(1)=3$ *or* -1.

Lemma 2.2 *For* $n\geq 3$*,* $\max\{\mathrm{FI}(\mathrm{CS}(n,3))\}=\max\{n,3n-8\}$.

Proof The graph $\mathrm{CS}(n,3)$ has $2n$ vertices and $3n$ edges. By Lemma 2.1, we know $\max|e(0)-e(1)|$ is attained if each K_3 subgraph of $\mathrm{CS}(n,3)$ contributes three or one 0-edge. If at most one K_3-subgraph contributes a 0-edge, such a labeling is not friendly. Therefore, at least two K_3 subgraphs of $\mathrm{CS}(n,3)$ contribute a 0-edge each. Hence, if exactly two K_3 subgraphs of $\mathrm{CS}(n,3)$ contribute a 0-edge, then $\max|e(0)-e(1)|=3(n-2)-2=3n-8$. If all K_3 subgraphs of $\mathrm{CS}(n,3)$ contribute a 0-edge, then $\max|e(0)-e(1)|=n$. It is easy to verify that a labeling with $|e(0)-e(1)|=3n-8$ or n exists. Consequently, $\max|e(0)-e(1)|=\max\{n,3n-8\}$. □

Theorem 2.3 *For* $n=3$*,* 4 *and* 5*,* $\mathrm{FI}(\mathrm{CS}(3,3))=\{1,3\}$*;* $\mathrm{FI}(\mathrm{CS}(4,3))=\{0,4\}$*;* $\mathrm{FI}(\mathrm{CS}(5,3))=\{1,3,5,7\}$.

Proof For $n=3$, the labelings are illustrated in Figure 4.

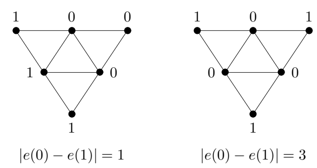

Figure 4 The friendly labelings of $\mathrm{CS}(3,3)$

For $n = 4$, the labelings are illustrated in Figure 5. Note that Lemma 2.1 implies that $2 \notin \mathrm{FI}(\mathrm{CS}(4,3))$.

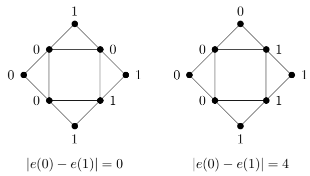

Figure 5 The friendly labelings of CS(4, 3)

For $n = 5$, the labelings are illustrated in Figure 6.

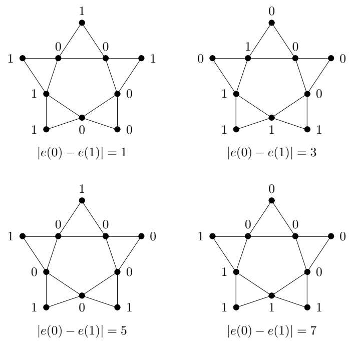

Figure 6 The friendly labelings of CS(5, 3)

Theorem 2.4 *In* $\mathrm{CS}_k(n,3)$, *if two vertex labels are exchanged, then we must get the labeling* $\mathrm{CS}_{|k+4t|}(n,3)$ *for* $t \in \{0, \pm 1, \pm 2, \pm 3, \pm 4\}$.

Proof In $\mathrm{CS}_k(n,3)$, if we exchange the labels of two vertices u and v, then by Lemma 2.1, it is routine to verify that one of the follow cases must exist:

if u and v are adjacent, then $e(0)$ changes by $\pm 0, \pm 2$ or ± 4;

if u and v are not adjacent, then $e(0)$ changes by $\pm 0, \pm 2, \pm 4, \pm 6$ or ± 8.

Since $e(1) - e(0) = q - 2e(0)$, the theorem holds. $\square$

Theorem 2.5 *For odd* $n \geq 7$, $\mathrm{FI}(\mathrm{CS}(n,3)) = \{1, 3, \ldots, n\} \cup \{n+2, n+6, n+10, \ldots, 3n-8\}$.

Proof By Theorem 1.4 and Lemma 2.2, $\mathrm{FI}(\mathrm{CS}(n,3)) \subseteq \{1, 3, 5, \ldots, 3n-8\}$. Theorem 2.4 then implies that the labelings with $e(0) - e(1) = 3n-10, 3n-14, 3n-18, \ldots$ do not exist if $e(0) - e(1) > 0$. Hence, it suffices to show that there exists labeling for $e(0) - e(1) \in \{3n-8, 3n-12, 3n-16, \ldots, 3, -1, -5, \ldots, -n\}$ or $e(0) - e(1) \in \{3n-8, 3n-12, 3n-16, \ldots, 1, -3, -7, \ldots, -n\}$. Let $G_k = \{\mathrm{CS}_{|3n-8k|}(n,3), \mathrm{CS}_{|3n-8k-4|}(n,3)\}$ $(k = 1, 2, \ldots, \frac{n-1}{2})$. We define

$$f(u_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n+1}{2}; \\ 1, & \text{for } i = \frac{n+1}{2}+1, \frac{n+1}{2}+2, \ldots, n, \end{cases}$$

and

$$f(w_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n-1}{2}; \\ 1, & \text{for } i = \frac{n-1}{2} + 1, \frac{n-1}{2} + 2, \ldots, n. \end{cases}$$

So, we get $\mathrm{CS}_{3n-8}(n,3)$ with $e(0) = 3n-4, e(1) = 4$. We now exchange the labels of u_1 and w_n in $\mathrm{CS}_{3n-8}(n,3)$ to decrease $e(0)$ by 2. So, we get $\mathrm{CS}_{3n-12}(n,3)$. We have obtained G_1. In the following four cases, we obtain the labeling in G_k $(2 \le k \le \frac{n-1}{2})$ successively. This shows that the graphs in G_k yield all the friendly indices of $\mathrm{CS}(n,3)$.

