Journal of Mathematical Research with Applications
Nov., 2015, Vol. 35, No. 6, pp. 653–658
DOI:10.3770/j.issn:2095-2651.2015.06.007
Http://jmre.dlut.edu.cn

Robust Exponential Stability of a Class of Fractional Order Hopfield Neural Networks

Xiaolei LIU, Mingjiu GAI*, Shiwei CUI
Institute of System Science and Mathematics, Naval Aeronautical and Astronautical University, Shandong 264001*, P. R. China*

Abstract In this paper, we investigate the robust exponential stability of a class of fractional order Hopfield neural network with Caputo derivative, and we get some sufficient conditions to guarantee its robust exponential stability. Finally, we use one numerical simulation example to illustrate the correctness and effectiveness of our results.

Keywords fractional order neural networks; Gronwall inequality; robust exponential stability

MR(2014) Subject Classification 34A08; 92B20; 93D09

1. Introduction

The subject of fractional calculus was planted over 300 years ago. In recent years, fractional calculus has played a significant role in many areas of science and engineering [1–3]. The necessary and sufficient stability conditions for linear fractional differential equations and linear time-delayed fractional differential equations have already been obtained in [4–6]. The stability of nonlinear fractional order system for Caputo's derivative was studied in [7,8]. Especially, some excellent results about fractional-order neural networks have been investigated in [9–11].

In this paper, by making the systems translate into the nonlinear Volterra integral equation of the second kind, and making use of the existence and uniqueness Theorem of the fractional differential equations, and the Gronwall inequality, we will study the robust stability for the fractional-order Hopfield neural networks as follows:

$$\mathrm{D}^{\alpha}x_i(t)=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}g_j(x_j(t))+I_i,\quad i=1,2,\ldots,n, \tag{1}$$

where $0<\alpha<1$, n corresponds to the number of units in the neural networks; $x_i(t)$ corresponds to the state of the i-th neuron at time t; $g_j(x_j)$ denotes the activation function of the j-th neuron; a_{ij} denotes the constant connection weight of the j-th neuron on the i-th neuron; $c_i>0$ represents the rate with which the i-th neuron will reset its potential to the resting state when disconnected from the network and I_i denotes external inputs. If we consider the influence of

Received November 7, 2014; Accepted December 12, 2014
Supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2011FQ002).
* Corresponding author
E-mail address: gaimingjiu@sina.com (Mingjiu GAI)

the disturbing function $c_i(t)$ and $a_{ij}(t)$ in the system, then the system (1) is changed into the system as follows:

$$\mathrm{D}^{\alpha} x_i(t) = -\left(c_i + c_i(t)\right) x_i(t) + \sum_{j=1}^{n} (a_{ij} + a_{ij}(t)) g_j(x_j(t)) + I_i, \quad i = 1, 2, \ldots, n. \tag{2}$$

The remainder of this paper is organized as follows: In Section 2, some necessary definitions and lemmas are presented. We give some sufficient conditions to guarantee robust exponential stability for a class of fractional order Hopfield neural networks in Section 3. In Section 4, one example and corresponding numerical simulation are used to illustrate the validity and feasibility of the results obtained in Section 3.

2. Preliminaries

There are several definitions of a fractional derivative of order α, which is the extended concept of integer order derivative. The commonly used definitions are Grunwald-Letnikov, Riemann-Liouville, and Caputo definitions. Firstly, we will recall the definition of Caputo fractional derivative and the several important lemmas.

Definition 2.1 ([12]) *The Caputo fractional derivative of order $\alpha \in \mathbf{R}^+$ of a function $x(t)$ is defined as*

$${}_{t_0}\mathrm{D}_t^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} \frac{\mathrm{d}^m x(\tau)}{\mathrm{d}\tau^m} (t-\tau)^{m-\alpha-1} \mathrm{d}\tau,$$

where $m \in \mathbf{N},\ m-1 \le \alpha < m$, $\frac{\mathrm{d}^m x(\tau)}{\mathrm{d}\tau^m}$ is the m-th derivative of $x(t)$ in the usual sense, $\Gamma(\cdot)$ is the gamma function, i.e., $\Gamma(\beta) = \int_0^{\infty} s^{\beta-1} e^{-s} \mathrm{d}s$.

Consider the Cauchy problem of the following Caputo fractional differential equation

$$\begin{cases} \mathrm{D}^{\alpha} x(t) = f(t, x(t)), & t \in [0, +\infty), \\ x(0) = x_0, & x_0 \in \mathbf{R}^n, \end{cases} \tag{3}$$

where $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbf{R}^n$, $0 < \alpha < 1$, $f : [0, +\infty \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous in t.

