Some Properties of Causal Linear Time-Invariant (LTI) Operators on the Weighted ℓ_2 Space

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Abstract We discuss the properties of causal LTI operators on weighted ℓ_2 spaces for different choices of the weighting sequence $\{w(t)\}_{t \in \mathbb{Z}}$. Problems of closability of unstable causal LTI convolution operators are also discussed. We shall provide a new type of argument concerning causal LTI operators and robust design that can be applied to a large class of weighted ℓ_2 spaces on $\mathbb{Z}$.

Keywords causal LTI operators; weighted ℓ_2 space; two-operator approach

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1. Introduction

Much of modern robust control is focused on the use of the graph of a system as a means of understanding its fundamental properties, especially those related to feedback stabilization. This approach is operator-theoretic method in order to study systems from an input-output point of view.

The basic input-output plant model is often taken to be the operator model $y = Pu$, where P is the linear plant convolution operator, y is the output and u is the input. Georgiou and Smith [1] have studied the properties such as causality and stabilizability of the systems over the signal space $\ell_2(\mathbb{Z})$ and they discovered intrinsic difficulties: A causal system could have a noncausal closure and a well-known stabilizable system seemed not to be stabilizable. The problem was analysed further by Mäkilä [2,3] in a series of papers. He was interested in the question of whether the graphs of linear systems are in fact closable (Operator closedness is well-known to be a minimum requirement in stabilization theory. Operator closures are studied extensively in various signal setting in [4]). Papers [5,6] have developed an input-output stabilization theory on $\ell_2(\mathbb{Z})^n$ for multi-input, multi-output systems based on the use of operator closures. In [7], transfer functions and symbols were studied and equivalent conditions for causality were given.

The two-operator, input-output model $Ay = Bu$, where A and B are bounded linear operators, avoids the intrinsic limitations of the one-operator model $y = Pu$ on the full time axis.

Such models were popular in time series analysis on $\mathbb{Z}$. Paper [8] indicated that input-output model definition $Ay = Bu$ was convenient if A and B were chosen as bounded linear operators on $\ell_\infty(\mathbb{Z})$. Paper [9] discussed the stabilization of linear system specified by an input-output relationship $Ay = Bu$ on the whole set of integers $\mathbb{Z}$. The two-operator approach is shown to lead to a generalization of robust H_∞ input-output control theory to a large class of $L_2(\mathbb{R}, w)$ spaces [10]. In [11], a two-operator approach is used to study robust stabilization of discrete time linear systems in a specific weighted ℓ_∞ space on $\mathbb{Z}$.

In this paper, we discuss some properties of causal LTI operators defined on the whole set of integers $\mathbb{Z}$ of linear systems which are specified by an input-output (I/O) relationship $Ay = Bu$. Our primary interest is in ℓ_2 signal spaces, which lead to H_∞ optimization in addition to avoiding the Georgiou-Smith paradox. We also discuss some signal setups in which unstable causal LTI convolution operators are not closable. This is achieved by weighted ℓ_2 spaces. So we cannot use the closure approach on $\ell_2(\mathbb{Z}, w)$.

This paper is organized as follows. In Section 2, we briefly describe mathematical background and notation. In Section 3, we discuss the weighted ℓ_2 signal spaces, which lead to H_∞ optimization in addition to avoiding the Georgiou-Smith paradox. Summable weights on $\mathbb{Z}$ are discussed in Section 4. Some conclusions are drawn in Section 5.

2. Mathematical preliminaries and notations

We use the standard notations $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}_-$, $\mathbb{N}$ for the complex numbers (or the complex plane), the reals, the integers, the positive integers, the negative integers and the non-negative numbers, respectively. Furthermore, $\mathbb{R}^n$ denotes the linear space of all real n-tuples $v = [v_1, \ldots, v_n]'$ equipped with the norm $|v| = (\sum_{i=1}^{n} v_i^2)^{\frac{1}{2}}$. The superscript $'$ denotes vector transpose.

