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Growth and Approximation of Generalized Bi-Axially Symmetric Potentials

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Abstract The paper deals with growth estimates and approximation (not necessarily entire) of solutions of certain elliptic partial differential equations. These solutions are called generalized bi-axially symmetric potentials (GBASP's). To obtain more refined measure of growth, we have defined q-proximate order and obtained the characterization of generalized q-type and generalized lower q-type with respect to q-proximate order of a GBASP in terms of approximation errors and ratio of these errors in sup norm.

Keywords generalized bi-axially symmetric potentials; q-proximate order; Jacobi polynomials; generalized q-type; generalized lower q-type; approximation errors

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1. Introduction and preliminaries

Regular solutions of the elliptic partial differential equations

$$\frac{\partial^2 H}{\partial x^2}+\frac{\partial^2 H}{\partial y^2}+\frac{2\alpha+1}{y}\frac{\partial H}{\partial y}+\frac{2\beta+1}{x}\frac{\partial H}{\partial x}=0,\ \alpha,\beta>-\frac{1}{2}, \tag{1.1}$$

which are even in x and y are known as generalized bi-axially symmetric potentials (GBASP's) [1]. A polynomial of degree n in x and y is said to be GBASP polynomial of degree n if it satisfies (1.1). A GBASP H regular about origin can be expanded as

$$H\equiv H(x,y)=H(r\cos\theta,r\sin\theta)=\sum_{n=0}^{\infty}a_n r^{2n}{P_n}^{(\alpha,\beta)}(\cos 2\theta), \tag{1.2}$$

where ${P_n}^{(\alpha,\beta)}(t)$ are Jacobi polynomials.

Let $D_R=\{(x,y): x^2+y^2<R^2\},\ 0<R\le\infty$ and $\overline{D}_R$ be the closure of D_R. A GBASP H is said to be regular in D_R if the series (1.2) converges uniformly on every compact subset of D_R. Let H_R be the class of all GBASP's H regular in D_R, for every $R'\le R$ but for no $R'>R$. The functions in the class H_∞ are called entire GBASP's.

Fryant [2] studied the function theoretic approach to the study of ultraspherical expansions and their conjugates of generalized axisymmetric potentials. Kumar [3] extended Fryant's results

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for GBASP's. McCoy [4] studied the fast growth of entire function solutions of the equation (1.1) in terms of order and type using the concept of index k and obtained bounds on the order and type of H that reflect their antecedents in the theory of analytic functions of a single complex variable. Kumar [5,6] refined McCoy [4] results and obtained some bounds on the growth parameters of entire function solutions of helmholtz equation in terms of coefficients and Chebyshev approximation errors in sup norm. In [7] Kumar studied the Chebyshev polynomial approximation of entire solutions of helmholtz equations in R^2 in Banach spaces. His results apply satisfactorily for slow growth.

McCoy [8] considered the approximation of an entire GBASP's H by GBASP polynomials and found the rate of decay of approximation errors

$$E_n(H,1) = \inf_{g\in\Pi_n} \|H-g\|_1 = \inf_{g\in\Pi_n} \{\max_{(x,y)\in\bar{D}_1} |H(x,y)-g(x,y)|\},$$

in terms of growth parameters associated with the maximum modulus function

$$M(r,H) = \max_{\theta} |H(r\cos\theta, r\sin\theta)|.$$

Also, McCoy [9] considered the approximation of pseudo-analytic functions on the disk. Pseudo-analytic functions are constructed as complex combinations of real-valued analytic solutions to the Stokes-Beltrami system. These solutions include GBASP's. McCoy obtained some coefficients and Bernstein type growth theorems on the disk. In this paper our results are different from all those of above results and generalize the results contained in [10–12].

A GBASP H is said to be regular on $\overline{D}_{R_0}$, $0 < R_0 < \infty$, the closure of D_{R_0}, if it is regular in $D_{R'}$, for some $R' \to R_0$. Let $\overline{H}_{R_0}$ be the class of all GBASP's H regular on $\overline{D}_{R_0}$. For $H \in \overline{H}_{R_0}$, the uniform norm $\|H\|_{R_0}$ of H on $\overline{D}_{R_0}$ (i.e., the space $\overline{H}_{R_0}$ is endowed with the uniform norm $\|.\|_{R_0}$) is defined by

$$\|H\|_{R_0} = \max_{(x,y)\in\overline{D}_{R_0}} |H(x,y)|, \tag{1.3}$$

and the approximation error $E_n(H,R_0)$ is defined as

$$E_n(H,R_0) = \inf_{g\in\Pi_n} \|H-g\|_{R_0}, \tag{1.4}$$

where Π_n consists of all GBASP polynomials of degree at most $2n$.

The concepts of index q and the q-order $\rho_R(q)$ are introduced by Bajpai et al. [13] in order to obtain a measure of growth of the maximum modulus, when it is rapidly increasing. Thus, let $M(r,H) \to \infty$ as $r \to R$ and for $q = 2, 3, \ldots$, set

$$\rho_R(q,H) = \lim_{r\to R} \sup \frac{\log^{[q]} M(r,H)}{\log(\frac{R}{(R-r)})},$$

where $\log^{[0]} M(r,H) = M(r,H)$ and $\log^{[q-1]} M(r,H) = \log\log^{[q-2]} M(r,H)$. The GBASP $H \in H_R$ is said to have the index q if $\rho_R(q,H) < \infty$ and $\rho_R(q-1,H) = \infty$. If q is the index of H, then $\rho_R(q,H)$ is called the q-order of H.