Definition 2.2 ([13]) *The constant x^* is an equilibrium point of Eq. (3) if and only if $f(t, x^*) = 0$ for any $t \in [0, +\infty)$.*

Definition 2.3 *The zero solution of the system (1) is said to be robust exponential stable for the disturbing function ΔC and ΔA if the equilibrium point of the system (2) is exponential asymptotically stable.*

Lemma 2.4 ([12]) *Let $0 \le \alpha < 1$ and $f(t, x) : [0, +\infty] \times \mathbf{R}^n \to \mathbf{R}^n$ be a function such that, for all $t \in [0, +\infty]$ and all $x_1, x_2 \in G \subset \mathbf{R}^n$,*

$$\left| f(t, x_1) - f(t, x_2) \right| \le L \left| x_1 - x_2 \right|, \tag{4}$$

where $L > 0$ does not depend on $t \in [0, +\infty]$. Then there exists a unique solution $x(t)$ to the Cauchy problem (3) in the $C[0, +\infty]$.

Lemma 2.5 ([14]) *Consider the following equation*

$$\begin{cases} \mathrm{D}^\alpha x(t) = f(t, x(t)), & m-1 < \alpha < m, m \in \mathbf{N}, t \in [0, +\infty), \\ x^{(k)}(0) = x_0^{(k)}, & x_0^{(k)} \in \mathbf{R}^n,\ k = 0, 1, \ldots, m-1. \end{cases} \tag{5}$$

The homotopy perturbation technique yields that the initial value problem (5) is equivalent to the nonlinear Volterra integral equation of the second kind

$$x(t) = \sum_{k=0}^{m-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau. \tag{6}$$

In particular, if $0 < \alpha < 1$, then Eq. (6) can be written in the following form

$$x(t) = x_0^{(0)} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau, x(\tau))}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau. \tag{7}$$

Lemma 2.6 ([15]) *Let a variable $x(t)$ satisfy $x(t) \le b(t) + \int_0^t a(\tau)x(\tau)\mathrm{d}\tau$ with $a(t)$ and $b(t)$ being known real functions. Then*

$$x(t) \le \int_0^t a(\tau)b(\tau) \exp\Big\{ \int_\tau^t a(r)\mathrm{d}r \Big\} \mathrm{d}\tau + b(t). \tag{8}$$

If $b(t)$ is differentiable, then

$$x(t) \le b(0) \exp\Big\{ \int_0^t a(\tau)\mathrm{d}\tau \Big\} + \int_0^t \dot{b}(\tau) \exp\Big\{ \int_\tau^t a(r)\mathrm{d}r \Big\} \mathrm{d}\tau. \tag{9}$$

In particular, if $b(t)$ is a constant, we simply have

$$x(t) \le b(0) \exp\Big\{ \int_0^t a(\tau)\mathrm{d}\tau \Big\}. \tag{10}$$

3. Main results

In this section, we suppose the fractional order Hopfield neural networks (1) satisfies the following assumptions:

(A1) g_j $(j = 1, 2, \ldots, n)$ are Lipschitz-continuous on $(-\infty, +\infty)$ with Lipschitz constants $L_j > 0$, i.e., $|g_j(\xi) - g_j(\eta)| \le L_j |\xi - \eta|$, for all $\xi, \eta \in (-\infty, +\infty)$;

(A2) $|c_i(t)| \le M$, $|a_{ij}(t)| \le N$;

(A3) $\lambda_i = c_i - M - \sum\limits_{j=1}^{n} (|a_{ji}| + N)L_i > 0$.

Let $\lambda = \min\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then we have $\lambda > 0$.

Theorem 3.1 *Under the assumptions (A1), (A2) and (A3), the system (1) is robust exponential stable.*

Proof Assume that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is a solution of the system (1) different from the equilibrium point x^*. Denote $e_i(t) = y_i(t) - x_i^*$, $i = 1, 2, \ldots, n$, then $e_i(0) \ne 0$, and

$$\mathrm{D}^\alpha e_i(t) = -\left(c_i + c_i(t)\right) e_i(t) + \sum_{j=1}^{n} \left(a_{ij} + a_{ij}(t)\right) \left(g_j(e_j(t) + x_j^*) - g_j(x_j^*)\right). \tag{11}$$

Let

$$f_i(t,e(t)) = -\left(c_i + c_i(t)\right)e_i(t) + \sum_{j=1}^{n}\left(a_{ij} + a_{ij}(t)\right)\left(g_j(e_j(t)+x_j^*) - g_j(x_j^*)\right), \quad i = 1,2,\ldots,n,$$

and $f(t,e(t)) = (f_1(t,e(t)), f_2(t,e(t)),\ldots,f_n(t,e(t)))^{\mathrm{T}}$. It can be proved that the vector function f is Lipschitz continuous according to the assumptions (A1) and (A2). By Lemma 2.4, there exists a unique solution of the system (11) associating with one initial value. It is easy to see that $e(t)\equiv 0$ is an equilibrium point of the system (11), therefore we have $e_i(t)e_i(0) > 0$ for $t\in[0,+\infty]$, $i = 1,2,\ldots,n$. We divide our discussion into two cases.

