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Nonlinear Maps Satisfying Derivability of a Class of Matrix Ring over Commutative Rings

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Abstract Let R be an arbitrary commutative ring with identity, and let $N_n(R)$ be the set consisting of all $n\times n$ strictly upper triangular matrices over R. In this paper, we give an explicit description of the maps (without linearity or additivity assumption) $\phi: N_n(R)\to N_n(R)$ satisfying $\phi(xy)=\phi(x)y+x\phi(y)$. As a consequence, additive derivations and derivations of $N_n(R)$ are also described.

Keywords maps satisfying derivability; derivations; strictly upper triangular matrices; commutative rings

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1. Introduction

Let R be a commutative ring with identity, and denote by $M_n(R)$ (resp., $T_n(R)$, $N_n(R)$ and $D_n(R)$) the set of all $n\times n$ matrices (resp., all $n\times n$ upper triangular matrices, all $n\times n$ strictly upper triangular matrices and all $n\times n$ diagonal matrices) over R.

Let $\mathscr{A}$ be an R-algebra. A map ϕ from $\mathscr{A}$ to itself is called an SD-map (means satisfying derivability) if

$$\phi(ab)=a\phi(b)+\phi(a)b,\quad \forall a,b\in\mathscr{A}.$$

It is well known that an SD-map ϕ is called an additive derivation (resp., a derivation) if it is additive (resp., R-linear).

In 1968, Johnson and Sinclair [1] initiated the study of additive derivations, which attracted series of authors to determine additive derivations on certain algebras. For instance, Coelho and Milies [2] characterized the additive derivations of $T_n(R)$ for R, an arbitrary ring with identity. Jøndrup [3] described the additive derivations of $T_n(\mathscr{A})$ and $M_n(\mathscr{A})$. See [4–7] for others. Some other authors [8–16] are interested in Lie derivations and Lie triple derivations. For example, Ou et al. [10] considered the Lie derivations on $N_n(R)$. Wang and Li [15] determined the Lie triple derivations of $N_n(R)$. Recently, Chen and Zhang [17] introduced nonlinear Lie derivations which may not satisfy linear conditions, and studied the nonlinear Lie derivation from $T_n(R)$ into $M_n(R)$ when R is a commutative unital algebra. Chen and Xiao [18] introduced the nonlinear Lie

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triple derivations on parabolic subalgebras of finite-dimensional simple Lie algebras. Motivated by the above works, we intend to investigate the SD-maps of $N_n(R)$.

Note that an SD-map ϕ of $\mathscr{A}$ is an additive derivation iff ϕ is additive, so the notion SD-map is a natural generalization of the notion additive derivation. But sometimes an SD-map of $\mathscr{A}$ may fail to be an additive derivation. The following is a counterexample.

Example 1.1 Let $\phi : N_3(R) \to N_3(R)$, defined by

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & a_{12}a_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is verified that ϕ is an SD-map of $N_3(R)$, but not an additive map.

Above example shows that it is interesting to characterize all SD-maps of $N_n(R)$. Before giving the main result of this paper, we introduce some preliminary notations.

For $1 \le i, j \le n$, we denote by rE_{ij} the $n \times n$ matrix, whose sole nonzero entry r is in the (i, j) position ($1E_{ij}$ is abbreviated to E_{ij}). Then, for any $X \in N_n(R)$, we may write $X = \sum_{1\le i<j\le n} x_{ij}E_{ij}$ with $x_{ij} \in R$. Set

$$\mathbb{Q}_{k+1} = \Big\{ \sum_{j-i\ge k} a_{ij}E_{ij} \in N_n(R) | a_{ij} \in R \Big\}, \quad 1 \le k \le n-1.$$

It is easy to see that each $\mathbb{Q}_k$ remains stable under any SD-maps of $N_n(R)$ and $\phi(O) = O$, where O is the $n \times n$ zero matrix. Denote

$$\mathbb{M}_k = \sum_{k+1\le j\le n} RE_{kj}, \quad 1 \le k \le n-1,$$

$$\mathbb{N}_k = \sum_{1\le i\le k-1} RE_{ik}, \quad 2 \le k \le n.$$

2. Standard SD-maps of $N_n(R)$

In this section, several standard SD-maps of $N_n(R)$ are given. They will be used to describe arbitrary SD-maps of $N_n(R)$ in the next section.

(1) Inner derivations

For $X \in N_n(R)$, the map $\operatorname{ad} X : N_n(R) \to N_n(R)$, $Y \mapsto XY - YX$ is a derivation of $N_n(R)$, called the inner derivation of $N_n(R)$ induced by X.