Case 1 $n \equiv 1 \pmod 8$. When $2 \le k \le \frac{3n-11}{8}$, the above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 7\}$. For $k = \frac{3n-3}{8}$, we get the $\mathrm{CS}_3(n,3)$ and $\mathrm{CS}_1(n,3)$ with $e(0) - e(1) = 3$ and -1, respectively. Theorem 2.4 then implies that the $\mathrm{CS}_a(n,3)$ with $e(0) - e(1) = -3, -7, -11, \ldots$ do not exist. So, when $\frac{3n+5}{8} \le k \le \frac{n-1}{2}$, we get the $\mathrm{CS}_a(n,3)$ for $-a = e(0) - e(1) \in \{-5, -9, -13, \ldots, 4-n, -n\}$. Hence, we have obtained all the possible friendly indices.

Case 2 $n \equiv 3 \pmod 8$. When $2 \le k \le \frac{3n-9}{8}$, the above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 9, 5\}$. For $k = \frac{3n-1}{8}$, we have the $\mathrm{CS}_1(n,3)$ and $\mathrm{CS}_3(n,3)$ with $e(0) - e(1) = 1$ and -3, respectively. Theorem 2.4 implies that the $\mathrm{CS}_a(n,3)$ with $e(0) - e(1) = -1, -5, -9, \ldots$ do not exist. So, when $\frac{3n+7}{8} \le k \le \frac{n-1}{2}$, we get the $\mathrm{CS}_a(n,3)$ for $-a = e(0) - e(1) \in \{-7, -11, -15, \ldots, 4-n, -n\}$. Hence, we have obtained all the possible friendly indices.

Case 3 $n \equiv 5 \pmod 8$. When $2 \le k \le \frac{3n-15}{8}$, the above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 7, 3\}$. Theorem 2.4 implies that the $\mathrm{CS}_a(n,3)$ with $e(0) - e(1) = -3, -7, -11, \ldots$ do not exist. So, when $\frac{3n-7}{8} \le k \le \frac{n-1}{2}$, we get the $\mathrm{CS}_a(n,3)$ for $-a = e(0) - e(1) \in \{-1, -5, -9, \ldots, 4-n, -n\}$. Hence, we have obtained all the possible friendly indices.

Case 4 $n \equiv 7 \pmod 8$. When $2 \le k \le \frac{3n-13}{8}$, the above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 5, 1\}$. Theorem 2.4 implies that the $\mathrm{CS}_a(n,3)$ with $e(0) - e(1) = -1, -5, -9, \ldots$ do not exist. So, when $\frac{3n-5}{8} \le k \le \frac{n-1}{2}$, we get the $\mathrm{CS}_a(n,3)$ for $-a = e(0) - e(1) \in \{-3, -7, -11, \ldots, 4-n, -n\}$. Hence, we have obtained all the possible friendly indices.

The proof is completed. □

Theorem 2.6 *For $n \ge 8$ and $n \equiv 0 \pmod 4$, $\mathrm{FI}(\mathrm{CS}(n,3)) = \{0, 4, 8, \ldots, 3n-8\}$.*

Proof By Theorem 1.4 and Lemma 2.2, $\mathrm{FI}(\mathrm{CS}(n,3)) \subseteq \{0, 2, 4, \ldots, 3n-8\}$. Theorem 2.4 implies that the labelings with $e(0) - e(1) = 3n-10, 3n-14, 3n-18, \ldots, 2, -2, -6, \ldots$ do not exist. It suffices to show that the friendly indices listed in the theorem are attainable. Let

$G_k = \{\mathrm{CS}_{3n-8k}(n,3),\ \mathrm{CS}_{3n-8k-4}(n,3)\}$. Define

$$f(u_i) = f(w_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n}{2}; \\ 1, & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n. \end{cases}$$

We get $\mathrm{CS}_{3n-8}(n,3)$ with $e(0) = 3n-4, e(1) = 4$. We now exchange the labels of u_1 and w_n in $\mathrm{CS}_{3n-8}(n,3)$ to decrease $e(0)$ by 2. So we get $\mathrm{CS}_{3n-12}(n,3)$. We have obtained G_1. We consider two cases.

Case 1 $n \equiv 0 \pmod 8$. To obtain G_k $(2 \le k \le \frac{3n}{8})$, we exchange the labels of w_k and w_{n-k+1} in G_{k-1}. This is attainable since $e(0)$ decreases by 4 after each exchange so that $e(0) - e(1)$ decreases by 8 successively. When $2 \le k \le \frac{3n}{8} - 1$, the above labeling process gives the graphs $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 4\}$. For $k = \frac{3n}{8}$, we have $\mathrm{CS}_0(n,3)$ and $\mathrm{CS}_4(n,3)$ with $e(0) - e(1) = 0$ and -4, respectively. Hence, we have obtained all the possible friendly indices.

Case 2 $n \equiv 4 \pmod 8$. To obtain G_k $(2 \le k \le \frac{3n-4}{8})$, we exchange the labels of w_k and w_{n-k+1} in G_{k-1}. As in Case 1, $e(0) - e(1)$ decreases by 8 successively. The above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 0\}$. Hence, we have obtained all the possible friendly indices.