To obtain a more refined measure of growth of GBASP $H \in H_R$, we consider a real-valued function $\rho_R(q,H,r)$ $(0 < r < R)$ having the following properties:

(i) $\rho_R(q,H,r)$ is positive, continuous and piecewise differentiable in $0 \le r_0 < r < R$;

(ii) $\lim_{r\to R^-} \rho_R(q,H,r) \to \rho_R(q,H)$, $(0 < \rho_R(q,H) < \infty)$;

(iii) $\lim_{r\to R^-} -\rho'_R(q,H,r)\triangle_{[q-1]}(\frac{R-r}{R}) \to 0$, where

$$\triangle_{[q-1]}(\frac{R-r}{R}) = \prod_{i=0}^{q-1} \log^{[i]}(\frac{R-r}{R})$$

and $\rho'_R(q,H,r)$ denotes the derivative of $\rho_R(q,H,r)$.

A function $\rho_R(q,H,r)$ satisfying above properties is said to be a q-proximate order. For a GBASP $H \in H_R$ having non-zero finite q-order $\rho_R(q,H)$, i.e., $\rho_R(q,H) < \infty$, and $\rho_R(q-1,H) = \infty$, let

$$\begin{aligned} T^*_R(q,H) &= \lim_{r\to R}\sup \frac{\log^{[q-1]} M(r,H)}{(R/(R-r))^{\rho_R(q,H,r)}}, \\ t^*_R(q,H) &= \lim_{r\to R}\inf \frac{\log^{[q-1]} M(r,H)}{(R/(R-r))^{\rho_R(q,H,r)}}, \end{aligned} \tag{1.5}$$

$0 < t^*_R(q,H) \le T^*_R(q,H) < \infty$.

The quantities $T^*_R(q,H)$ and $t^*_R(q,H)$ are known as generalized q-type and generalized lower q-type of a GBASP H with respect to q-proximate order $\rho_R(q,H,r)$. If these quantities are different from zero and infinity, then $\rho_R(q,H,r)$ is said to be q-proximate order of a given GBASP H with index q.

Since $(R/(R-r))^{\rho_R(q,H,r)}$ is monotonically increasing, we can define the function $\phi(x)$, $x > x_0$ to be the unique solution of the equation

$$x = (R/(R-r))^{\rho_R(q,H,r)+A(q,H)} \Leftrightarrow R/(R-r) = \phi(x),$$

where $A(q,H) = 1$ if $q = 2$ and $A(q,H) = 0$ if $q = 3,4,\ldots$.

The growth of GBASP $H \in H_R$ has been studied in terms of q-orders and q-types by Kasana and Kumar [12]. However, these parameters are inadequate for comparing the growth of those GBASP $H \in H_R$ which are of same q-orders but of infinite q-type. To refine this scale, we have used here the concept of q-proximate order for GBASP H with index q. Moreover, we obtain the characterization of generalized q-type and generalized lower q-type with respect to q-proximate order of a GBASP $H \in H_R$ in terms of approximation errors in supnorm defined by (1.4). Some results are also obtained in terms of the ratio of these approximation errors.

2. Auxiliary results

Now we shall prove some auxiliary results which shall be used in proving the main theorems.

Lemma 2.1 *Let $H \in \overline{H}_{R_0}$. Then for $n \ge 1$, we have*

$$|a_n| R_0^{2n} \le 2\big((2n+\alpha+\beta+1)P(\alpha,\beta)P(n,\alpha,\beta)\big)^{1/2} E_{n-1}(H,R_0), \tag{2.1}$$

where

$$P(n,\alpha,\beta) = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)},\ P(\alpha,\beta) = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}. \tag{2.2}$$

Proof The proof follows from [10].

Lemma 2.2 *Let $H \in H_R,\ 0 < R < \infty$ and $R > R_0$. Then,*

$$M(r,H) \le |a_0| + \frac{2\sqrt{P(\alpha,\beta)}}{\Gamma(\eta+1)} m(r,h),\ \ \eta = \max(\alpha,\beta), \tag{2.3}$$

where $m(r,h)$ denotes the maximum term of h defined by

$$h(u) = \sum_{n=1}^{\infty} \Big((2n+\alpha+\beta+1)P(n,\alpha,\beta)\Big)^{1/2} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)} E_{n-1}(H,R_0)(u/R_0)^{2n} \tag{2.4}$$

and $P(n,\alpha,\beta)$, $P(\alpha,\beta)$ are given in (2.2).