(2) Diagonal derivations

Let $D \in D_n(R)$. Then the map $D^\# : N_n(R) \to N_n(R)$, $Y \mapsto DY - YD$ is a derivation of $N_n(R)$, called the diagonal derivation of $N_n(R)$ induced by $D \in D_n(R)$.

(3) Ring derivations

Let σ be an additive derivation of R. Then the map

$$\sigma^\# : N_n(R) \to N_n(R), \quad \sum_{1\le i<j\le n} a_{ij}E_{ij} \mapsto \sum_{1\le i<j\le n} \sigma(a_{ij})E_{ij},$$

is an additive derivation of $N_n(R)$, which is called the ring derivation of $N_n(R)$ induced by σ.

(4) Induced SD-maps ($n = 3$)

Let $\theta : R \to R$ be an SD-map of R. We define the map

$$\theta^{\#} : N_3(R) \to N_3(R), \quad \sum_{1\le i<j\le 3} a_{ij}E_{ij} \mapsto \sum_{1\le i<j\le 3} \theta(a_{ij})E_{ij}.$$

It is easy to verify that $\theta^{\#}$ is an SD-map, which is called an induced SD-map of $N_3(R)$.

Remark 2.1 Let $\theta^{\#}$ be defined as above with θ an SD-map of R. Then $\theta^{\#}$ is an additive derivation of $N_3(R)$ iff θ is an additive map. Since, for any $a, b \in R$, $\theta^{\#}$ is an additive derivation of $N_3(R) \Leftrightarrow \theta(a+b)E_{13} = \theta^{\#}((a+b)E_{13}) = \theta^{\#}((aE_{12})E_{23} + E_{12}(bE_{23})) = (\theta(a) + \theta(b))E_{13} \Leftrightarrow \theta(a+b) = \theta(a) + \theta(b)$. Moreover, $\theta^{\#}$ is a ring derivation when θ is additive.

(5) Central derivations ($n \ge 4$)

Let $\alpha = (r_1, r_2, \dots, r_{n-3}) \in R^{n-3}$. Then the map $\alpha^{\#} : N_n(R) \to N_n(R)$, defined by

$$\alpha^{\#}(\sum_{1\le i<j\le n} a_{ij}E_{ij}) = (r_1a_{23} + r_2a_{34} + \cdots + r_{n-3}a_{n-2,n-1})E_{1n},$$

is a derivation of $N_n(R)$, which is called a central derivation of $N_n(R)$ induced by $\alpha \in R^{n-3}$.

(6) Central SD-maps ($n \ge 3$)

Let $f(x_1, x_2, \dots, x_{n-1})$ be an R-value function on variables $x_1, x_2, \dots, x_{n-1}$ satisfying $f(0, 0, \dots, 0) = f(1, 0, \dots, 0) = f(0, 1, \dots, 0) = \cdots = f(0, 0, \dots, 1) = 0$. We define $f^{\#} : N_n(R) \to N_n(R)$ by

$$f^{\#}(\sum_{1\le i<j\le n} a_{ij}E_{ij}) = f(a_{12}, a_{23}, \dots, a_{n-1,n})E_{1n}.$$

It is checked that $f^{\#}$ is an SD-map of $N_n(R)$, which is called a central SD-map of $N_n(R)$ induced by f.

Remark 2.2 Let $f^{\#}$ be a central SD-map defined as above. Then $f^{\#}$ is an additive derivation of $N_n(R)$ iff f is an additive function:

$$f(x_1 + y_1, x_2 + y_2, \dots, x_{n-1} + y_{n-1}) = f(x_1, x_2, \dots, x_{n-1}) + f(y_1, y_2, \dots, y_{n-1}).$$

(7) Almost zero SD-maps

Let $\xi : N_n(R) \to N_n(R)$ be an SD-map. We call ξ an almost zero SD-map of $N_n(R)$ if ξ sends any elements of the set $\{rE_{ij} | r \in R, 1 \le i < j \le n\}$ to O, i.e.,

$$\xi(rE_{ij}) = O \quad \text{for any} \quad r \in R,\ 1 \le i < j \le n.$$

Lemma 2.3 *Let $\xi : N_n(R) \to N_n(R)$ be an almost zero SD-map. For any $X \in N_n(R)$, assume that $\xi(X) = \sum_{1\le i<j\le n} a_{ij}E_{ij}$. Then $a_{12} = 0$, $a_{n-1,n} = 0$ and $a_{ij} = 0$ for $2 \le i < j \le n-1$.*