Let $\ell_q(\mathbb{Z})$ denote the linear normed space of all sequences $x = \{x(t) \in \mathbb{R}\}_{t \in \mathbb{Z}}$ such that

$$\| x \|_q = (\sum_{t \in \mathbb{Z}} |x(t)|^q)^{\frac{1}{q}} < \infty.$$

For $q = \infty$, the space $\ell_\infty(\mathbb{Z})$ is defined analogously using the ∞-norm defined as

$$\|x\|_\infty = \sup_{t \in \mathbb{Z}} |x(t)|.$$

We define $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. $H^\infty(\mathbb{D})$ denotes the space of bounded analytic functions on the unit disc $\mathbb{D}$.

Let X and Y be real linear spaces. Then for an operator A mapping from a subspace of X into Y, $D(A; X)$ denotes the domain of A. $R(A; Y) = \{y = Ax : x \in D(A; X)\} \subset Y$ denotes the image of A. The kernel space $N(A; X)$ of A is defined as $N(A; X) = \{x \in D(A; X) : Ax = 0\}$. Let $V \subset D(A; X)$. We denote $AV = \{Av : v \in V\} \subset Y$. Denote by $\| \cdot \|_X$ the norm of the linear normed spaces X.

We say that the operator $A : X \to Y$ is bounded if there exists $K > 0$ such that

$$\|Ax\|_Y \leq K\|x\|_X, \quad \forall x \in X.$$

For a bounded operator $A : X \rightarrow Y$, the quantity

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

is called the induced norm of A.

The graph of A is defined as $G(A) = \{(u, Au) : u \in D(A; X)\}$. An operator A_E is called an extension of A if $D(A; X) \subset D(A_E; X) \subseteq X$ and $A_E u = Au$ for any $u \in D(A; X)$. An operator A is called closed if its graph $G(A)$ is a closed set. An operator A is called closable if it has a minimal closed extension. The (minimal) closed extension of A is called the closure of the operator A.

Proposition 2.1 *A is closable if and only if the following condition holds: If $x_n \in D(A; X)$, $x_n \rightarrow 0$ and $Ax_n \rightarrow y$, then $y = 0$.*

Let $s(\mathbb{Z})$ denote the linear space of double-sided real sequences $x = \{x(t) \in \mathbb{R}\}_{t \in \mathbb{Z}}$.

By $\widehat{\cdot}$, we denote the z-transform which is given by

$$\widehat{u}(z) = \sum_{n \in \mathbb{Z}} u(n) z^n, \quad u \in l_2(\mathbb{Z}).$$

The z-transform is a linear, bounded mapping from $l_2(\mathbb{Z})$ to $L_2(\mathbb{T})$.

Definition 2.2 *The quadruple (A, B, Y, X), where*

$$A : D(A; Y) \rightarrow Y$$

$$B : D(B; X) \rightarrow Y$$

are linear operators, is called a discrete linear system, consisting of the set of trajectories

$$T(A, B, Y, X) \equiv \{(u \ y)' \in D(B; X) \times D(A; Y) : Ay = Bu\}.$$

Here $Y \subseteq s(\mathbb{Z})$ and $X \subseteq s(\mathbb{Z})$ are linear spaces.

For each $k \in \mathbb{Z}$, denote the truncation operator $P_k : s(\mathbb{Z}) \rightarrow s(\mathbb{Z})$, $k \in \mathbb{Z}$ by

$$(P_k x)(t) = \begin{cases} x(t), & t \leq k, \\ 0, & t > k. \end{cases}$$

Definition 2.3 *The linear operator $P : D(P; X) \rightarrow Y$ is said to be causal if*

$$P_k(Y)P = P_k(Y)PP_k(X), \quad \text{for } k \in \mathbb{Z},$$

where $P_k(Y)$ and $P_k(X)$ denote the truncation operators on Y and X, respectively.

Definition 2.4 *Let $P : D(P; s(\mathbb{Z})) \rightarrow s(\mathbb{Z})$ denote the linear operator*

$$(Px)(t) \equiv \sum_{k \in \mathbb{Z}} H(t, k) x(t - k) = \lim_{K, L \rightarrow \infty} \sum_{-K \leq k \leq L} H(t, k) x(t - k),$$

where the $H(t, k)$ are real matrices. The operator P is called a linear convolution operator.