Proof It is given in [14, p.168] that

$$\max_{-1\le t\le 1} |P_k^{(\alpha,\beta)}(t)| = \frac{\Gamma(k+\eta+1)}{\Gamma(k+1)\Gamma(\eta+1)}.$$

So we have

$$\begin{aligned}
|\sum_{n=0}^{\infty} a_n r^{2n} P_n^{(\alpha,\beta)}(\cos 2\theta)| \le & |a_0| + \sum_{n=1}^{\infty} |a_n| r^{2n} \frac{\Gamma(n+\eta+1)}{\Gamma(n+1)\Gamma(\eta+1)} \\
\le & |a_0| + \frac{2\sqrt{P(\alpha,\beta)}}{\Gamma(\eta+1)} \sum_{n=0}^{\infty} E_{n-1}(H,R_0)(r/R_0)^{2n} \times \\
& \big((2n+\alpha+\beta+1)P(n,\alpha,\beta)\big)^{1/2} \frac{\Gamma(\eta+n+1)}{\Gamma(n+1)} \text{ (using Lemma 2.1)}
\end{aligned}$$

or,

$$M(r,H) \le |a_0| + \frac{2\sqrt{P(\alpha,\beta)}}{\Gamma(\eta+1)} m(r,h),$$

where

$$h(u) = \sum_{n=1}^{\infty} \Big((2n+\alpha+\beta+1)P(n,\alpha,\beta)\Big)^{1/2} \frac{\Gamma(\eta+n+1)}{\Gamma(n+1)} E_{n-1}(H,R_0)(u/R_0)^{2n}.$$

Lemma 2.3 *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < R$ $(0 < R < \infty)$ with q-order $\rho_R(q)$ $(\rho_R(q) > 0)$ and a q-proximate order $\rho_R(q,r)$. If $\varphi(n) = \log^+ |a_n/a_{n+1}|$ forms a non-decreasing function of n for all large n, then the generalized q-type $T_R^*(q)$ of $f(z)$ with respect to a q-proximate order $\rho_R(q,r)$ is given by*

$$\limsup_{n\to\infty} \big[\phi(\log^{[q-2]} n) \log^+ |a_n/a_{n+1}|R\big]^{\rho_R(q)+A(q)} = T_R^*(q) B_R(q), \tag{2.5}$$

where

$$B_R(q) = \begin{cases} \frac{(\rho_R(q)+1)^{(\rho_R(q)+1)}}{(\rho_R(q))^{(\rho_R(q))}}, & \text{if } q = 2 \\ 1, & \text{if } q = 3,4,\ldots, \end{cases}$$

$\rho_R(q) > 0,\ q = 2,3,\ldots$.

$$A(q) = \begin{cases} 1, & \text{if } q = 2 \\ 0, & \text{if } q = 3,4,\ldots. \end{cases}$$

Proof Let

$$\limsup_{n\to\infty} \big[\phi(\log^{[q-2]} n) \log^+ |a_n/a_{n+1}|R\big]^{\rho'_R(q)} = Q,$$

where $\rho'_R(q) = \rho_R(q) + A(q)$. We first assume that $0 \le Q < \infty$. Then for $\epsilon > 0$ and sufficiently large $n > n(\epsilon)$, we have

$$\log |a_n/a_{n-1}|R < \big[(Q+\epsilon)/\phi(\log^{[q-2]} n)\big]^{1/\rho'_R(q)}.$$

Writing the above inequality for $n = N+1, N+2, \ldots, k$ and adding them, we obtain

$$\begin{aligned}\log |a_k/a_N|R^{k-N} &< \sum_{n=N+1}^{k} \big[(Q+\epsilon)/\phi(\log^{[q-2]} n)\big]^{1/\rho'_R(q)}\\ &< (k-N)\big[(Q+\epsilon)/\phi(\log^{[q-2]} k)\big]^{1/\rho'_R(q)}.\end{aligned}$$

Hence for all large k, we get

$$\log^+ |a_k|R^k < O(1) + (1+o(1))k\big[(Q+\epsilon)/\phi(\log^{[q-2]} k)\big]^{1/\rho'_R(q)},$$

$$\Big[\frac{\log^+ |a_k|R^k}{k}\Big]^{\rho'_R(q)} < \big[(Q+\epsilon)/\phi(\log^{[q-2]} k)\big] + o(1),$$

$$\limsup_{k\to\infty} \Big\{\phi(\log^{[q-2]} k)\Big[\frac{\log^+ |a_k|R^k}{k}\Big]\Big\}^{\rho'_R(q)} < Q+\epsilon. \tag{2.6}$$

Using [15, Thm. 2.1], we get $T^*_R(q)B_R(q) \le Q$. The inequality (2.6) holds if $Q = \infty$. To prove the reverse inequality, let us put the left hand side of (2.6) equal to $S_R(q)$ and let $0 \le S_R(q) < \infty$. Then for arbitrary small $\epsilon > 0$, we have for all large value of $n > N(\epsilon)$,

$$\log^+ |a_n|R^n < n\big[(S_R(q)+\epsilon)^{1/\rho'_R(q)}/\phi(\log^{[q-2]} n)\big]. \tag{2.7}$$

Next we assume $\varphi(n) = \log |a_n/a_{n+1}|$ is nondecreasing function of n. Then for large n,

$$\log |a_N/a_n| = \log |a_N/a_{N+1}| + \cdots + \log |a_{n-1}/a_n|,$$

$$\log |a_N/a_n| = (n-N)\varphi(n-1),$$

$$\log^+ |a_n|R^n > \log |a_N| + (n-N)\log^+ |a_n/a_{n-1}| + n\log R. \tag{2.8}$$

In view of (2.7) and (2.8), we obtain

$$\big[\log^+ |a_n/a_{n-1}|R(1-o(1))\big]^{\rho'_R(q)} < (S_R(q)+\epsilon)/[\phi((\log^{[q-2]}(n))]^{\rho'_R(q)}, \quad n > N,$$

$$Q = \limsup_{n\to\infty}[\phi((\log^{[q-2]}(n))\log^+ |a_n/a_{n-1}|R]^{\rho'_R(q)} \le S_R(q), \tag{2.9}$$

which obviously holds if $S_R(q) = \infty$. Combining (2.6) and (2.9), we get

$$\begin{aligned}&\limsup_{n\to\infty} \big[\phi(\log^{[q-2]}(n))\big(\frac{\log^+ |a_n|R^n}{n}\big)\big]^{\rho_R(q)+A(q)}\\ &= \limsup_{n\to\infty} \big[\phi(\log^{[q-2]}(n))(\log^+ |a_n/a_{n-1}|R)\big]^{\rho_R(q)+A(q)}.\end{aligned}$$

Now the proof follows in view of [15, Thm. 2.1].