Proof Let $X = \sum_{1\le i<j\le n} x_{ij}E_{ij} \in N_n(R)$. Since

$$\begin{cases} XE_{2n} = x_{12}E_{1n}, \\ E_{1,n-1}X = x_{n-1,n}E_{1n}, \\ E_{i-1,i}XE_{j,j+1} = x_{ij}E_{i-1,j+1}, \quad 2 \le i < j \le n-1, \end{cases}$$

it follows that

$$\begin{cases} \xi(X)E_{2n} = O, \\ E_{1,n-1}\xi(X) = O, \\ E_{i-1,i}\xi(X)E_{j,j+1} = O, \quad 2 \le i < j \le n-1. \end{cases}$$

By a direct computation, we get

$$\begin{cases} a_{12} = 0, \\ a_{n-1,n} = 0, \\ a_{ij} = 0, \quad 2 \le i < j \le n-1. \quad \square \end{cases}$$

Remark 2.4 An almost zero SD-map ξ is just the zero mapping of $N_n(R)$ if it is an additive map (since $\{rE_{i,i+1} \mid r \in R,\ 1 \le i \le n-1\}$ generates the ring $N_n(R)$ and all such $rE_{i,i+1}$ are sent to O by ξ). Sometimes an almost zero SD-map is not the zero mapping (see Example 1.1 in Section 1).

3. SD-maps of $N_n(R)$

We prove, in this section, the main result of this paper. If $n = 1$ or $n = 2$, there is nothing to do on the SD-maps of $N_n(R)$, so we only consider the case when $n \ge 3$. As a beginning, we give a lemma.

Lemma 3.1 *Let ϕ be an SD-map of $N_n(R)$. If $\phi(E_{1t}) = O$ for any $2 \le t \le n$, then $\phi(E_{ij}) = \mathbb{M}_1$, $1 \le i < j \le n$.*

Proof Suppose that

$$\phi(E_{ij}) = \sum_{1 \le k < l \le n} a_{kl}^{(ij)} E_{kl} \in N_n(R), \quad 2 \le i < j \le n. \tag{3.1}$$

Since $E_{1t}E_{ij} = \delta_{ti}E_{1j}$, where δ is the Kronecker delta symbol, it follows that $E_{1t}\phi(E_{ij}) = O$. This forces that in (3.1)

$$a_{t,t+1}^{(ij)} = a_{t,t+2}^{(ij)} = \cdots = a_{tn}^{(ij)} = 0, \quad 2 \le t \le n-1,$$

leading to $\phi(E_{ij}) = \mathbb{M}_1$, $2 \le i < j \le n$. Thus $\phi(E_{ij}) = \mathbb{M}_1$ for all $1 \le i < j \le n$. $\square$

The following theorem is the main result of this paper.

Theorem 3.2 *Let R be an arbitrary commutative ring with identity, ϕ an SD-map of the ring $N_n(R)$. Then ϕ may be uniquely written as*

(1) $\phi = \mathrm{ad}\,X + D^{\#} + \alpha^{\#} + \sigma^{\#} + f^{\#} + \xi$ when $n \ge 4$,

(2) $\phi = \mathrm{ad}\,X + D^{\#} + \theta^{\#} + f^{\#} + \xi$ when $n = 3$,

where $\mathrm{ad}\,X$, $D^{\#}$, $\alpha^{\#}$, $\sigma^{\#}$, $\theta^{\#}$, $f^{\#}$ and ξ are the inner derivation, diagonal derivation, central derivation, ring derivation, induced SD-map, central SD-map and almost zero SD-map, respectively.

Proof Let ϕ be an SD-map of $N_n(R)$.

(1) If $n \ge 4$, the proof will be given by steps.

Step 1. There exist $X_1 \in N_n(R)$ and $D \in D_n(R)$ such that $(\phi - \mathrm{ad}\, X_1 - D^{\#})(E_{1j}) = O$, $2 \le j \le n$.

