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Monotonicity Formulas of E_F-Critical Maps with Potential

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Abstract In this paper, we introduce the notion of E_F-critical map with potential with respect to the functional $E_{F,H}(u)$. By using the stress-energy tensor, we obtain some monotonicity formulas and vanishing results for these maps under conditions on H.

Keywords E_F-critical map with potential; stress-energy tensor; monotonicity formula

MR(2010) Subject Classification 58E20; 53C21

1. Introduction

Let (M^m, g) and (N^n, J, h) be Riemannian manifolds, the second being endowed with a Kähler structure with the second fundamental 2-form $\omega(\cdot,\cdot) = h(J\cdot,\cdot)$. Let $u : (M^m, g) \to (N, J, h)$ be a smooth map. Motivated by the strong coupling limit of Faddeev-Niemi model [1], Speight and Svensson [2,3] introduced a functional related to the 2-from $u^*\omega$ as follows:

$$E(u) = \frac{1}{2}\int_M \|u^*\omega\|^2 \mathrm{d}v_g, \tag{1}$$

where $\|u^*\omega\|$ is given by

$$\|u^*\omega\|^2 = \langle u^*\omega, u^*\omega\rangle = \frac{1}{2!}\sum_{i,j=1}^m u^*\omega(e_i, e_j)u^*\omega(e_i, e_j) = \frac{1}{2}\sum_{i,j=1}^m [\omega(\mathrm{d}u(e_i), \mathrm{d}u(e_j))]^2$$

with respect to a local orthonormal frame $(e_1, \ldots, e_m)$ on (M, g). Any map u for which $E(u) = 0$, the minimum possible, will be called a vacuum solution or vacuum of the theory. Clearly u is a vacuum if and only if $u^*\omega = 0$ everywhere, that is, if u is isotropic. The map u is an E-critical map for the functional $E(u)$ if it is a critical point of $E(u)$ with respect to any compact supported variation of u and u is stable if the second variation for the functional $E(u)$ is non-negative. Slobodeanu [4] showed the non-existence of non-isotropic stable E-critical map for $E(u)$ from S^m $(m \geq 5)$ to any Kähler manifold.

The theory of harmonic maps has been developed by many researchers so far, and a lot of results have been obtained [5,6]. In 1999, Ara [7] introduced the notion of F-harmonic maps, which is a generalization of harmonic maps, p-harmonic maps or exponentially harmonic maps.

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Since then, there have been many results for F-harmonic maps such as [8–10]. On the other hand, Fardon and Ratto in [11] introduced generalized harmonic maps of a certain kind, harmonic maps with potential, which had its own mathematical and physical background, for example, the static Landu-Lifschitz equation. They discovered some properties quite different from those of ordinary harmonic maps due to the presence of the potential. After this, there are many results for harmonic map with potential such as [12,13], p-harmonic map with potential such as [14], F-harmonic map with potential such as [15], f-harmonic map with potential such as [16] and F-stationary map with potential such as [17].

In this paper, we define the functional $E_{F,H}(u)$ by

$$E_{F,H}(u) = \int_M [F(\frac{\|u^*\omega\|^2}{2}) - H \circ u]\mathrm{d}v_g = E_F(u) - \int_M H \circ u \mathrm{d}v_g, \tag{2}$$

where $F : [0,\infty) \to [0,\infty)$ is a C^2 function such that $F(0) = 0$, $F'(t) > 0$ on $[0,\infty)$ and H a smooth function on N^n. If $H = 0$, then we have $E_{F,H}(u) = E_F(u)$. If $H = 0$ and $F(t) = t$, then we have $E_{F,H}(u) = E(u)$. We call u an E_F-critical map with potential for $E_{F,H}(u)$, if

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{F,H}(u_t)|_{t=0} = 0$$

for any compactly supported variation $u_t : M \to N$ with $u_0 = u$. We will use the stress-energy tensor to obtain the monotonicity formulas and vanishing results for E_F-critical map with potential under some conditions on H.

2. First variation formula

In this section we give the first variation formula for the functional $E_{F,H}(u)$. Let ∇ and ${}^N\nabla$ always denote the Levi-Civita connections of M and N, respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_X W = {}^N \nabla_{\mathrm{d}u(X)} W$, where X is a tangent vector of M and W is a section of $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ on M. We define an $u^{-1}TN$-valued 1-form α_u, which plays an important role in our argument, as follows:

$$\alpha_u(X) = \sum_{j=1}^m \omega(\mathrm{d}u(X), \mathrm{d}u(e_j))\mathrm{d}u(e_j) \tag{3}$$

for any vector field X on M, which gives

$$\|\omega\|^2 = \frac{1}{2}\sum_i \omega(\mathrm{d}u(e_i), \alpha_u(e_i)) = \frac{1}{2}\sum_{ij} [\omega(\mathrm{d}u(e_i), \mathrm{d}u(e_j))]^2.$$