The proof is completed. □

Theorem 2.7 *For $n \ge 6$ and $n \equiv 2 \pmod 4$,* $\mathrm{FI}(\mathrm{CS}(n,3)) = \{2, 6, 10, \ldots, 3n-8\}$.

Proof By Theorem 1.4 and Lemma 2.2, $\mathrm{FI}(\mathrm{CS}(n,3)) \subseteq \{0, 2, 4, \ldots, 3n-8\}$. Theorem 2.4 implies that the labelings with $e(0) - e(1) = 3n-10, 3n-14, 3n-18, \ldots, 4, 0, -4, \ldots$ do not exist. It suffices to show that the values are attainable. Let $G_k = \{\mathrm{CS}_{3n-8k}(n,3),\ \mathrm{CS}_{3n-8k-4}(n,3)\}$. Define

$$f(u_i) = f(w_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n}{2}; \\ 1, & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n. \end{cases}$$

We get $\mathrm{CS}_{3n-8}(n,3)$ with $e(0) = 3n-4, e(1) = 4$. We now exchange the labels of u_1 and w_n in $\mathrm{CS}_{3n-8}(n,3)$ to decrease $e(0)$ by 2. So we get $\mathrm{CS}_{3n-12}(n,3)$. We consider two cases.

Case 1 $n \equiv 2 \pmod 8$. To obtain G_k $(2 \le k \le \frac{3n-6}{8})$, we exchange the labels of w_k and w_{n-k+1} in G_{k-1}. This is attainable since $e(0)$ decreases by 4 after each exchange so that $e(0) - e(1)$ decreases by 8 successively. The above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 2\}$. Hence, we have obtained all the possible friendly indices.

Case 2 $n \equiv 6 \pmod 8$. To obtain G_k $(2 \le k \le \frac{3n-2}{8})$, we exchange the labels of w_k and w_{n-k+1} in G_{k-1}. As in Case 1, $e(0) - e(1)$ decreases by 8 successively. When $2 \le k \le \frac{3n-10}{8}$, the above labeling process gives the $\mathrm{CS}_a(n,3)$ for $a = e(0) - e(1) \in \{3n-16, 3n-20, 3n-24, \ldots, 6\}$. For $k = \frac{3n-2}{8}$, we have the $\mathrm{CS}_a(n,3)$ with $e(0) - e(1) = 2$ and -2, respectively. Hence, we have obtained all the possible friendly indices.

The proof is completed. □

Corollary 2.8 *The graph* $\mathrm{CS}(n,3)$ *is cordial if and only if* n *is odd or* $n \equiv 0 \pmod 4$. *Moreover, the friendly indices form an arithmetic sequence if and only if* n *is even.*

3. The friendly index sets of $\mathrm{CS}(n,4)$

When $m = 4$, $\mathrm{CS}(n,4)$ is shown in Figure 7 with the K_4 subgraphs given by vertices in $\{u_1, u_n, v_n, w_n\}$ and in $\{u_i, u_{i+1}, v_i, w_i\}$ for $1 \le i \le n-1$.

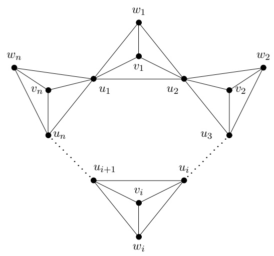

Figure 7 Graph $\mathrm{CS}(n,4)$

Each vertex of a K_4 can be labeled with 0 or 1, it is easy to verify that $e(0)$ is either 6, 3 or 2. The following lemma then follows.

Lemma 3.1 *In all possible* $(0,1)$*-labelings of the vertices of a* K_4*, we have* $e(0) - e(1) = 6, 0$ *or* -2.

Lemma 3.2 *For* $n \ge 3$, $\max\{\mathrm{FI}(\mathrm{CS}(n,4))\} \le 6(n-2)$.

Proof The graph $\mathrm{CS}(n,4)$ has $3n$ vertices and $6n$ edges. We first show that $\max|e(1) - e(0)| \le 6(n-2)$. By Lemma 3.1, we know $\max|e(0) - e(1)|$ is attained if each K_4 subgraph of $\mathrm{CS}(n,4)$ contributes six or three 0-edges. If at most one K_4-subgraph contributes three 0-edges, such a labeling is not friendly. Hence, at least two K_4 subgraphs of $\mathrm{CS}(n,4)$ contribute three 0-edges. Therefore, $\max|e(0) - e(1)| \le 6(n-2)$. □

A K_4 subgraph is of Type 1 (respectively, Types 2 and 3) if it has six (respectively, three and two) 0-edges.

Lemma 3.3 *For odd* $n > 3$ *(respectively even* $n \ge 4$*), the* $\mathrm{CS}_{6(n-2)-4}(n,4)$ *(respectively* $\mathrm{CS}_{6(n-2)-2}(n,4)$*) does not exist.*

Proof Consider the $\mathrm{CS}_{6(n-2)-2t}(n,4), t = 1,2$. Suppose the number of Types 1 and 3 subgraphs are y and z, respectively, and all other K_4 subgraphs are of Type 2. Note that $0 \le y \le n-2$ and $0 \le z \le n$. Hence, we must have $|6y - 2z| = 6(n-2) - 2t$. We first consider $6y - 2z = 6(n-2) - 2t$.