Suppose that

$$\phi(E_{12}) = \sum_{1 \le k < l \le n} a_{kl}^{(2)} E_{kl} \in N_n(R). \tag{3.2}$$

For $2 \le k \le n-1$, by applying ϕ on $E_{1k}E_{12} = O$, we get $E_{1k}\phi(E_{12}) = O$, following that in (3.2) $a_{kl}^{(2)} = 0$, $k+1 \le l \le n$. Set $X_{11} = -\sum_{3 \le t \le n} a_{1t}^{(2)} E_{2t}$ and $D_1 = -a_{12}^{(2)} E_{22}$. Then $(\phi - \mathrm{ad}\, X_{11} - D_1^{\#})(E_{12}) = O$. Denote $\phi - \mathrm{ad}\, X_{11} - D_1^{\#}$ by ϕ_1.

Now we consider the action of ϕ_1 on E_{1j}, $3 \le j \le n$. Operating ϕ_1 to $E_{1j} = E_{12}E_{2j}$, we get that $\phi_1(E_{1j}) = E_{12}\phi_1(E_{2j}) \in \mathbb{M}_1$. On the other hand, by $E_{1j} \in \mathbb{Q}_j$ we have $\phi_1(E_{1j}) \in \mathbb{Q}_j$, $3 \le j \le n$. Thus, we may assume that

$$\phi_1(E_{1j}) = \sum_{j \le l \le n} a_{1l}^{(j)} E_{1l} \in \mathbb{M}_1 \cap \mathbb{Q}_j, \quad 3 \le j \le n. \tag{3.3}$$

Set $X_{22} = -\sum_{3 \le k \le n-1} \sum_{k+1 \le t \le n} a_{1t}^{(k)} E_{kt}$ and $D_2 = -\mathrm{diag}(0, 0, a_{13}^{(3)}, a_{14}^{(4)}, \ldots, a_{1,n-1}^{(n-1)}, a_{1n}^{(n)})$. Then by (3.3) we see that $(\phi_1 - \mathrm{ad}\, X_{22} - D_2^{\#})(E_{1j}) = O$, $3 \le j \le n$. In the following, we denote $\phi_1 - \mathrm{ad}\, X_{22} - D_2^{\#}$ by ϕ_2.

Step 2. There exist $X_2 \in \mathbb{M}_1$ and $\alpha \in R^{n-3}$ such that $(\phi_2 - \mathrm{ad}\, X_2 - \alpha^{\#})(E_{i,i+1}) = O$, $2 \le i \le n-1$.

By Step1 and Lemma 3.1, we may assume that

$$\phi_2(E_{i,i+1}) = \sum_{2 \le l \le n} a_{1l}^{(i)} E_{1l} \in \mathbb{M}_1, \quad 2 \le i \le n-1. \tag{3.4}$$

For $2 \le t \le n-1$ and $t \ne i+1$, by applying ϕ_2 on $E_{i,i+1}E_{tn} = O$, we get that

$$\phi_2(E_{i,i+1})E_{tn} + E_{i,i+1}\phi_2(E_{tn}) = O.$$

Since $E_{i,i+1}\phi_2(E_{tn}) = O$ (by Lemma 3.1), $\phi_2(E_{i,i+1})E_{tn} = O$. This implies that $a_{1t}^{(i)} = 0$ for $2 \le t \le n-1$ and $t \ne i+1$. Thus (3.4) may be rewritten as

$$\begin{aligned} &\phi_2(E_{i,i+1}) = a_{1,i+1}^{(i)} E_{1,i+1} + a_{1n}^{(i)} E_{1n}, \quad 2 \le i \le n-2, \\ &\phi_2(E_{n-1,n}) = a_{1n}^{(n-1)} E_{1n}. \end{aligned} \tag{3.5}$$

Choose $X_2 = \sum_{2 \le t \le n-1} a_{1,t+1}^{(t)} E_{1t} \in \mathbb{M}_1$ and $\alpha = (a_{1n}^{(2)}, a_{1n}^{(3)}, \ldots, a_{1n}^{(n-2)}) \in R^{n-3}$, then by (3.5) we obtain that $(\phi_2 - \mathrm{ad}\, X_2 - \alpha^{\#})(E_{i,i+1}) = O$, $2 \le i \le n-1$. Now we denote $\phi_3 = \phi_2 - \mathrm{ad}\, X_2 - \alpha^{\#}$.

Step 3. $\phi_3(RE_{i,i+1}) \subseteq RE_{i,i+1} + RE_{1n}$, $1 \le i \le n-1$.