The convolution operator P is causal if $H(t, k) = 0$ for all $k < 0$. The operator P is called anti-causal if $H(t, k) = 0$ for all $k > 0$. If $H(t, k) = G(k)$ for some sequence of real matrices

$\{G(k)\}_{k \in \mathbb{Z}}$, then the linear convolution operator

$$(Px)(t) \equiv \sum_{k \in \mathbb{Z}} G(k)x(t-k) = \lim_{K,L \to \infty} \sum_{-K \leq k \leq L} G(k)x(t-k),$$

is called an LTI convolution operator.

3. Weighted ℓ_2 spaces on $\mathbb{Z}$

Let $\ell_2(\mathbb{Z})$ denote the space of all square summable sequences on the whole set of integers $\mathbb{Z}$, $\ell_2(k, \infty) \cap \mathbb{Z}$ denote the space of all square summable sequences on $(k, \infty) \cap \mathbb{Z}$. The $\ell_2(\mathbb{Z})$ signal space setup is that here unstable causal LTI convolution operators may have noncausal closures. This phenomenon is called the Georgiou-Smith paradox [4,5,12]. We shall discuss among other signal setups in which unstable causal LTI convolution operators are not closable. This is achieved by weighted ℓ_2 spaces.

3.1. ℓ_2 spaces with decreasing weights

We shall consider a class of weighted $\ell(\mathbb{Z})$ spaces which lead to H_∞ optimization in addition to avoiding the Georgiou-Smith paradox. Let $\{w(t)\}_{t \in \mathbb{Z}}$ be a positive decreasing sequences on $\mathbb{Z}$.

$$\ell_2(\mathbb{Z}, w) = \{\{x(t)\}_{t \in \mathbb{Z}} | \{x(t)\sqrt{w(t)}\}_{t \in \mathbb{Z}} \in \ell_2(\mathbb{Z})\}$$

with

$$\|x\|_w = \left(\sum_{t=-\infty}^{\infty} x^2(t)w(t) \right)^{\frac{1}{2}} < \infty.$$

Suppose $u \in \ell_2(\mathbb{Z}, w)$. Then for each $k \in \mathbb{N}$, the shifted sequence $S_k u$ defined by $(S_k u)(t) = u(t-k)$ also belongs to $\ell_2(\mathbb{Z}, w)$ and $\|S_k u\|_w \leq \|u\|_w$.

An operator P on $\ell_2(\mathbb{Z}, \omega)$ is shift-invariant if $PS_k = S_k P$ for all $k > 0$. If P is shift-invariant and both u and $S_{-k}u$ belong to $\ell_2(\mathbb{Z}, \omega)$ for some $k > 0$, then $PS_{-k}u = S_{-k}Pu$.

Note that, for w decreasing, each unweighted space $l_2(k, \infty) \cap \mathbb{Z}$ for $k \in \mathbb{Z}$ embeds as a subspace of $l_2(\mathbb{Z}, w)$. The following result shows that in many cases the use of the space $\ell_2(\mathbb{Z}, w)$ leads to an H_∞ norm.

Lemma 3.1 ([10]) *Let $\{w(t)\}_{t \in \mathbb{Z}}$ be a positive decreasing sequence such that either (i) w is bounded or (ii) w is bounded below. Let P denote the causal convolution operator defined by*

$$(Pu)(t) = \sum_{k=0}^{\infty} g(k)u(t-k),$$

where $\{g(k)\}_{k \in \mathbb{N}}$ is a real sequence. Then P can extend to a bounded operator defined on the whole of $\ell_2(\mathbb{Z}, w)$ if and only if the function G defined by

$$G(z) = \sum_{k=0}^{\infty} g(k)z^k$$

belongs to $H^\infty(\mathbb{D})$. In this case $\|P\| = \|G\|_\infty$.