Case 1 $n > 3$ is odd. Suppose $t = 2$, then $6y - 2z = 6(n-2) - 4$ or $3(n-2-y) = 2 - z \ge 0$. Hence, $z = 2$ and $y = n-2$. Moreover, there exist no Type 2 subgraphs. Clearly, the two Type 3 subgraphs do not have any common vertex. Hence, we may assume $f(u_i) = f(v_i) = x$ $(1 \le i \le (n+1)/2)$ and $f(w_n) = f(w_i) = x$ $(1 \le i \le (n-1)/2)$ whereas the remaining vertices are labeled with $1-x$. However, this labeling is not friendly, a contradiction.

Case 2 $n \geq 4$ is even. Suppose $t=1$, then $6y-2z=6(n-2)-2$ or $3(n-2-y)=1-z \geq 0$. Hence, $z=1$ and $y=n-2$. Since n is even, so is y. So, the $\mathrm{CS}_{6(n-2)-2}(n,4)$ has exactly one Type 2 subgraph. Clearly, the Type 2 and the Type 3 subgraph do not have any common vertex. Hence, we may assume $f(w_n)=f(u_i)=f(v_i)=f(w_i)=x$ $(1 \leq i \leq n/2)$ and the remaining vertices are labeled with $1-x$. However, the labeling is not friendly, a contradiction.

We now consider $2z-6y=6(n-2)-2t$. If $n>3$ is odd, we have $6(n-2)-4>2n \geq 2z \geq 2z-6y$, a contradiction. If $n \geq 4$ is even, we have $6(n-2)-2>2n \geq 2z \geq 2z-6y$, also a contradiction. $\Box$

Theorem 3.4 *For $n=3$ and 4,* $\mathrm{FI}(\mathrm{CS}(3,4))=\{0,2,4,6\}$*;* $\mathrm{FI}(\mathrm{CS}(4,4))=\{0,2,4,6,8,12\}$.

Proof For $n=3$, the labelings are illustrated in Figure 8.

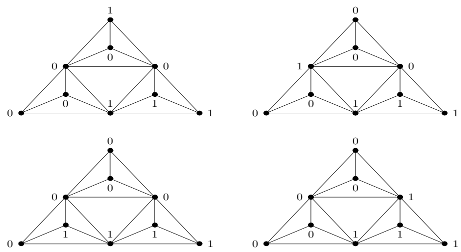

Figure 8 The friendly labelings of CS(3,4)

For $n=4$, the labelings are illustrated in Figure 9.

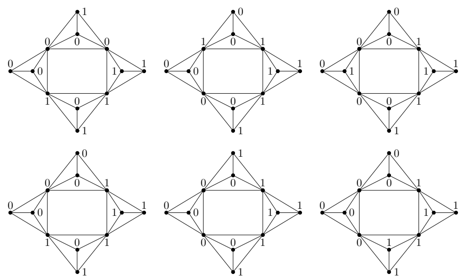

Figure 9 The friendly labelings of CS(4,4)

Theorem 3.5 *For odd $n \geq 5$,* $\mathrm{FI}(\mathrm{CS}(n,4))=\{0,2,4,\ldots,6(n-2)-6\} \cup \{6(n-2)-2, 6(n-2)\}$.

Proof By Theorem 1.4 and Lemmas 3.2 and 3.3, it suffices to show that all the friendly indices listed in the theorem are attainable. Define

$$f(u_i)=\begin{cases} 1, & \text{for } i=1,3,\ldots,n; \\ 0, & \text{for } i=2,4,\ldots,n-1, \end{cases}$$

$$f(v_i)=f(w_i)=\begin{cases} 1, & \text{for } i=1,3,\ldots,n-2; \\ 0, & \text{for } i=2,4,\ldots,n-1, \end{cases}$$

and $f(v_n) = 1, f(w_n) = 0$. We have $v(1) - v(0) = 1$ and each K_4 subgraph is of Type 2. Hence, we get $\mathrm{CS}_0(n, 4)$. Now, exchanging the labels of u_2 and v_1 to decrease $e(0)$ by 1. Hence, we get the $\mathrm{CS}_2(n, 4)$ with $e(1) - e(0) = 2$.

Next, we divide the vertices v_1 to v_{n-1} into $(n-1)/2$ pairs of vertices v_i, v_{i+1} for $i = 1, 3, 5, \ldots, n-2$. Beginning with $\mathrm{CS}_0(n, 4)$, we now exchange the labels of v_1 and v_2 to decrease $e(0)$ by 2. Hence, we get $\mathrm{CS}_4(n, 4)$ with $e(1) - e(0) = 4$. Using $\mathrm{CS}_4(n, 4)$, we exchange the labels of v_3 and v_4 to decrease $e(0)$ by 2 again. Hence, we get $\mathrm{CS}_8(n, 4)$. Repeating the same process for each pair v_i, v_{i+1}, $i = 5, 7, \ldots, n-2$. After exchanging the labels of v_i and v_{i+1}, $i \in \{5, 7, \ldots, n-2\}$, we get $\mathrm{CS}_{2(i+1)}(n, 4)$ with $e(1) - e(0) = 2(i+1)$. In this process, we obtained $\mathrm{CS}_a(n, 4)$ for $a \in \{4, 8, 12, \ldots, 2(n-1)\}$. Finally, we change the vertex label of v_n to 0 to get $\mathrm{CS}_{2n}(n, 4)$.