Given $r \in R$, assume that

$$\phi_3(rE_{i,i+1}) = \sum_{1 \le k < l \le n} a_{kl}^{(i)} E_{kl} \in N_n(R), \quad 1 \le i \le n-1.$$

We first consider the action of ϕ_3 on $rE_{i,i+1}$, $1 \le i \le n-2$. For $2 \le s \le n-1$ and $s \ne i$, by applying ϕ_3 on $E_{s-1,s}(rE_{i,i+1}) = O$, we have $E_{s-1,s}\phi_3(rE_{i,i+1}) = O$, which leads to $a_{sl}^{(i)} = 0$, $s+1 \le l \le n$. For $2 \le t \le n-1$ and $t \ne i+1$, by applying ϕ_3 on $(rE_{i,i+1})E_{t,t+1} = O$, we get $\phi_3(rE_{i,i+1})E_{t,t+1} = O$, which shows that $a_{1t}^{(i)} = a_{it}^{(i)} = 0$. Thus

$$\phi_3(rE_{12}) = a_{12}^{(1)} E_{12} + a_{1n}^{(1)} E_{1n},$$

$$\phi_3(rE_{i,i+1}) = a^{(i)}_{1,i+1}E_{1,i+1} + a^{(i)}_{1n}E_{1n} + a^{(i)}_{i,i+1}E_{i,i+1} + a^{(i)}_{in}E_{in}, \quad 2 \le i \le n-2. \tag{3.6}$$

Operating ϕ_3 to $(rE_{i,i+1})E_{i+1,n} = E_{i,i+1}(rE_{i+1,n})$, we get

$$\phi_3(rE_{i,i+1})E_{i+1,n} = E_{i,i+1}\phi_3(rE_{i+1,n}) \in \mathbb{M}_i.$$

This implies that $a^{(i)}_{1,i+1} = 0$, $2 \le i \le n-2$. Operating ϕ_3 to $E_{1i}(rE_{i,i+1}) = (rE_{12})E_{2,i+1}$, $2 \le i \le n-2$, we have $E_{1i}\phi_3(rE_{i,i+1}) = \phi_3(rE_{12})E_{2,i+1} \in RE_{1,i+1}$, which shows that $a^{(i)}_{in} = 0$, $2 \le i \le n-2$. Thus (3.6) may be rewritten as

$$\phi_3(rE_{i,i+1}) = a^{(i)}_{i,i+1}E_{i,i+1} + a^{(i)}_{1n}E_{1n}, \quad 1 \le i \le n-2.$$

We next consider the action of ϕ_3 on $rE_{n-1,n}$. For $2 \le s \le n-2$, by applying ϕ_3 to $E_{s-1,s}(rE_{n-1,n}) = O$, we obtain that $a^{(n-1)}_{sl} = 0$, $s+1 \le l \le n$. For $2 \le t \le n-1$, by operating ϕ_3 to $(rE_{n-1,n})E_{t,t+1} = O$, we obtain that $a^{(n-1)}_{1t} = 0$. So $\phi_3(rE_{n-1,n}) = a^{(n-1)}_{n-1,n}E_{n-1,n} + a^{(n-1)}_{1n}E_{1n}$.

Now, we define $\sigma_i : R \to R$, $f_i : R \to R$, $1 \le i \le n-1$ such that

$$\phi_3(rE_{i,i+1}) = \sigma_i(r)E_{i,i+1} + f_i(r)E_{1n}, \quad r \in R, \ 1 \le i \le n-1. \tag{3.7}$$

Obviously, $f_i(0) = f_i(1) = 0$, $1 \le i \le n-1$. Let $f(x_1, x_2, \ldots, x_{n-1})$ be an R-value function satisfying $f(x_1, x_2, \ldots, x_{n-1}) = f_1(x_1) + f_2(x_2) + \cdots + f_{n-1}(x_{n-1})$. Then

$$f(0,0,\ldots,0) = f(1,0,\ldots,0) = f(0,1,\ldots,0) = \cdots = f(0,0,\ldots,1) = 0.$$

Denote $\phi_3 - f^{\#}$ by ϕ_4, where $f^{\#}$ is the central SD-map induced by f. By (3.7) we see that

$$\phi_4(rE_{i,i+1}) = \sigma_i(r)E_{i,i+1}, \quad 1 \le i \le n-1. \tag{3.8}$$

Step 4. There exists an SD-map σ of R such that

$$\phi_4(rE_{kl}) = \sigma(r)E_{kl} \ \text{ for all } \ r \in R, \quad 1 \le k < l \le n. \tag{3.9}$$