A condition such as boundedness or boundedness away from zero is necessary here if we wish the induced norm to be the standard H_∞ norm. Define the function

$$w_e(k) = \begin{cases} 1, & \text{if } k \in \mathbb{Z}_-, \\ e^{-k^2}, & \text{if } k \in \mathbb{N}. \end{cases}$$

We write $\ell_2^e(\mathbb{Z})$ for the space $\ell_2(\mathbb{Z}, w_e)$ and $\|\cdot\|_e$ for the norm $\|\cdot\|_{w_e}$.

Corollary 3.2 *Let $P : D(P; \ell_2^e(\mathbb{Z})) \to \ell_2^e(\mathbb{Z})$ be the causal LTI convolution operator defined by*

$$(Pu)(t) = \sum_{k=0}^{\infty} g(k)u(t-k),$$

$\{g(k)\}_{k \in \mathbb{N}}$ is a real sequence on $\mathbb{Z}$ with $g(k) = 0$ for $k \in \mathbb{Z}_-$. Then $D(P; \ell_2^e(\mathbb{Z})) = \ell_2^e(\mathbb{Z})$ if and only if $\|G\|_\infty < \infty$, where the function G defined by $G(z) = \sum_{k=0}^{\infty} g(k)z^k$ lies in $H_\infty(\mathbb{D})$. In this case, $\|P\| = \|G\|_\infty$.

Proof This result can be treated as a special case of Lemma 3.1.

Another interesting weight that leads to H_∞ optimization is the function

$$w_s(k) = \begin{cases} e^{k^2}, & \text{if } k \in \mathbb{Z}_-, \\ 1, & \text{if } k \in \mathbb{N}. \end{cases}$$

The use of this weighted signal space also allows to avoid the Georgiou-Smith paradox discussed in [1].

The sequence $\{g(k)\}_{k \in \mathbb{Z}}$ is exponentially bounded if there exist $M, a > 0$ such that $|g(k)| \leq Me^{ak}$ for $k \in \mathbb{Z}$. $\square$

Theorem 3.3 *Let $\{g(k)\}_{k \in \mathbb{Z}}$ be an exponentially bounded sequence and let P be the convolution operator defined on $\ell_2(\mathbb{Z}, w_s)$ by*

$$(Pu)(t) = \sum_{k=0}^{\infty} g(k)u(t-k). \tag{3.1}$$

Then P is a closed operator.

Proof Suppose that $\{u(t)\}_{t \geq 1}$ is a sequence in $\ell_2(\mathbb{Z}, w_s)$ with $y_n = Pu_n$ also lying in $\ell_2(\mathbb{Z}, w_s)$, and that $\|u(t) - u\| \to 0$ and $\|y(t) - y\| \to 0$ for some $u, y \in \ell_2(\mathbb{Z}, w_s)$. We show that $y = Pu$ and hence the graph of P is closed.

For $n \in \mathbb{Z}$, we have $\hat{y}_n(z) = \hat{g}(z)\hat{u}_n(z)$, where $\hat{x}(z) = \sum_{n \in z} x(n)z^n$. Now it is easily verified that $\|u_n - u\| \to 0$ implies that $\hat{u}_n(s) \to u(s)$ for $n \in \mathbb{Z}$. Thus $\hat{y}(s) = \hat{g}(s)\hat{u}(s)$.

Now, (3.1) converges for all $t \in \mathbb{Z}$, since g is exponentially bounded and u has rapid decrease at $-\infty$, although (3.1) does not a priori define a function in $u \in \ell_2(\mathbb{Z}, w_s)$. However, the uniqueness of the z transform implies that indeed $y = pu$, as required. $\square$

Theorem 3.4 *Let $a > 0$, $b \neq 0$ and P_a denote the causal convolution operator*

$$(P_a u)(t) = \sum_{k=0}^{\infty} be^{ak}u(t-k)$$

defined on a subdomain of the space $\ell_2(\mathbb{Z}, w)$, where w is a positive bounded weight. Suppose that $e_a(t) = e^{at}$ lies in $\ell_2(\mathbb{Z}, w)$. Then P_a is not closable.