Case 1 If $e_i(0) > 0$, then $e_i(t) > 0$ for $t \in [0,+\infty]$. From (7) in Lemma 2.5, we have

$$\begin{aligned}
&e_i(t)\\
&\quad= e_i(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{-\left(c_i + c_i(t)\right)e_i(\tau) + \sum_{j=1}^{n}\left(a_{ij}+a_{ij}(t)\right)\left(g_j(e_j(\tau)+x_j^*) - g_j(x_j^*)\right)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau\\
&\quad\le |e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{-\left(c_i - M\right)|e_i(\tau)| + \sum_{j=1}^{n}\left(|a_{ij}|+N\right)L_j\,|e_j(\tau)|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau. \qquad (12)
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n}|e_i(t)| &\le \sum_{i=1}^{n}|e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\sum_{i=1}^{n}\left(-(c_i-M) + \sum_{j=1}^{n}\left(|a_{ij}|+N\right)L_i\right)|e_i(\tau)|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau\\
&= \sum_{i=1}^{n}|e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\sum_{i=1}^{n}\left(-\lambda_i\right)|e_i(\tau)|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau\\
&\le \sum_{i=1}^{n}|e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\sum_{i=1}^{n}\left(-\lambda\right)|e_i(\tau)|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau. \qquad (13)
\end{aligned}$$

Case 2 If $e_i(0) < 0$, then $e_i(t) < 0$ for $t\in[0,+\infty]$. Similarly to Case 1, we have

$$\begin{aligned}
&-e_i(t)\\
&= -e_i(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{-\left(c_i + c_i(t)\right)\left(-e_i(\tau)\right) + \sum_{j=1}^{n}\left(-a_{ij}-a_{ij}(t)\right)\left(g_j(e_j(\tau)+x_j^*) - g_j(x_j^*)\right)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau\\
&\le |e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{-\left(c_i - M\right)|e_i(\tau)| + \sum_{j=1}^{n}\left(|a_{ij}|+N\right)L_j\,|e_j(\tau)|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau. \qquad (14)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=1}^{n}|e_i(t)| &= \sum_{i=1}^{n}\left(-e_i(t)\right)\\
&\le \sum_{i=1}^{n}|e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\sum_{i=1}^{n}\left(-(c_i-M)|e_i(\tau)| + \sum_{j=1}^{n}\left(|a_{ij}|+N\right)L_j\,|e_j(\tau)|\right)}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau\\
&\le \sum_{i=1}^{n}|e_i(0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\sum_{i=1}^{n}\left(-\lambda\right)|e_i(\tau)|}{(t-\tau)^{1-\alpha}}\mathrm{d}\tau. \qquad (15)
\end{aligned}$$

From Cases 1 and 2, we get the inequality

$$\|e(t)\| = \sum_{i=1}^{n} |e_i(t)| \leq \sum_{i=1}^{n} |e_i(0)| \cdot \exp\left\{ \int_0^t \frac{-\lambda}{\Gamma(\alpha)(t-\tau)^{1-\alpha}} \mathrm{d}\tau \right\} = \|e(0)\| \cdot \exp\left[\frac{-\lambda}{\Gamma(\alpha+1)} t^{\alpha} \right],$$

which shows that the system (1) is robust exponential stable.

4. Illustrative examples

In the system (1), let

$$\alpha = 0.8, C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5 \end{pmatrix}, A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 3 \\ -1 & 3 & -4 \end{pmatrix},$$

$$x(0) = (x_1(0), x_2(0), x_3(0))^{\mathrm{T}} = [1, 1, 1]^{\mathrm{T}},$$

$$G(x) = (g_1(x_1), g_2(x_2), g_3(x_3))^{\mathrm{T}} = \left[\frac{1}{2\pi} \arctan x_1, \frac{1}{2\pi} \arctan x_2, \frac{1}{2\pi} \arctan x_3 \right]^{\mathrm{T}},$$

$$\Delta C = \begin{pmatrix} -3\sin 6t & 0 & 0 \\ 0 & 3\cos t & 0 \\ 0 & 0 & -3\sin t \end{pmatrix}, \Delta A = \begin{pmatrix} \cos t & \sin t & \sin t \\ \sin t & \cos 2t & -\cos 3t \\ \sin t & -\cos 3t & -\sin 2t \end{pmatrix}.$$

Then the system (1) satisfies the condition of Theorem, therefore it is robust stable (see Figure 1).