3. Main results

Now we shall prove our main results.

Theorem 3.1 *Let $H \in H_R$, $0 < R < \infty$, have q-order $\rho_R(q,H)$ (> 0) and a q-proximate order*

$\rho_R(q,H,r)$. *Then the generalized q-type* $T^*_R(q,H)$ *of* H *with respect to a q-proximate order* $\rho_R(q,H,r)$ *is given by*

$$T^*_R(q,H) = B_R(q,H)V_R(q,H), \tag{3.1}$$

where

$$V_R(q,H) = \limsup_{n\to\infty} \Big[\frac{\phi(\log^{[q-2]} n)\log^+ E_n(H,R_0)(R/R_0)^{2n}}{n}\Big]^{\rho_R(q,H)+A(q,H)} \tag{3.2}$$

and

$$B_R(q,H) = \begin{cases} \frac{(\rho_R(q,H)/2)^{\rho_R(q,H)}}{(\rho_R(q,H)+1)^{\rho_R(q,H)+1}}, & \text{if } q=2 \\ 1, & \text{if } q=3,4,\dots, \end{cases}$$

$\rho_R(q,H) > 0,\ q = 2,3,\dots.$ $A(q,H) = 1$ *if* $q = 2$ *and* $A(q,H) = 0$ *if* $q = 3,4,\dots.$ *If* $V_R(q,H) = 0$ *or* $\infty,\ H \in H_R$ *is respectively of growth* $(\rho_R(q,H),0)$ *or of growth* $(\rho_R(q,H),\infty)$.

Proof Let $0 < T^*_R(q,H) < \infty$ and H have q-order $\rho_R(q,H)$. Then for arbitrary $\epsilon > 0$, equality (1.5) gives that there exists $r_0 = r_0(\epsilon)$ such that

$$\log M(r,H) \le \exp^{[q-2]}\big\{(T^*_R(q,H)+\epsilon)(R/(R-r))^{\rho_R(q,H,r)}\big\} \tag{3.3}$$

for $r_0 < r < R$. Using [15, Lemma 1] and (3.3), we get

$$\begin{aligned}\log^+ E_n(H,R_0)(R/R_0)^{2n} \le &\exp^{[q-2]}\big\{(T^*_R(q,H)+\epsilon)(R/(R-r))^{\rho_R(q,H,r)}\big\}+ \\ &2n\log(R/r) + (\eta+1/2)\log(n+1) + \log^+ K\end{aligned} \tag{3.4}$$

for all r sufficiently near to R and all sufficiently large values of n.

For $q = 2$, let r be given by the equation

$$\big(R/(R-r)\big) = \phi\Big(\frac{2n}{\rho_R(2,H)(T^*_R(2,H)+\epsilon)}\Big). \tag{3.5}$$

From (3.3), using $R/(R-r) \sim \log R/r$ for sufficiently large r near to R, we obtain

$$\begin{aligned}\log^+ E_n(H,R_0)(R/R_0)^{2n} &\le (T^*_R(2,H)+\epsilon)(R/(R-r))^{\rho_R(2,H,r)} + 2n(R-r)/R+ \\ &\quad (\eta+1/2)\log(n+1) + \log^+ K \\ &\le \frac{2n}{\rho_R(2,H)R/(R-r)} + 2n(R-r)/R + (\eta+1/2)\log(n+1) + \log^+ K\end{aligned}$$

or,

$$\begin{aligned}\frac{1}{n}\log^+ E_n(H,R_0)(R/R_0)^{2n} &< 2(R-r)/R\big[1 + \frac{1}{\rho_R(2,H)} + O(1)\big] \\ &= \frac{2}{\phi\big(\frac{2n}{\rho_R(2,H)(T^*_R(2,H)+\epsilon)}\big)}\Big[\frac{\rho_R(2,H)+1}{\rho_R(2,H)} + O(1)\Big] \\ &= \frac{2\big(\rho_R(2,H)(T^*_R(2,H)+\epsilon)\big)^{1/\rho_R(2,H)+1}}{\phi(2n)}\Big[\frac{\rho_R(2,H)+1}{\rho_R(2,H)} + O(1)\Big]\end{aligned}$$

or,

$$\begin{aligned}&\Big[\frac{\phi(2n)\log^+ E_n(H,R_0)(R/R_0)^{2n}}{n}\Big]^{\rho_R(2,H)+1} \\ &\quad \le 2^{(\rho_R(2,H)+1)}\frac{(\rho_R(2,H)+1)^{\rho_R(2,H)+1}}{(\rho_R(2,H))^{\rho_R(2,H)}}.(T^*_R(2,H)+\epsilon) + o(1)\end{aligned}$$

or,

$$\Big[\frac{\phi(n)\log^{+} E_n(H,R_0)(R/R_0)^{2n}}{n}\Big]^{\rho_R(2,H)+1}$$
$$\leq \Big(\frac{2}{\rho_R(2,H)}\Big)^{\rho_R(2,H)}(\rho_R(2,H)+1)^{(\rho_R(2,H)+1)}T^{*}_R(2,H). \tag{3.6}$$