We first assert that the σ_i's in (3.8) may be chosen to be identical. For any $r \in R$, by applying ϕ_4 on $(rE_{12})E_{2n} = \cdots = E_{1i}(rE_{i,i+1})E_{i+1,n} = \cdots = E_{1,n-1}(rE_{n-1,n})$, we get

$$\phi_4(rE_{12})E_{2n} = \cdots = E_{1i}\phi_4(rE_{i,i+1})E_{i+1,n} = \cdots = E_{1,n-1}\phi_4(rE_{n-1,n}),$$

which follows that $\sigma_1(r) = \cdots = \sigma_i(r) = \cdots = \sigma_{n-1}(r)$. By the arbitrariness of r, we get $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1}$, as required. Denote σ_i $(1 \le i \le n-1)$ by σ, then (3.8) may be rewritten as $\phi_4(rE_{i,i+1}) = \sigma(r)E_{i,i+1}$, $1 \le i \le n-1$. For $i+2 \le j$, by applying ϕ_4 on $rE_{ij} = (rE_{i,i+1})E_{i+1,j}$, we obtain that $\phi_4(rE_{ij}) = \sigma(r)E_{ij}$. Thus $\phi_4(rE_{ij}) = \sigma(r)E_{ij}$ for all $1 \le i < j \le n$.

We next prove that σ is an SD-map of R. For any $a, b \in R$, by operating ϕ_4 to $(ab)E_{1n} = (aE_{12})(bE_{2n})$, we have $\sigma(ab)E_{1n} = \phi_4(aE_{12})(bE_{2n}) + (aE_{12})\phi_4(bE_{2n})$. Comparing the $(1,n)$-entry of the two sides, we see that $\sigma(ab) = \sigma(a)b + a\sigma(b)$.

Step 5. σ is an additive derivation of R.

It suffices to prove that $\sigma(a+b) = \sigma(a) + \sigma(b)$ for any $a, b \in R$. Denote $A = E_{12} + aE_{13}$, $B = bE_{24} + E_{34}$, and suppose that

$$\phi_4(A) = a_{12}E_{12} + a_{13}E_{13} + a_{23}E_{23} + A_1 \in N_n(R), \tag{3.10}$$

$$\phi_4(B) = \sum_{3\leq j\leq n} b_{2j}E_{2j} + \sum_{4\leq j\leq n} b_{3j}E_{3j} + B_1 \in N_n(R), \tag{3.11}$$

where $A_1 \in \bigcup_{s=4}^{n} \mathbb{N}_s$, $B_1 \in \mathbb{M}_1 \bigcup(\bigcup_{t=4}^{n-1} \mathbb{M}_t)$. Applying ϕ_4 on $AE_{23} = E_{13}$, $E_{12}A = O$ and $AE_{34} = aE_{14}$, respectively, we get

$$\phi_4(A)E_{23} = O, \quad E_{12}\phi_4(A) = O \text{ and } \phi_4(A)E_{34} = \sigma(a)E_{14},$$

which shows that $a_{12} = 0$, $a_{23} = 0$ and $a_{13} = \sigma(a)$. Thus, (3.10) may be rewritten as

$$\phi_4(A) = \sigma(a)E_{13} + A_1. \tag{3.12}$$

Similarly, by operating ϕ_4 to $E_{12}B = bE_{14}$ and $E_{13}B = E_{14}$, respectively, we have

$$E_{12}\phi_4(B) = \sigma(b)E_{14} \quad \text{and} \quad E_{13}\phi_4(B) = O.$$

This implies that $b_{24} = \sigma(b)$, $b_{2s} = 0$ for $s \neq 4$, $b_{3t} = 0$ for $4 \leq t \leq n$. Then (3.11) may be rewritten as

$$\phi_4(B) = \sigma(b)E_{24} + B_1. \tag{3.13}$$

By (3.12) and (3.13), we obtain that $\phi_4((a+b)E_{14}) = \phi_4(AB) = \phi_4(A)B + A\phi_4(B) = (\sigma(a)E_{13} + A_1)B + A(\sigma(b)E_{24} + B_1) = (\sigma(a) + \sigma(b))E_{14}$. On the other hand, by (3.9) we know that $\phi_4((a + b)E_{14}) = \sigma(a + b)E_{14}$. Thus $\sigma(a + b) = \sigma(a) + \sigma(b)$, as desired.

Since σ is an additive derivation of R, we may construct the ring derivation $\sigma^{\#}$ of $N_n(R)$. Then by (3.9) we get $(\phi_4 - \sigma^{\#})(rE_{kl}) = O$ for all $r \in R$ and all $1 \leq k < l \leq n$. This shows that $\phi_4 - \sigma^{\#}$ is an almost zero SD-map of $N_n(R)$, which is denoted by ξ. Above discussion shows that $\phi = \operatorname{ad} X + D^{\#} + \alpha^{\#} + \sigma^{\#} + f^{\#} + \xi$, where $X = X_1 + X_2$.