We define the $E_{F,H}$-tension field $\tau_{F,H}(u)$ of u by

$$\begin{aligned}\tau_{F,H}(u) &= \mathrm{div}_g(F'(\frac{\|u^*\omega\|^2}{2})\alpha_u) + J({}^N\nabla H \circ u)\\ &= F'(\frac{\|u^*\omega\|^2}{2})\mathrm{div}_g(\alpha_u) + \alpha_u(\mathrm{grad}\, F'(\frac{\|u^*\omega\|^2}{2})) + J({}^N\nabla H \circ u).\end{aligned} \tag{4}$$

Under the notation above we have the following:

Theorem 2.1 (First variation formula) *Let $u : M \to N$ be a C^2 map. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{F,H}(u_t)|_{t=0} = -\int_M \omega(V, \tau_{F,H}(u))\mathrm{d}v_g, \tag{5}$$

where $V = \frac{\mathrm{d}}{\mathrm{d}t}u_t|_{t=0}$.

Proof Let $\Psi : (-\varepsilon, \varepsilon) \times M \to N$ be defined by $\Psi(t, x) = u_t(x)$, where $(-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$ on $(-\varepsilon, \varepsilon)$, X on M naturally on $(-\varepsilon, \varepsilon) \times M$, and denote those also by $\frac{\partial}{\partial t}$, X. Then

$$V = \mathrm{d}\Psi(\frac{\partial}{\partial t})|_{t=0}. \tag{6}$$

We shall use the same notations ∇ and $\widetilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon, \varepsilon) \times M$ and the induced connection on $\Psi^{-1}TN$, respectively.

$$\begin{aligned}\frac{\partial}{\partial t}[F(\frac{\|u_t^*\omega\|^2}{2}) - H \circ u_t] &= \frac{\partial}{\partial t}F(\frac{\|u_t^*\omega\|^2}{2}) - \frac{\partial}{\partial t}H \circ u_t \\ &= \frac{\partial}{\partial t}F(\frac{\|u_t^*\omega\|^2}{2}) - h({}^N\nabla H \circ u_t, \mathrm{d}\Psi(\frac{\partial}{\partial t})).\end{aligned} \tag{7}$$

Now we calculate

$$\begin{aligned}\frac{\partial}{\partial t}F(\frac{\|u_t^*\omega\|^2}{2}) =&F'(\frac{\|u_t^*\omega\|^2}{2})\frac{1}{2}\frac{\partial}{\partial t}\|u_t^*\omega\|^2 \\ =&F'(\frac{\|u_t^*\omega\|^2}{2})\frac{1}{4}\frac{\partial}{\partial t}[\sum_{ij}\omega^2(\mathrm{d}\Psi(e_i), \mathrm{d}\Psi(e_j))] \\ =&F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i,j=1}^m \omega(\widetilde{\nabla}_{\frac{\partial}{\partial t}}\mathrm{d}\Psi(e_i), \mathrm{d}\Psi(e_j))\omega(\mathrm{d}\Psi(e_i), \mathrm{d}\Psi(e_j)) \\ =&F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i,j=1}^m \omega(\widetilde{\nabla}_{e_i}\mathrm{d}\Psi(\frac{\partial}{\partial t}), \mathrm{d}u_t(e_j))\omega(\mathrm{d}u_t(e_i), \mathrm{d}u_t(e_j)) \\ =&F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m \omega(\widetilde{\nabla}_{e_i}\mathrm{d}\Psi(\frac{\partial}{\partial t}), \sigma_{u_t}(e_i)) \\ =&F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m [e_i\omega(\mathrm{d}\Psi(\frac{\partial}{\partial t}), \sigma_{u_t}(e_i)) - \omega(\mathrm{d}\Psi(\frac{\partial}{\partial t}), \widetilde{\nabla}_{e_i}\sigma_{u_t}(e_i))] \\ =&-F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m [e_ih(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\sigma_{u_t}(e_i)) - h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\widetilde{\nabla}_{e_i}\sigma_{u_t}(e_i))],\end{aligned}$$

where we use

$$\widetilde{\nabla}_{\frac{\partial}{\partial t}}\mathrm{d}\Psi(e_i) - \widetilde{\nabla}_{e_i}\mathrm{d}\Psi(\frac{\partial}{\partial t}) = \mathrm{d}\Psi[\frac{\partial}{\partial t}, e_i] = 0$$

for the third equality. Let X_t be a compactly supported vector field on M such that $g(X_t, Y) = h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\sigma_{u_t}(Y))$ for any vector field Y on M. Then

$$\begin{aligned}&-\frac{\partial}{\partial t}F(\frac{\|u_t^*\omega\|^2}{2}) = F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m e_ig(X_t, e_i) - F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\widetilde{\nabla}_{e_i}\sigma_{u_t}(e_i))] \\ &= F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m [g(\nabla_{e_i}X_t, e_i) + g(X_t, \nabla_{e_i}e_i)]-\end{aligned}$$