Now consider the case $q = 3, 4, \ldots$. Choose r such that

$$R/(R-r) = \phi\Big(\frac{\log^{[q-2]}(n/\rho_R(q,H))}{T^{*}_R(q,H)+\epsilon}\Big) \text{ as } n \to \infty.$$

For $n > n_0$ and $q = 3, 4, \ldots$, (3.4) becomes

$$\log^{+} E_n(H,R_0)(R/R_0)^{2n} \leq \exp^{[q-2]}\{\log^{[q-2]}(n/\rho_R(q,H))\} + n\log(R/r) +$$
$$(\eta+1/2)\log(n+1) + \log^{+} K$$
$$= \frac{n}{\rho_R(q,H)} + n(R-r)/R + (\eta+1/2)\log(n+1) + \log^{+} K$$

or,

$$\frac{1}{n}\log^{+} E_n(H,R_0)(R/R_0)^{2n}$$
$$\leq \frac{1}{\rho_R(q,H)} + (R-r)/R + O(1) \text{ for sufficiently large values of } n$$
$$\leq \frac{1}{\rho_R(q,H)} + \frac{1}{\phi\big(\frac{\log^{[q-2]}(n/\rho_R(q,H))}{T^{*}_R(q,H)+\epsilon}\big)} + O(1)$$
$$= \frac{(T^{*}_R(q,H)+\epsilon)^{1/\rho_R(q,H)}}{\phi\big(\log^{[q-2]}(n/\rho_R(q,H))\big)}\Big[1 + \frac{\phi\big(\log^{[q-2]}(n/\rho_R(q,H))\big)}{(T^{*}_R(q,H)+\epsilon)^{1/\rho_R(q,H)}}.\rho_R(q,H) + O(1)\Big]$$

or,

$$\Big[\frac{\phi(\log^{[q-2]} n)\log^{+} E_n(H,R_0)(R/R_0)^{2n}}{n}\Big]^{\rho_R(q,H)} \leq (T^{*}_R(q,H)+\epsilon)[1+o(1)]. \tag{3.7}$$

Proceeding to limits in the inequalities (3.6), (3.7) and combining, we get

$$\limsup_{n\to\infty}\Big[\frac{\phi(\log^{[q-2]} n)\log^{+} E_n(H,R_0)(R/R_0)^{2n}}{n}\Big]^{\rho_R(q,H)+A(q,H)}$$
$$\leq T^{*}_R(q,H)/B_R(q,H). \tag{3.8}$$

To prove the reverse inequality in (3.8), we use Lemma 2.2 and apply [15, Thm. 2.1] to the function $h(u)$ defined by (2.4) with simple manipulation.

If $V_R(q,H) = 0$, then GBASP H is of q-order at most $\rho_R(q,H)$ and equality (3.1) gives if GBASP H is of q-order $\rho_R(q,H)$, then its q-type is zero. If $V_R(q,H) = 0$, then GBASP H has the growth $(\rho_R(q,H), 0)$.

Similarly, if $V_R(q,H) = \infty$, then a GBASP H is of q-order at least $\rho_R(q,H)$ and (3.8) implies that if H is of q-order $\rho_R(q,H)$, then $T^{*}_R(q,H) = \infty$. Thus if $V_R(q,H) = \infty$, GBASP H is of growth not less than $(\rho_R(q,H), \infty)$. Hence the proof follows.

Remark 3.2 Theorem 3.1 generalizes the Theorem 3.1 of [11].

It is known that a theorem analogous to Theorem 3.1 does not always hold for the generalized lower q-type of a GBASP $H \in H_R$. Hence we shall prove some results on generalized lower q-type of a GBASP $H \in H_R$, in terms of approximation error defined by (1.4).

Theorem 3.3 *Let $H \in H_R$, $0 < R < \infty$, have q-order $\rho_R(q,H)$ (> 0) and a q-proximate order $\rho_R(q,H,r)$. Let n_k be an increasing sequence of natural numbers. Then the generalized lower q-type $t^*_R(q,H)$ of a GBASP H with respect to a q-proximate order $\rho_R(q,H,r)$ is given by*

$$\begin{aligned} &B_R(q,H)t^*_R(q,H) \\ &\geq \liminf_{k\to\infty} \big\{\phi(\log^{[q-2]} n_{k-1})\big(\frac{\log^+ E_{n_k}(H,R_0)(R/R_0)^{2n_k}}{n_k}\big)\big\}^{\rho_R(q,H)+A(q,H)}. \end{aligned} \tag{3.9}$$

Proof Let us assume that

$$\begin{aligned} \beta^*(q,H) &\equiv \beta^*(\{n_k,H\}) \\ &= \liminf_{k\to\infty} \big\{\phi\big(\frac{\log^{[q-2]} n_{k-1}}{B_R(q,H)}\big)\big(\frac{\log^+ E_{n_k}(H,R_0)(R/R_0)^{2n_k}}{n_k}\big)\big\}^{\rho_R(q,H)+A(q,H)}. \end{aligned}$$