We now begin with $\mathrm{CS}_2(n, 4)$. We divide the vertices v_3 to v_{n-1} into $(n-3)/2$ pairs of vertices $v_i v_{i+1}$. Repeating the same process as above will decrease $e(0)$ by 2. Hence, after exchanging the labels of v_i and v_{i+1}, $i = 3, 5, \ldots, n-2$, we get $\mathrm{CS}_{2i}(n, 4)$ with $e(1) - e(0) = 2i$. In this process, we obtain $\mathrm{CS}_a(n, 4)$ for $a \in \{6, 10, 14, \ldots, 2n-4\}$. Hence, $\{0, 2, 4, \ldots, 2n\} \subseteq \mathrm{FI}(\mathrm{CS}(n, 4))$.

We now give the labeling graphs $\mathrm{CS}_a(n, 4)$, $a \in \{2n+2, 2n+4, \ldots, 6(n-2)-6\} \cup \{6(n-2)-2, 6(n-2)\}$. We define

$$f(u_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n+1}{2}; \\ 1, & \text{for } i = \frac{n+1}{2}+1, \frac{n+1}{2}+2, \ldots, n, \end{cases}$$

and

$$f(v_i) = f(w_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n-1}{2}; \\ 1, & \text{for } i = \frac{n+1}{2}, \frac{n+1}{2}+1, \ldots, n. \end{cases}$$

So, $v(0) = \frac{n+1}{2} + n - 1, v(1) = \frac{n+1}{2} + n$, $e(0) - e(1) = 6(n-2)$. Now exchange the label of v_n in $\mathrm{CS}_{6(n-2)}(n, 4)$ to 0 to get $\mathrm{CS}_{6(n-2)-2}(n, 4)$ with $v(0) - v(1) = 1$. To complete the proof, we need the six labeling graphs $\mathrm{CS}_a(n, 4)$, $a = 6(n-2)-6, 6(n-2)-8, \ldots, 6(n-2)-16$. They can be obtained as follows:

(1) In $\mathrm{CS}_{6(n-2)}(n, 4)$, exchange the labels of u_1 and v_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-6}(n, 4)$.

(2) In $\mathrm{CS}_{6(n-2)-2}(n, 4)$, exchange the labels of u_1 and w_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-8}(n, 4)$.

(3) In $\mathrm{CS}_{6(n-2)-6}(n, 4)$, exchange the labels of w_1 and w_n so that $e(1)$ increases by 2 and $e(0)$ decreases by 2. We get $\mathrm{CS}_{6(n-2)-10}(n, 4)$.

(4) In $\mathrm{CS}_{6(n-2)-6}(n, 4)$, exchange the labels of u_n and v_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-12}(n, 4)$.

(5) In $\mathrm{CS}_{6(n-2)-8}(n, 4)$, exchange the labels of u_n and w_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-14}(n)$.

(6) In $\mathrm{CS}_{6(n-2)-10}(n, 4)$, exchange the labels of u_n and v_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-16}(n)$.

We have now obtained $\mathrm{CS}_{6(n-2)-6}(n, 4)$, $\mathrm{CS}_{6(n-2)-8}(n, 4), \ldots$, $\mathrm{CS}_{6(n-2)-16}(n, 4)$. Note that the above labelings can give us the friendly index sets for $n = 5, 7$. We now assume $n \geq 9$. Observe that in (1) to (6) above, only the labels of u_1, u_n, v_n, w_1 or w_n are changed so that each labeling obtained has at least $\frac{n-3}{2}$ Type 1 subgraphs with all vertices labeled with 0, and at least $\frac{n-5}{2}$ Type 1 subgraph with all vertices labeled with 1. Moreover, the number of former subgraphs is more than the number of latter subgraphs.

Let $G_k = \{\mathrm{CS}_{6(n-2)-6k}(n,4), \mathrm{CS}_{6(n-2)-2-6k}(n,4), \ldots, \mathrm{CS}_{6(n-2)-10-6k}(n,4)\}$, where $k \geq 1$ is odd. To obtain the G_k ($3 \leq k \leq \frac{n-5}{2}$ is odd), we change the labels of v_k and $v_{(n-1)/2+k}$ in G_{k-2}. This is attainable since $e(0)$ decreases by 6 after each change so that $e(0) - e(1)$ decreases by 12 successively.

Note that there are $2n - 9$ even numbers from $2n + 2$ to $6(n-2) - 18$ inclusive. We divide them into $\lfloor \frac{2n-9}{6} \rfloor$ groups of six successive even numbers. Since $6(\frac{n-5}{2}) > \lceil \frac{2n-9}{6} \rceil$, we must eventually obtain the $\mathrm{CS}_{2n+2}(n,4)$. Therefore, $\mathrm{FI}(\mathrm{CS}(n,4)) = \{0, 2, 4, \ldots, 6(n-2)-6\} \cup \{6(n-2)-2, 6(n-2)\}$. □

Theorem 3.6 *When $n \geq 6$ is even,* $\mathrm{FI}(\mathrm{CS}(n,4)) = \{0, 2, 4, \ldots, 6(n-2)-4\} \cup \{6(n-2)\}$.

Proof By Theorem 1.4 and Lemmas 3.2 and 3.3, it suffices to show that all the values are attainable. First, let

$$f(u_i) = \begin{cases} 1, & \text{for } i = 1, 3, \ldots, n-1; \\ 0, & \text{for } i = 2, 4, \ldots, n, \end{cases}$$

and

$$f(v_i) = f(w_i) = \begin{cases} 1, & \text{for } i = 1, 3, \ldots, n-1; \\ 0, & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

Then we have $v(1) = v(0)$ and each K_4 subgraph is of Type 2. Hence, we get $\mathrm{CS}_0(n,4)$. Now, exchanging the labels of u_2 and v_1 to decrease $e(0)$ by 1. Hence, we get the $\mathrm{CS}_2(n,4)$ with $e(1) - e(0) = 2$.