Proof We construct a sequence $\{u(t)\}_{t \in \mathbb{N}}$ of inputs such that $\|u(t)\|_w \to 0$ and $\|P_a u(t) - b e_a\|_w \to 0$. This is sufficient to show that P_a is not closable. To do this, take

$$u(t) = \begin{cases} 0, & \text{if } t < -n - 1, \\ h_n, & \text{if } -n - 1 \le t \le -n, \\ 0, & \text{if } t > -n. \end{cases}$$

Now $P_a u(t) = 0$ for $t < -n - 1$. For $-n - 1 \le t \le -n$, we have

$$(P_a u)(t) = \sum_{k=0}^{t+n+1} b e^{ak} h_n = \frac{e^{a(t+n+2)} - 1}{e^a - 1} \cdot b h_n.$$

Then for $t > -n$ we have

$$(P_a u)(t) = \sum_{k=t+n}^{t+n+1} b e^{ak} h_n = \frac{e^{a(t+n)}(e^{2a} - 1)}{e^a - 1} \cdot b h_n.$$

We now choose $h_n = \frac{e^a - 1}{e^{an}(e^{2a} - 1)}$. Since $\{w(t)\}$ is bounded, there exists $M > 0$ such that $|w(t)| \le M$ for all $t \in \mathbb{Z}$, so we have

$$\|u(t)\|_w^2 = \sum_{t \in \mathbb{Z}} u^2(t) w(t) = \sum_{t=-n-1}^{-n} \left(\frac{e^a - 1}{e^{an}(e^{2a} - 1)} \right)^2 w(t) \le \frac{2M}{(e^a + 1)^2} \cdot \frac{1}{e^{2an}} \to 0, \quad \text{for } n \to \infty.$$

Moreover

$$\begin{aligned} \|P_a u - b e_a\|_w^2 &= \sum_{t \in \mathbb{Z}} (P_a u - b e^{at})^2 w(t) \\ &= \sum_{t=-\infty}^{-(n+2)} (-b e^{at})^2 w(t) + \sum_{t=-(n+1)}^{-n} \left(\frac{e^{a(t+n+2)} - 1}{e^a - 1} \cdot b h_n - b e^{at} \right)^2 w(t) + \\ &\quad \sum_{t=-n+1}^{+\infty} \left(\frac{e^{a(t+n)}(e^{2a} - 1)}{e^a - 1} \cdot b h_n - b e^{at} \right)^2 w(t) \\ &\le \frac{M b^2 e^{2a}}{e^{2a} - 1} \cdot \frac{1}{e^{2a(n+2)}} + \frac{M b^2 (1 - e^a)^2}{e^{2a}(e^{2a} - 1)^2} \cdot \frac{1}{e^{2an}} \to 0 \quad \text{for } n \to \infty. \end{aligned}$$

Example 3.5 We consider the causal convolution operator $P : D_P \subseteq l_2(\mathbb{Z}, w) \to l_2(\mathbb{Z}, w)$ given by

$$(Pu)(t) = \sum_{k=0}^{\infty} 3^k u(t - k), \quad u \in D_P,$$

$$D_P := \{u \,|\, u \in l_2(\mathbb{Z}, w), Pu \in l_2(\mathbb{Z}, w)\},$$

where $w(t)$ is bounded. We construct a sequence $\{u(t)\}_{t \in \mathbb{N}}$

$$u(t) = \begin{cases} 0, & \text{if } t < -n - 1, \\ \dfrac{1}{3^n}, & \text{if } -n - 1 \le t \le -n, \\ 0, & \text{if } t > -n. \end{cases}$$

by Theorem 3.4, the LTI system P is not closable. In fact, it is easy to see that $\| u(t) \|_w^2 \to 0$, but $\| (Pu)(t) \|_w^2 \nrightarrow 0$.