If $\beta^*(q,H) = 0$, then $t^*_R(q,H) \geq \beta^*(q,H)$ is obvious. So $\beta^*(q,H) > \epsilon > 0$. Then for sufficiently large $k > k_0 = k_0(\epsilon)$, we have

$$\frac{n_k\big[(\beta^*(q,H)-\epsilon)B_R(q,H)\big]^{\frac{1}{(\rho_R(q,H)+A(q,H))}}}{\phi(\log^{[q-2]} n_{k-1})} < \log^+ E_{n_k}(H,R_0)(R/R_0)^{2n_k}. \tag{3.10}$$

In view of [11, Lemma 2.1] with (3.10) we get

$$\begin{aligned} \log M(r,H) \geq & n_k \frac{\big[(\beta^*(q,H)-\epsilon)B_R(q,H)\big]^{\frac{1}{(\rho_R(q,H)+A(q,H))}}}{\phi(\log^{[q-2]} n_{k-1})} + \\ & 2n_k \log(R_0/R) - (\eta+1/2)\log(n_k+1) + \\ & (2n_k+2)\log(r/R_0) - \log^+ K \end{aligned}$$

or,

$$\begin{aligned} \log M(r,H) \geq & \frac{n_k}{\phi(\log^{[q-2]} n_{k-1})}\big[(\beta^*(q,H)-\epsilon)B_R(q,H)\big]^{\frac{1}{(\rho_R(q,H)+A(q,H))}} - \\ & 2n_k\big(\frac{R-r}{R}\big) - 2\log(r/R_0) - (\eta+1/2)\log(n_k+1) - \log^+ K. \end{aligned}$$

Let us choose a sequence $\{r_{n_k}\}$ such that

$$r_{n_k}/R = \exp\big\{-\frac{1}{2}\big[\frac{\big((\beta^*(q)-\epsilon)C(q,H)\big)^{\frac{1}{(\rho_R(q,H)+A(q,H))}}}{\phi(\log^{[q-2]} n_{k-1})}\big]\big\}, \quad k = 1,2,\ldots, \tag{3.11}$$

where

$$C(q,H) = \rho_R(2,H) \text{ if } q = 2 \text{ and } C(q,H) = C,\ 0 < C < 1 \text{ if } q = 3,4,\ldots.$$

If $k > k_0$ and $r_k \leq r \leq r_{k+1}$, then

$$\begin{aligned} \log M(r,H) \geq & \log^+ E_{n_k}(H,R_0) + 2n_k\log(r_{n_k}/R) - 2\log(r/R_0) + \\ & (\eta+1/2)\log(n_k+1) - \log^+ K \\ = & \log^+ E_{n_k}(H,R_0) - 2n_k\frac{(R-r_{n_k})}{R} - 2\log(r/R_0) - \\ & (\eta+1/2)\log(n_k+1) - \log^+ K \end{aligned}$$

$$>n_k\Big[\frac{\big[(\beta^*(q,H)-\epsilon)\big]^{\frac{1}{(\rho_R(q,H)+A(q,H))}}}{\phi(\log^{[q-2]}n_{k-1})}\Big].$$
$$\Big[\big\{(B_R(q,H))^{\frac{1}{(\rho_R(q,H)+A(q,H))}}-(C(q,H))^{\frac{1}{(\rho_R(q,H)+A(q,H))}}\big\}-O(1)\Big].$$

Using (3.11) in above inequality, we get for $k>k_0$,

$$\log M(r,H)>\frac{\exp^{[q-2]}\{(\beta^*(q,H)-\epsilon)C(q,H)2(\frac{R-r_{n_k-1}}{R})\}^{-\rho_R(q,H)-A(q,H)}}{(C(q,H))^{\frac{1}{(\rho_R(q,H)+A(q,H))}}\big(2(\frac{R-r_{n_k}}{R})\big)^{-1}}\times$$
$$\big\{(B_R(q,H))^{\frac{1}{(\rho_R(q,H)+A(q,H))}}-(C(q,H))^{\frac{1}{(\rho_R(q,H)+A(q,H))}}-O(1)\big\}$$
$$>\frac{\exp^{[q-2]}\{(\beta^*(q,H)-\epsilon)C(q,H)2(\frac{R-r_{n_k-1}}{R})\}^{-\rho_R(q,H)-A(q,H)}}{\big(2(\frac{R-r_{n_k}}{R})\big)^{-1}}\times$$
$$\Big[\big(\frac{B_R(q,H)}{C(q,H)}\big)^{\frac{1}{\rho_R(q,H)+A(q,H)}}-1-o(1)\Big].$$

For the case $q=2$, it follows by the lower estimate at $\log M(r,H)$ that

$$t_R^*(2,H)=\liminf_{r\to R}\frac{\log M(r,H)}{\big((R-r)/R\big)^{-\rho_R(2,H,r)}}\geq\beta^*(2,H)$$

and for $q=3,4,\ldots$, we get

$$t_R^*(q,H)=\liminf_{r\to R}\frac{\log^{[q-1]}M(r,H)}{\big((R-r)/R\big)^{-\rho_R(q,H,r)}}\geq\beta^*(q,H)C.$$

The above inequality holds for every C such that $0<C<1$. Making C tend to 1, we have $t_R^*(q,H)\geq\beta^*(q,H)$ for $q=3,4,\ldots$.