Next, we divide the vertices v_1 to v_n into $n/2$ pairs of vertices v_i, v_{i+1} for $i = 1, 3, 5, \ldots, n-1$. Beginning with $\mathrm{CS}_0(n,4)$, we now exchange the labels of v_1 and v_2 to decrease $e(0)$ by 2. Hence, we get $\mathrm{CS}_4(n,4)$ with $e(1) - e(0) = 4$. Using $\mathrm{CS}_4(n,4)$, we exchange the labels of v_3 and v_4 to decrease $e(0)$ by 2 again. Hence, we get $\mathrm{CS}_8(n,4)$. Repeating the same process for each pair v_i, v_{i+1}, $i = 5, 7, \ldots, n-1$. After exchanging the labels of v_i and v_{i+1}, $i \in \{5, 7, \ldots, n-1\}$, we get $\mathrm{CS}_{2(i+1)}(n,4)$ with $e(1) - e(0) = 2(i+1)$. In this process, we obtain $\mathrm{CS}_a(n,4)$ for $a \in \{4, 8, 12, \ldots, 2n\}$.

We now begin with $\mathrm{CS}_2(n,4)$. We divide the vertices v_3 to v_n into $(n-2)/2$ pairs of vertices v_i, v_{i+1}. Repeating the same process as above will decrease $e(0)$ by 2. Hence, after exchanging the labels of v_i and v_{i+1}, $i = 3, 5, \ldots, n-1$, we get $\mathrm{CS}_{2i}(n,4)$ with $e(1) - e(0) = 2i$. In this process, we obtain $\mathrm{CS}_a(n,4)$ for $a \in \{6, 10, 14, \ldots, 2n-2\}$. Hence, $\{0, 2, 4, \ldots, 2n\} \subseteq \mathrm{FI}(\mathrm{CS}(n,4))$.

We now give the labeling $\mathrm{CS}_a(n,4)$, $a \in \{2n+2, 2n+4, \ldots, 6(n-2)-4\} \cup \{6(n-2)\}$. Define

$$f(u_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n}{2}; \\ 1, & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n, \end{cases}$$

and

$$f(v_i) = f(w_i) = \begin{cases} 0, & \text{for } i = 1, 2, \ldots, \frac{n}{2}; \\ 1, & \text{for } i = \frac{n}{2}+1, \frac{n}{2}+2, \ldots, n. \end{cases}$$

So $v(0) = v(1) = \frac{n}{2} + n$, $e(0) - e(1) = 6(n-2)$, in $\mathrm{CS}_{6(n-2)}(n,4)$. Now exchange the label of $v_{\frac{n}{2}}, v_n$ in $\mathrm{CS}_{6(n-2)}(n,4)$ to get $\mathrm{CS}_{6(n-2)-4}(n,4)$. To complete the proof, we shall need the 5 labeling graphs $\mathrm{CS}_a(n,4)$, $a = 6(n-2)-6, 6(n-2)-8, \ldots, 6(n-2)-14$ that can be obtained as follows:

(1) In $\mathrm{CS}_{6(n-2)}(n,4)$, exchange the labels of u_1 and v_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-6}(n,4)$.

(2) In $\mathrm{CS}_{6(n-2)-4}(n,4)$, exchange the labels of v_1 and $v_{\frac{n}{2}}$ so that $e(1)$ increases by 2 and $e(0)$ decreases by 2. We get $\mathrm{CS}_{6(n-2)-8}(n,4)$.

(3) In $\mathrm{CS}_{6(n-2)-4}(n,4)$, exchange the labels of u_1 and w_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-10}(n)$.

(4) In $\mathrm{CS}_{6(n-2)-6}(n,4)$, exchange the labels of u_n and v_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-12}(n,4)$.

(5) In $\mathrm{CS}_{6(n-2)-8}(n,4)$, exchange the labels of u_n and v_n so that $e(1)$ increases by 3 and $e(0)$ decreases by 3. We get $\mathrm{CS}_{6(n-2)-14}(n,4)$.

We have now obtained the labeling graphs $\mathrm{CS}_{6(n-2)-4}(n,4), \mathrm{CS}_{6(n-2)-6}(n,4), \mathrm{CS}_{6(n-2)-8}(n,4), \ldots, \mathrm{CS}_{6(n-2)-14}(n,4)$. Note that the above labelings can give us the friendly index sets for $n = 6$. We now assume $n \geq 8$. Observe that in $\mathrm{CS}_{6(n-2)-4}(n,4)$ and (1) to (5) above, only the labels of $u_1, u_n, v_{\frac{n}{2}}, v_n, w_n$ are changed so that each labeling graph obtained has at least $\frac{n-4}{2}$ Type 1 subgraphs with all vertices labeled with 0, and at least $\frac{n-4}{2}$ Type 1 subgraph with all vertices labeled with 1.

Let $G_k = \{\mathrm{CS}_{6(n-2)+2-6k}(n,4), \mathrm{CS}_{6(n-2)-6k}(n,4), \ldots, \mathrm{CS}_{6(n-2)-8-6k}(n,4)\}$, where $k \geq 1$ is odd. To obtain the labeling graphs in G_k ($3 \leq k \leq \frac{n-4}{2}$ is odd), we change the labels of v_k and $v_{n/2+k}$ in G_{k-2}. This is attainable since $e(0)$ decreases by 6 after each change so that $e(0)-e(1)$ decreases by 12 successively.