Step 6. The uniqueness of the decomposition.

It suffices to prove that if $adX + D^{\#} + \alpha^{\#} + \sigma^{\#} + f^{\#} + \xi = O$, then $\operatorname{ad} X = D^{\#} = \alpha^{\#} = \sigma^{\#} = f^{\#} = \xi = O$. Assume that $\phi = \operatorname{ad} X + D^{\#} + \alpha^{\#} + \sigma^{\#} + f^{\#} + \xi = O$. Using $\phi(E_{i,i+1}) = O$, $2 \leq i \leq n-2$, we see that $\alpha = (0, 0, \ldots, 0)$, which leads to $\alpha^{\#} = O$. Thus $\phi = \operatorname{ad} X + D^{\#} + \sigma^{\#} + f^{\#} + \xi = O$. By $\phi(E_{i,i+1}) = O$, $1 \leq i \leq n-1$, we get $X \in RE_{1n}$ and $D = aE$ for some $a \in R$, forcing $\operatorname{ad} X = D^{\#} = O$. So $\phi = \sigma^{\#} + f^{\#} + \xi = O$. Then by making use of $\phi(rE_{i,i+1}) = O$ for any $r \in R$ and any $1 \leq i \leq n-1$, we see that $\sigma(r) = 0$, which shows that $\sigma^{\#} = O$. Therefore, $\phi = f^{\#} + \xi = O$. For any $x_1, x_2, \ldots, x_{n-1} \in R$, by $\phi(\sum_{1\leq i\leq n-1} x_iE_{i,i+1}) = O$, we obtain that $f(x_1, x_2, \ldots, x_{n-1}) = 0$. Thus $f^{\#} = O$, and so $\xi = O$.

(2) If $n = 3$, we first consider the action of ϕ on E_{12} and E_{23}. Suppose that

$$\phi(E_{i,i+1}) = a_{12}^{(i)}E_{12} + a_{13}^{(i)}E_{13} + a_{23}^{(i)}E_{23} \in N_3(R), \quad i = 1, 2.$$

Applying ϕ to $E_{12}^2 = O$ and $E_{23}^2 = O$, we get $E_{12}\phi(E_{12}) = O$ and $\phi(E_{23})E_{23} = O$, respectively. This shows that $a_{23}^{(1)} = 0$ and $a_{12}^{(2)} = 0$. Choose $X = a_{13}^{(2)}E_{12} - a_{13}^{(1)}E_{23}$ and $D = (a_{12}^{(1)} + a_{23}^{(2)})E_{11} + a_{23}^{(2)}E_{22} \in D_3(R)$, then we have that $(\phi - \operatorname{ad} X - D^{\#})(E_{i,i+1}) = O$, $i = 1, 2$. Denote $\phi - \operatorname{ad} X - D^{\#}$ by ϕ_1.

Next, we consider the action of ϕ_1 on $rE_{i,i+1}$ for any $r \in R$ and $i = 1, 2$. Assume that

$$\phi_1(rE_{i,i+1}) = b_{12}^{(i)}E_{12} + b_{13}^{(i)}E_{13} + b_{23}^{(i)}E_{23} \in N_3(R), \quad i = 1, 2. \tag{3.14}$$

Operating ϕ_1 to $E_{12}(rE_{12})=O$ and $(rE_{23})E_{23}=O$, we get $b_{23}^{(1)}=0$ and $b_{12}^{(2)}=0$, respectively. Thus, (3.14) may be rewritten as

$$\phi_1(rE_{i,i+1})=\theta_i(r)E_{i,i+1}+f_i(r)E_{13},\quad i=1,2,\tag{3.15}$$

where $\theta_i:R\to R$, $f_i:R\to R$ satisfying $f_i(0)=f_i(1)=0$, $i=1,2$. Let $f(x_1,x_2)$ be an R-value function satisfying $f(x_1,x_2)=f_1(x_1)+f_2(x_2)$. Then $f(0,0)=f(1,0)=f(0,1)=0$. Denote $\phi_1-f^{\#}$ by ϕ_2, where $f^{\#}$ is the central SD-map induced by f, then by (3.15) we obtain that

$$\phi_2(rE_{i,i+1})=\theta_i(r)E_{i,i+1},\quad i=1,2.\tag{3.16}$$

Operating ϕ_2 to $(rE_{12})E_{23}=E_{12}(rE_{23})$, we get $\theta_1(r)=\theta_2(r)$ for any $r\in R$. This shows that $\theta_1=\theta_2$. Denote θ_i $(i=1,2)$ by θ.