If $\beta^*(q,H)=0$, the equality (3.9) follows trivially. If $\beta^*(q,H)=\infty$, the above arguments with an arbitrary large number in place of $(\beta^*(q,H)-\epsilon)$ would give $t_R^*(q,H)=\infty$.

Remark 3.4 Theorem 3.3 generalizes the Theorem 3.3 of [11].

Theorem 3.5 *Let $H\in H_R$, $0<R<\infty$, have q-order $\rho_R(q,H)$ and q-proximate order $\rho_R(q,H,r)$. If $\varphi(n)=\big[\frac{E_n(H,R_0)(R/R_0)^{2n}}{E_{n+1}(H,R_0)(R/R_0)^{2(n+1)}}\big]$ forms a non-decreasing sequence of n for $n>n_0$, then*

$$B_R(q,H)t_R^*(q,H)\leq\liminf_{n\to\infty}\big\{\phi(\log^{[q-2]}n)\big(\frac{2\log^+E_n(H,R_0)(R/R_0)^{2n}}{n}\big)\big\}^{\rho_R(q,H)+A(q,H)}.$$

Proof The proof follows by using Lemma 2.2 and applying [15, Thm. 2.3] to the function $h(u)$ defined by (2.4). In view of Theorems 3.3 and 3.5, we obtain the following result on generalized lower q-type for a subclass of GBASP $H\in H_R$ in terms of approximation error defined by (1.4).

Theorem 3.6 *Let $H\in H_R$, $0<R<\infty$, have q-order $\rho_R(q,H)$ and generalized lower q-type $t_R^*(q,H)$. Let $\varphi(k)=\big[E_{n_k}(H,R_0)/E_{n_{k+1}}(H,R_0)(R/R_0)^2\big]$ form a non-decreasing function of k for $k>k_0$ and $\log^{[q-2]}n_k\sim\log^{[q-2]}n_{k+1}$ as $k\to\infty$. Then*

$$B_R(q,H)t_R^*(q,H)$$
$$=\liminf_{k\to\infty}\big\{\phi(\log^{[q-2]}n_{k-1})\big(\frac{\log^+E_{n_k}(H,R_0)(R/R_0)^{2n_k}}{n_k}\big)\big\}^{\rho_R(q,H)+A(q,H)}. \quad (3.12)$$

Remark 3.7 Using $\phi(x)=x^{\frac{1}{\rho_R(q,H)+A(q,H)}}$ in (3.12), we get Theorem 3.6 of [11].

4. Growth of GBASP $H\in H_R$ in terms of the ratio of approximation errors

In this section we shall study some results related to generalized type and generalized lower type of GBASP $H\in H_R$ in terms of the ratio of approximation errors defined by (1.4).

Theorem 4.1 *Let $H\in H_R,\ 0<R<\infty\ (R_0<R)$, have q-order $\rho_R(q,H)$ and a q-proximate order $\rho_R(q,H,r)$. If $\varphi(n)=\log^+\{(E_n(H,R_0)/E_{n+1}(H,R_0)(R_0/R)^2\}$ forms a non-decreasing function of n for all large n, then the generalized q-type $T_R^*(q,H)$ of H with respect to a q-proximate order $\rho_R(q,H,r)$ is given by*

$$G_R^*(q,H)=B_R(q,H)T_R^*(q,H),$$

where

$$G_R^*(q,H)=\limsup_{n\to\infty}\big[\phi(\log^{[q-2]}n)\big(\log^+\frac{E_n(H,R_0)(R/R_0)^{2n}}{E_{n-1}(H,R_0)}\big)\big]^{\rho_R(q,H)+A(q,H)}. \tag{4.1}$$

Proof Let the right hand side of (4.1) be denoted by S^*. Following the lines of Lemma 2.3, we obtain

$$\limsup_{k\to\infty}\big[\phi(\log^{[q-2]}k)\big(\frac{\log^+E_n(H,R_0)(R/R_0)^{2k}}{k}\big)\big]^{\rho_R(q,H)+A(q,H)}\le S^*. \tag{4.2}$$

Using Theorem 3.1, we get

$$T_R^*(q,H)B_R(q,H)\le S^*. \tag{4.3}$$

The inequality (4.3) obviously holds if $S^*=\infty$. To prove the reverse inequality, we use Lemma 2.2 and apply Lemma 2.3 to the function $h(u)$ given by (2.4). Hence the theorem follows.

Remark 4.2 Theorem 4.1 is the generalization of the Theorem 3.1 contained in [12].

Theorem 4.3 *Let $H\in H_R,\ 0<R<\infty\ (R_0<R)$, have q-order $\rho_R(q,H)$ and a q-proximate order $\rho_R(q,H,r)$ and generalized lower q-type $t_R^*(q,H)$. Let $\{n_k\}$ be the increasing sequence of natural numbers. Then*

$$B_R(q,H)t_R^*(q,H)\ge\liminf_{k\to\infty}\big[\phi(\log^{[q-2]}n_{k-1})\big(\frac{\log^+\frac{E_n(H,R_0)(R/R_0)^{2n_k}}{E_{n_{k-1}}(H,R_0)^{2n_{k-1}}}}{(n_k-n_{k-1})}\big)^{\rho'_R(q,H)}\big], \tag{4.4}$$

where $\rho'_R(q,H)=\rho_R(q,H)+A(q,H)$.