Note that there are $2n-8$ even numbers from $2n+2$ to $6(n-2)-16$ inclusive. We divide them into $\lfloor\frac{2n-8}{6}\rfloor$ groups of six successive even numbers. Since $6(\frac{n-4}{2}) > \lceil\frac{2n-8}{6}\rceil$, we must eventually obtain the labeling graphs $\mathrm{CS}_{2n+2}(n,4)$. Therefore, $\mathrm{FI}(\mathrm{CS}(n,4)) = \{0, 2, 4, \ldots, 6(n-2)-4\} \cup \{6(n-2)\}$. □

Corollary 3.7 *The graph* $\mathrm{CS}(n,4)$ *is cordial for all* $n \geq 3$. *Moreover, the friendly indices form an arithmetic sequence if and only if* $n = 3$.

4. Discussion on the friendly index sets of $\mathrm{CS}(n,m)$ $(m \geq 5)$

Theorem 4.1 *In all possible* $(0,1)$*-labelings of the vertices of a* K_m*, we have* $|e(0)-e(1)| \leq \frac{m(m-1)}{2}$.

Proof In a K_m, assume that there are i vertices labeled with x and $m-i$ vertices labeled with

$1-x$ ($x \in \{0,1\}$), then $e(0) = \frac{i(i-1)}{2} + \frac{(m-i)(m-i-1)}{2} = \frac{m(m-1)}{2} - i(m-i)$ and $e(1) = i(m-i)$. So, $e(0)-e(1) = \frac{m(m-1)}{2} + 2i(i-m)$. We consider two cases.

Case 1 When $i=0$ or m, $e(0) = \frac{m(m-1)}{2}$, $e(1)=0$ so $e(0)-e(1) = \frac{m(m-1)}{2}$;

Case 2 When $1 \le i \le m-1$, $m-1 \le e(1) \le \frac{m^2}{4}$ for even m, and $m-1 \le e(1) \le \frac{m^2-1}{4}$ for odd m. We consider 2 subcases.

Subcase (1). When $i=1$ or $m-1$, $e(0)-e(1) = \frac{m(m-1)}{2} - 2(m-1) = \frac{m^2-5m+4}{2}$;

Subcase (2). When $i = \frac{m}{2}$ for even m, $e(1)-e(0) = \frac{m}{2}$. When $i = \frac{m-1}{2}$ or $i = \frac{m+1}{2}$ for odd m, $e(1)-e(0) = \frac{m-1}{2}$.

So, $|e(0)-e(1)| \le \frac{m(m-1)}{2}$. □

Theorem 4.2 *For $m \ge 5$, $\max\{\mathrm{FI}(\mathrm{CS}(n,m))\}$ equals*

(1) $\frac{m(m-1)(n-2)}{2} + m^2 - 5m + 4$ *if n is even;*

(2) $\frac{m(m-1)(n-3)}{2} + \frac{2m^2-7m+5}{2}$ *if n, m are odd;*

(3) $\frac{m(m-1)(n-3)}{2} + \frac{2m^2-7m+8}{2}$ *if n is odd and m is even.*

Proof By Theorem 4.1, we know $|e(0)-e(1)|$ is maximum when the number of subgraphs K_m that contain only 0-edges is $n-2$. Let the n subgraphs K_m be denoted by K_m^t ($t = 1,2,\ldots,n$).

Case 1 $n \ge 4$ is even. Let the vertices in K_m^t ($t = 1,2,\ldots,(n-2)/2$) be labeled with 0 and the vertices in K_m^t ($t = (n+2)/2, (n+4)/2, \ldots, n-1$) be labeled with 1. Now, label all the $(m-2)$ unlabeled vertices in $K_m^{n/2}$ with 0 and all the $m-2$ unlabeled vertices in K_m^n with 1. We then get a friendly labeling with $\max|e(0)-e(1)| = \frac{nm(m-1)}{2} - 4(m-1) = \frac{m(m-1)(n-2)}{2} + m^2 - 5m + 4$.

Case 2 Let $n=3$. We consider two subcases.

Subcase (1). $m \ge 5$ is odd. Label all the vertices in K_m^1 with x. Recall that u_3 is the common vertex of K_m^2 and K_m^3. For the remaining $m-1$ vertices in K_m^2 and in K_m^3, let the number of vertices labeled with $1-x$ be i and j, respectively, such that $i \le j$ satisfying:

$$i = (m-1)/2, j = m-1; \tag{1.1}$$

and

$$(m+1)/2 \le i \le j \le m-2, \quad i+j = 3(m-1)/2. \tag{1.2}$$

Since the labeling is friendly, all the i and j vertices are distinct.

In (1.1), we get $e(0)-e(1) = \frac{2m^2-7m+5}{2}$. In (1.2), we consider three cases:

(I). Vertex u_3 is labeled with x. We have $e(1) = i(m-i) + j(m-j)$. So, $e(0)-e(1) = 3m(m-1)/2 - 2e(1) = 3m(m-1)/2 + 2[i(i-m) + j(j-m)] = 2(i^2+j^2) - 3m(m-1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-11m+17}{2}$ when $i = (m+1)/2, j = m-2$.

(II). u_3 is one of the i vertices labeled with $1-x$. We have $e(1) = i(m-i) + (j+1)(m-j-1)$. So, $e(0)-e(1) = 3m(m-1)/2 + 2[i(i-m) + (j+1)(j+1-m)] = 2[i^2 + (j+1)^2] - m(3m+1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-7m+5}{2}$ when $i = (m+1)/2, j = m-2$ as in (1.1) above.