For any $a,b\in R$, by applying ϕ_2 on $((ab)E_{12})E_{23}=(aE_{12})(bE_{23})$, we get $\theta(ab)=\theta(a)b+a\theta(b)$, which shows that θ is an SD-map of R. Then by (3.16) we have $(\phi_2-\theta^{\#})(rE_{i,i+1})=O$, $i=1,2$, where $\theta^{\#}$ is an induced SD-map of $N_3(R)$. It follows that $(\phi_2-\theta^{\#})(rE_{13})=(\phi_2-\theta^{\#})(rE_{12})E_{23}+(rE_{12})(\phi_2-\theta^{\#})(E_{23})=O$. Thus $\phi_2-\theta^{\#}$ is an almost zero SD-map of $N_3(R)$, which is denoted by ξ. So

$$\phi=\operatorname{ad}X+D^{\#}+\theta^{\#}+f^{\#}+\xi.$$

The proof of the uniqueness is similar to that when $n\geq 4$, thus, is omitted. The proof is completed. □

4. Applications

As an application of Theorem 3.2, we consider the additive derivations of $N_n(R)$. In [6], Driss et al. gave an decomposition of any additive derivations of $N_n(R)$. However, the decomposition in [6] is not unique. In the following, by using the result of Theorem 3.2, we give a unique decomposition of the additive derivations of $N_n(R)$.

Theorem 4.1 *Let R be an arbitrary commutative ring with identity, ϕ an additive derivation of $N_n(R)$. Then ϕ may be uniquely written as*

(1) $\phi=\operatorname{ad}X+D^{\#}+\sigma^{\#}+\alpha^{\#}+f^{\#}$ when $n\geq 4$,

(2) $\phi=\operatorname{ad}X+D^{\#}+\sigma^{\#}+f^{\#}$ when $n=3$,

where $\operatorname{ad}X$, $D^{\#}$, $\sigma^{\#}$, $\alpha^{\#}$ and $f^{\#}$ are additive derivations of $N_n(R)$, defined as in Section 2.

Proof Any additive derivation ϕ of $N_n(R)$ is also an SD-map. If $n\geq 4$, by Theorem 3.2, we have a unique decomposition: $\phi=\operatorname{ad}X+D^{\#}+\alpha^{\#}+\sigma^{\#}+f^{\#}+\xi$. Since adX, $D^{\#}$, $\sigma^{\#}$ and $\alpha^{\#}$ are additive derivations of $N_n(R)$, so does $f^{\#}+\xi$. For $1\leq i\leq n-1$ and $x_i,y_i\in R$, by applying $f^{\#}+\xi$ to $(x_i+y_i)E_{i,i+1}=x_iE_{i,i+1}+y_iE_{i,i+1}$, we get

$$f(0,\ldots,0,x_i+y_i,0,\ldots,0)=f(0,\ldots,0,x_i,0,\ldots,0)+f(0,\ldots,0,y_i,0,\ldots,0),$$

which shows that f is additive. Thus, $f^{\#}$ is an additive derivation and so $\xi=O$. In the same way, we can prove that the theorem is true for $n=3$. □

By Theorem 4.1, one can obtain the following result.

Corollary 4.2 ([6]) *Let R be an arbitrary commutative ring with identity, ϕ a derivation of $N_n(R)$. Then ϕ may be uniquely written as*

(1) $\phi = \operatorname{ad} X + D^{\#} + \alpha^{\#}$ *when* $n \geq 4$,

(2) $\phi = \operatorname{ad} X + D^{\#}$ *when* $n = 3$,

where $\operatorname{ad} X$, $D^{\#}$ *and* $\alpha^{\#}$ *are derivations of* $N_n(R)$, *defined as in Section 2.*

For a long time, linear preserving problem attracted a lot of attention. Recently, some authors are interested in non-linear preserving problem on matrix algebras or operator algebras. Sometimes, it seems much difficult to determine non-linear maps on the algebra in question. An effective method of simplifying the non-linear preserving problem is to turn to study its linear object, SD-map of the algebra. In view of this point, we think that the main result of this paper is a foundation for further works on non-linear preserving problem.

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