Proof Let

$$\liminf_{k\to\infty}\big[\phi(\log^{[q-2]}n_{k-1})\big(\frac{\log^+\big(E_{n_k}(H,R_0)/E_{n_{k-1}}(H,R_0)\big)(R/R_0)^{2(n_k-n_{k-1})}}{n_k-n_{k-1}}\big)\big]^{\rho'_R(q,H)}=C.$$

The inequality in (4.4) obviously holds if $C=0$. Hence we assume that $0<C<\infty$. Then for $\epsilon>0$ and for all sufficiently large values of k, we have

$$\log^+\big\{\big(E_{n_k}(H,R_0)/E_{n_{k-1}}(H,R_0)\big)(R/R_0)^{2(n_k-n_{k-1})}\big\}$$

$$> (n_k - n_{k-1})\big[(C-\epsilon)^{1/\rho'_R(q,H)}/\phi(\log^{[q-2]} n_{k-1})\big]. \tag{4.5}$$

Substituting $k = N+1, N+2, \dots,\ j$ in above inequality and adding them, we get

$$\log^+ \big\{\big(E_{n_j}(H,R_0)/E_{n_N}(H,R_0)\big)(R/R_0)^{2(n_j-n_N)}\big\}$$
$$> \sum_{k=N+1}^{j} \big[(C-\epsilon)^{1/\rho'_R(q,H)}/\phi(\log^{[q-2]} n_{k-1})\big]. \tag{4.6}$$

To find the maximum value of right hand side, we put $n(t) = n_j$ for $n_{j-1} < t \le n_j$ and

$$F(t) = \big[(C-\epsilon)^{1/\rho'_R(q,H)}/\phi(\log^{[q-2]} t)\big].$$

Hence right hand side of (4.6) can be written as

$$\begin{aligned}
&\sum_{k=N+1}^{j} F(n_{k-1})(n_k - n_{k-1})\\
&= (n_{N+1}-n_N)F(n_N) + (n_N - n_{N-1})F(n_{N-1}) + \cdots + (n_j - n_{j-1})F(n_{j-1})\\
&= n_jF(n_{j-1}) - n_{j-1}\{F(n_{j-1}) - F(n_{j-2})\} -\\
&\quad n_{N+1}\{F(n_{N+1}) - F(n_N)\} - n_NF(n_N)\\
&= n_jF(n_{j-1}) - \sum_{k=N+1}^{j} n_{k-1}\{F(n_{k-1}) - F(n_{k-2})\} - n_NF(n_N)\\
&= n_jF(n_{j-1}) - \int_{n_N}^{n_{j-1}} n(t)dF(t) - n_NF(n_N)\\
&= n_jF(n_{j-1}) + \frac{1}{\rho'_R(q,H)} \int_{n_N}^{n_{j-1}} n(t)\frac{dF(t)}{t\prod_{m=1}^{q-2}\log^{[m]} t} + O(1).
\end{aligned}$$

Since $\frac{n(t)}{t} \ge 1$, we have by substituting the above expression in (4.6),

$$\begin{aligned}
&\log^+ E_{n_j}(H,R_0)(R/R_0)^{2(n_j-n_N)}\\
&> n_jF(n_{j-1}) + \frac{(n_{j-1}-n_N)F(n_{j-1})}{\rho'_R(q,H)\prod_{m=1}^{q-2}\log^{[m]} n_{j-1}} - O(1)\\
&= \frac{(C-\epsilon)^{1/\rho'_R(q,H)}}{\phi(\log^{[q-2]} n_{j-1})} n_j +\\
&\quad n_j\frac{(C-\epsilon)^{1/\rho'_R(q,H)}(n_{j-1}-nN)}{\rho'_R(q,H)\phi(\log^{[q-2]} n_{j-1})\prod_{m=1}^{q-2}\log^{[m]} n_{j-1} - O(1)}\\
&= \frac{(C-\epsilon)^{1/\rho'_R(q,H)}}{\phi(\log^{[q-2]} n_{j-1}} n_j)\Big[1 + \frac{n_{j-1}(1-O(1))}{\rho'_R(q,H)n_j\prod_{m=1}^{q-2}\log^{[m]} n_{j-1}}\Big] - O(1)\\
&= \frac{(C-\epsilon)^{1/\rho'_R(q,H)}}{\phi(\log^{[q-2]} n_{j-1}} n_j(1+o(1)-o(1)), \quad \text{for large } j.
\end{aligned}$$

Since $n_{j-1}/n_j < 1$, it gives

$$\frac{\log^+ E_{n_j}(H,R_0)(R/R_0)^{2n_j}}{n_j} > (C-\epsilon)^{1/\rho'_R(q,H)}/\phi(\log^{[q-2]} n_{j-1}) + o(1).$$

Proceeding to limits, we get

$$\liminf_{j\to\infty}\big[\phi(\log^{[q-2]} n_{j-1})\big(\frac{\log^+ E_{n_j}(H,R_0)(R/R_0)^{2n_j}}{n_j}\big)\big]^{\rho'_R(q,H)} \geq C.$$

Now using Theorem 3.3 in above inequality, the required result follows.

Remrak 4.4 The above theorem generalizes the Theorem 3.3 of [12].

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