(III). Vertex u_3 is one of the j vertices labeled with $1-x$. We have $e(1) = (i+1)(m-i-$

$1) + j(m-j)$. So, $e(0)-e(1) = 3m(m-1)/2 + 2[(i+1)(i+1-m) + j(j-m)] = 2[(i+1)^2 + j^2] - m(3m+1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-11m+25}{2}$ when $i = (m+1)/2, j = m-2$.

Therefore, for odd $m \geq 5$, $\max\{e(0)-e(1)\} = \frac{2m^2-7m+5}{2}$.

Subcase (2). $m \geq 6$ is even. We label the vertices as in Subcase (1) above satisfying:

$$i = m/2, j = m-1 \text{ or } i = (m-2)/2, j = m-1, \tag{2.1}$$

$$(m+2)/2 \leq i \leq j \leq m-2, \quad i+j = (3m-2)/2; \tag{2.2}$$

or

$$m/2 \leq i \leq j \leq m-2, \quad i+j = (3m-4)/2. \tag{2.3}$$

Recall that all the i and j vertices are distinct.

In (2.1), we get $e(0)-e(1) = \frac{2m^2-7m+4}{2}$ or $\frac{2m^2-7m+8}{2}$. In (2.2), we consider three cases:

(IV). Vertex u_3 is labeled with x. We have $e(1) = i(m-i) + j(m-j)$. So, $e(0)-e(1) = 2(i^2+j^2) - m(3m-1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-11m+20}{2}$ when $i = (m+2)/2, j = m-2$.

(V). Vertex u_3 is one of the i vertices labeled with $1-x$. We have $e(1) = i(m-i)+(j+1)(m-j-1)$. So, $e(0)-e(1) = 3m(m-1)/2+2[i(i-m)+(j+1)(j+1-m)] = 2[i^2+(j+1)^2]-3m(m+1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-7m+8}{2}$ when $i = (m+2)/2, j = m-2$.

(VI). Vertex u_3 is one of the j vertices labeled with $1-x$. We have $e(1) = (i+1)(m-i-1) + j(m-j)$. So, $e(0)-e(1) = 3m(m-1)/2 + 2[(i+1)(i+1-m) + j(j-m)] = 2[(i+1)^2 + j^2] - 3m(m+1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-11m+32}{2}$ when $i = (m+2)/2$, $j = m-2$.

In (2.3), we also consider three cases.

(VII). Vertex u_3 is labeled with x. We have $e(1) = i(m-i) + j(m-j)$. So, $e(0)-e(1) = 2(i^2+j^2) - m(3m-5)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-11m+16}{2}$ when $i = m/2, j = m-2$.

(VIII). Vertex u_3 is one of the i vertices labeled with $1-x$. We have $e(1) = i(m-i) + (j+1)(m-j-1)$. So, $e(0)-e(1) = 3m(m-1)/2 + 2[i(i-m) + (j+1)(j+1-m)] = 2[i^2 + (j+1)^2] - m(3m-1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-7m+4}{2}$ when $i = m/2, j = m-2$.

(IX). Vertex u_3 is one of the j vertices labeled with $1-x$. We have $e(1) = (i+1)(m-i-1) + j(m-j)$. So, $e(0)-e(1) = 3m(m-1)/2 + 2[(i+1)(i+1-m) + j(j-m)] = 2[(i+1)^2+j^2] - m(3m-1)/2$. Hence, $\max\{e(0)-e(1)\} = \frac{2m^2-11m+20}{2}$ when $i = m/2$, $j = m-2$.

Therefore, for even $n \geq 6$, $\max\{e(0)-e(1)\} = \frac{2m^2-7m+8}{2}$.

Case 3 $n \geq 5$ is odd. We consider two subcases.

Subcase (3). $m \geq 5$ is odd. By Theorem 4.1, we seek to maximize the number of subgraph K_m with all 0-edges only. Without loss of generality, we label the vertices of K_m^t by x for $2 \leq t \leq (n+1)/2$, and by $1-x$ for $(n+5)/2 \leq t \leq n$. We then label all the remaining $(m-1)$ vertices in K_m^1 by $1-x$. For $K_m^{(n+3)/2}$, we label $(m-1)/2$ of the unlabeled vertices by $(1-x)$ and the rest by x. We now have a friendly labeling with maximum 0-edges. By Subcase (1), we can get the maximum of $|e(0)-e(1)|$ is $\frac{m(m-1)(n-3)}{2} + \frac{2m^2-7m+5}{2}$.

Subcase (4). $m \geq 6$ is even. Similarly to Subcase (3) above, by Theorem 4,1 and Subcase (2), we can get the maximum of $|e(0) - e(1)|$ is $\frac{m(m-1)(n-3)}{2} + \frac{2m^2-7m+8}{2}$. $\square$

Theorem 4.3 *If $n \equiv 0 \pmod 4$ and $m = k^2$ (k an integer), then* CS(n, m) *is cordial.*

Proof Suppose CS(n, m) is cordial, then $e(0) - e(1) = 0$. Assume every subgraph K_m has i 1-vertices and $i(m-i)$ 1-edges. By the given condition, we have $\frac{m(m-1)}{2} - i(m-i) = i(m-i)$ so that $i = \frac{m \pm \sqrt{m}}{2}$. Define a friendly labeling such that the number of subgraphs K_m having $\frac{m \pm \sqrt{m}}{2}$ 0-vertices and the number of subgraphs having $\frac{m \pm \sqrt{m}}{2}$ 1-vertices are equal. Now, CS(n, m) is cordial since the labeling is attainable. $\square$

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