3.2. Closed systems on $\ell_2(\mathbb{Z}, w)$

Let $\{w(t)\}_{t \in \mathbb{Z}}$ be a positive decreasing sequence on $\mathbb{Z}$. Then a closed LTI system on $\ell_2(\mathbb{Z}, w)$ is a causal operator P with $D(P) \subseteq \ell_2(\mathbb{Z}, w)$ and closed graph $G(P)$ such that if $(u, Pu) \in G(P)$ and $\tau \in \mathbb{Z}_+$, then

(1) $(S_\tau u, S_\tau Pu) \in G(P)$; (2) $(S_{-\tau} u, S_{-\tau} Pu) \in G(P)$ whenever $(S_\tau u, S_\tau Pu) \in \ell_2(\mathbb{Z}, w) \times \ell_2(\mathbb{Z}, w)$.

Proposition 3.6 *Let $\{w(t)\}_{t \in \mathbb{Z}}$ be a positive decreasing sequence on $\mathbb{Z}$ and let P be a closed LTI system on $\ell_2(\mathbb{Z}, w)$. Then the restriction of $G(P)$ to $\ell_2(k, \infty) \times \ell_2(k, \infty)$ defines a closable operator.*

Proof Suppose that $(u(t), y(t)) \in G(P) \cap \ell_2(k, \infty) \times \ell_2(k, \infty)$ and that $\|u(t)\|_w \to 0$ whereas $\|y(t) - y\|_w \to 0$ (all in the norm of $\ell_2(k, \infty)$). Then we also have $\|u(t)\|_w \to 0$ and $\|y(t) - y\|_w \to 0$ in the norm of $\ell_2(\mathbb{Z}, w)$, since $\{w(t)\}_{t \in \mathbb{Z}}$ is bounded. Hence $(0, y) \in G(P)$. Since $G(P)$ is closed and we include that $y = 0$. This implies closability of the operator obtained by restricting the graph of P. $\square$

4. ℓ_2 space with summable weights on $\mathbb{Z}$

In this section, we shall study what happens when at least bilaterally bounded, persistent, signals are included.

Theorem 4.1 *Let $\{w(t)\}_{t \in \mathbb{Z}}$ be a positive weight on $\mathbb{Z}$ such that*

$$\sum_{t \in \mathbb{Z}} w(t) < \infty. \tag{4.1}$$

Let the operator $P_L : D(P_L; \ell_2(\mathbb{Z}, w)) \to \ell_2(\mathbb{Z}, w)$ be given by

$$(P_L u)(t) = \sum_{n=0}^{+\infty} u(t - L - n),$$

where $L \geq 0$ is a non-negative number (a delay). Then P_L is not closable.

Proof Take the sequence of input signals $\{u(t)\}_{t \geq 1}$, where

$$u(t) = \begin{cases} 1, & t = -i, \\ 0, & \text{otherwise.} \end{cases}$$

From (4.1), we have that

$$\|u(t)\|_w^2 = \sum_{t \in \mathbb{Z}} (u(t))^2 w(t) = \sum_{t=-\infty}^{-i-1} (u(t))^2 w(t) + (u(-i))^2 w(-i) + \sum_{t=-i+1}^{+\infty} (u(t))^2 w(t)$$
$$= w(-i) \to 0,$$

when $i \to \infty$ and $u(t) \in D(P_L)$ for all $i \in \mathbb{N}$.

Define $y(t) = (P_L u)(t)$. Clearly, $y(t) = 0$ for $t \leq -i + L - 1$ and $y(t) = 1$ for $t \geq -i + L$. From (4.1), we see that

$$\|y(t) - u\|_w^2 = \sum_{t \in \mathbb{N}} (y(t) - u(t))^2 w(t)$$

$$= \sum_{t=-i+L}^{+\infty} (1 - u(t))^2 w(t) + \sum_{t=-\infty}^{-i+L-1} (0 - u(t))^2 w(t))$$

$$\leq \sum_{t=-i+L}^{+\infty} w(t) + \sum_{t=-\infty}^{-i+L-1} w(t) \to 0,$$

for $i \to \infty$. So $Pu(t) \to u(t) \neq 0$ while $w(t) \to 0$, we can get p_L is not closable. $\square$

Theorem 4.2 Let $\{w(t)\}_{t \in \mathbb{Z}}$ be a positive weight on $\mathbb{Z}$ such that $\sum_{t \in \mathbb{Z}} w(t) < \infty$. Let the operator $F_L : D(F_L; \ell_2(Z, w)) \to \ell_2(\mathbb{Z}, w)$ be given by

$$(F_L u)(t) = \sum_{k=0}^{\infty} k u(t - L - k),$$

where $L \geq 0$ is a non-negative number (a delay). Then F_L is not closable.