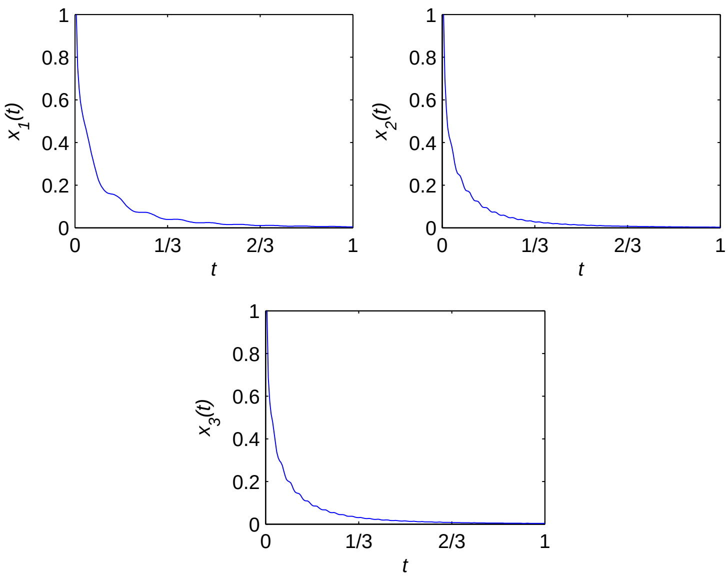

Figure 1 Phase plot of the fractional order Hopfield neural networks: (a) $x_1(t)$ plane, (b) $x_2(t)$ plane, (c) $x_3(t)$ plane.

References

[1] I. PODLUBNY. *Geometric and physical interpretation of fractional integration and fractional differentiation*. Fract. Calc. Appl. Anal., 2002, **5**(4): 367–386.

[2] J. T. MACHADO, V. KIRYAKOVA, F. MAINARDI. *Recent history of fractional calculus*. Commun. Nonlinear Sci. Numer. Simul., 2011, **16**(3): 1140–1153.

[3] G. SAMKO, A. A. KILBAS, O. I. MARICHEV. *Fractional Integrals and Derivatives: Theory and applications*. Gordon and Breach Science Publishers, Yverdon, 1993.

[4] C. BONNET, J. R. PARTINGTON. *Coprime factorizations and stability of fractional differential systems*. Systems Control Lett., 2000, **41**(3): 167–174.

[5] WeiHua DENG, Cangpin LI, Jinhu LÜ. *Stability analysis of linear fractional differential system with multiple time-delays*. Nonlinear Dynam., 2007, **48**(4): 409–416.

[6] Cangpin LI, WeiHua DENG. *Remarks on fractional derivatives*. Appl. Math. Comput., 2007, **187**(2): 777–784.

[7] Liqiong LIU, Shouming ZHONG. *Finite-time stability analysis of fractional order with multi-state time delay*. Sci., Eng. Technol., 2011, **76**: 874–877.

[8] M. LAZAREVIC. *Finite-time stability analysis of fractional order time delay systems: Bellman-Grronwall's approach*. Sci. Tech. Rev., 2007, **7**: 8–15.

[9] H. DELAVARI, D. BALEANU, J. SADATI. *Stability analysis of Caputo fractional-order nonlinear systems revisited*. Nonlinear Dynam., 2012, **67**(4): 2433–2439.

[10] Cangpin LI, Yi CHAI, Ranchao WU. *Dynamic analysis of a class of fractional-order neural networks with delay*. Neurocomputing, 2013, **111**: 190–194.

[11] Juan YU, Cheng HU, Haijun JIANG. *α-stability and α-synchronization for fractional-order neural networks*. Neural Networks, 2012, **35**: 82–87.

[12] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO. *Theory and Applications of Fractional Differential Equations*. Elsevier Science B.V., Amsterdam, 2006.

[13] H. DELAVARI, D. BALEANU, J. SADATI. *Stability analysis of Caputo fractional-order nonlinear systems revisited*. Nonlinear Dynam., 2012, **67**(4): 2433–2439.

[14] K. SAYEVAND, A. GOLBABAI, A. YILDIRIM. *Analysis of differential equations of fractional order*. Appl. Math. Model., 2012, **36**(9): 4356–4364.

[15] J. E. SLOTINE, Weiping LI. *Applied Nonlinear Control*. Prentice Hall, Englewood Cliffs, 1991.