Proof Consider the sequence of input signals $\{u(t)\}_{t \geq 1}$, where

$$u(t) = \begin{cases} 1, & t = -i - 1, \\ -1, & t = -i, \\ 0, & \text{otherwise.} \end{cases}$$

From (4.1), we have that

$$\|u(t)\|_w^2 = \sum_{t \in \mathbb{Z}} (u(t))^2 w(t) = \sum_{t=-\infty}^{-i-2} (u(t))^2 w(t) + (u(-i-1))^2 w(-i-1) + (u(-i))^2 w(-i) +$$

$$\sum_{t=-i+1}^{+\infty} (u(-i+1))^2 w(-i+1) \to 0,$$

for $j \to \infty$.

Now $(F_L u)(t) = 0$ for $t < -i - 2 + L$. By computing $(F_L u)(t)$, we get $(F_L u)(t) = 0$ for $t = -i + L - 1$ and $(F_L u)(t) = 1$ for $t \geq -i + L$. From (5) we have that

$$\|(F_L u)(t) - u(t)\|_w^2 = \sum_{t \in \mathbb{Z}} ((F_L u)(t) - u(t))^2 w(t)$$

$$= \sum_{t=-\infty}^{-i+L-2} ((F_L u)(t) - u(t))^2 w(t) +$$

$$((F_L u)(-i+L-1) - u(-i+L-1))^2 w(-i+L-1) +$$

$$\sum_{t=-i+L}^{+\infty} ((F_L u)(t) - u(t))^2 w(t)$$

$$\leq \sum_{t=-\infty}^{-i+L-2} w(t) + w(-i+L-1) + 4 \sum_{t=-i+L}^{+\infty} w(t) \to 0,$$

for $i \to \infty$. So $(F_L u)(t) \to u(t) \neq 0$ while $i \to \infty$. Thus, F_L is not closable. $\square$

The operator $M_L : D(M_L; \ell_2(\mathbb{Z}, w)) \to \ell_2(\mathbb{Z}, w)$ given by

$$(M_L u)(t) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} u(t - L - n - k),$$

satisfies $(M_L u)(t) = (F_L u)(t)$ for the input sequence $\{u(t)\}_{t \geq 1}$ used in the proof of the previous result. Hence, M_L is not closable on $\ell_2(\mathbb{Z}, w)$.

Note that the operator $P_L : D(P_L; \ell_2(\mathbb{Z}, w)) \to \ell_2(\mathbb{Z}, w)$ is closed. And $F_L : D(F_L; \ell_2(\mathbb{Z}, w)) \to \ell_2(\mathbb{Z}, w)$ is closable with a causal closure on $\ell_2(\mathbb{Z})$.

The above results imply that we cannot use the closure approach on $\ell_2(\mathbb{Z}, w)$. A considerable technical complication follows from the fact that $\ell_2(\mathbb{Z}, w)$ allows signals which do not tend to zero when $t \to -\infty$.

5. Conclusions

The two-operator plant model $Ay = Bu$ has been used to develop a robust input-output stabilization on the full time axis $\mathbb{Z}$. Both plant and controller uncertainty can be handled within the proposed framework which uses bounded causal LTI operators only in the plant and controller modeling, the unbounded operator models must be avoided. Robust design can be applied to weighted ℓ_2 spaces on $\mathbb{Z}$. A large class of weighted ℓ_2 spaces on $\mathbb{Z}$ have been shown to lead to H_∞ optimization. The unstable causal LTI convolution operators are not closable in weighted ℓ_2 spaces.

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