$$F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\widetilde{\nabla}_{e_i}\sigma_{u_t}(e_i)) = F'(\frac{\|u_t^*\omega\|^2}{2})\mathrm{div}_g(X_t)-$$
$$F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\widetilde{\nabla}_{e_i}\sigma_{u_t}(e_i) - J\sigma_{u_t}(\nabla_{e_i}e_i))$$
$$= \mathrm{div}(F'(\frac{\|u_t^*\omega\|^2}{2})X_t) - g(X_t, \mathrm{grad}(F'(\frac{\|u_t^*\omega\|^2}{2})))-$$
$$F'(\frac{\|u_t^*\omega\|^2}{2})\sum_{i=1}^m h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J\widetilde{\nabla}_{e_i}\sigma_{u_t}(e_i) - J\sigma_{u_t}(\nabla_{e_i}e_i))$$
$$= \mathrm{div}(F'(\frac{\|u_t^*\omega\|^2}{2})X_t) - h(\mathrm{d}\Psi(\frac{\partial}{\partial t}), F'(\frac{\|u_t^*\omega\|^2}{2})J\mathrm{div}_g\sigma_{u_t} + J\sigma_{u_t}(\mathrm{grad}(F'(\frac{\|u_t^*\omega\|^2}{2})))). \quad (8)$$

By (7), (8) and Green's theorem, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{F,H}(u_t)|_{t=0} = \int_M \frac{\partial}{\partial t}[F(\frac{\|u_t^*h\|^2}{4}) - H\circ u_t]|_{t=0}\mathrm{d}v_g$$
$$= -\int_M \{\omega(\mathrm{d}\Psi(\frac{\partial}{\partial t}), F'(\frac{\|u_t^*h\|^2}{4})\mathrm{div}_g\sigma_{u_t} + \sigma_{u_t}(\mathrm{grad}(F'(\frac{\|u_t^*h\|^2}{4})))+$$
$$\omega(\mathrm{d}\Psi(\frac{\partial}{\partial t}), J({}^N\nabla H\circ u_t))\}|_{t=0}\mathrm{d}v_g$$
$$= -\int_M \omega(V, \tau_{\Phi_{F,H}}(u))\mathrm{d}v_g.$$

This completes the proof. □

The first variation formula allows us to define the notion of E_F-critical map with potential for the functional $E_{F,H}$.

Definition 2.2 *A smooth map u is called E_F-critical map with potential for the functional $E_{F,H}$ if it is a solution of the Euler-Lagrange equation $\tau_{F,H}(u)=0$.*

3. Stress energy tensor

Following Baird [18], for a smooth map $u:(M,g)\to(N,J,h)$, we associate a symmetric 2-tensor $S_{F,H}$ to the functional $E_{F,H}$ called the stress energy tensor

$$S_{F,H}(X,Y) = [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]g(X,Y) - F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(X), \alpha_u(Y)), \quad (9)$$

where X,Y are vector fields on M.

Proposition 3.1 *Under the notation above, we have*

$$(\mathrm{div}\, S_{F,H})(X) = -\omega(\mathrm{d}u(X), \tau_{F,H}(u)) \quad (10)$$

for any vector field X on M.

Proof Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N, respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}$ around a point P on M with $\nabla_{e_i}e_j|_P=0$.

Let X be a vector field on M. At P, we compute

$$
\begin{aligned}
(\operatorname{div} S_{F,H})(X) &= \sum_{i=1}^{m}(\nabla_{e_i} S_{F,H})(e_i, X)\\
&= \sum_{i=1}^{m}\{e_i(S_{F,H}(e_i, X)) - S_{F,H}(\nabla_{e_i} e_i, X) - S_{F,H}(e_i, \nabla_{e_i} X)\}\\
&= \sum_{i=1}^{m}\{e_i([F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]g(e_i, X)) - e_i(F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(X), \alpha_u(e_i)))-\\
&\quad [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]g(e_i, \nabla_{e_i} X) + F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i} X), \alpha_u(e_i))\}\\
&= \sum_{i=1}^{m}\{e_i(F(\frac{\|u^*\omega\|^2}{2}))g(e_i, X) - e_i(F'(\frac{\|u^*\omega\|^2}{2}))\omega(\mathrm{d}u(X), \alpha_u(e_i))-\\
&\quad F'(\frac{\|u^*\omega\|^2}{2})\omega(\widetilde{\nabla}_{e_i}\mathrm{d}u(X), \alpha_u(e_i)) - F'(\frac{\|u^*\omega\|^2}{4})\omega(\mathrm{d}u(X), \widetilde{\nabla}_{e_i}\alpha_u(e_i))+\\
&\quad F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i} X), \alpha_u(e_i))\} - h(\mathrm{d}u(X), {}^N\nabla H\circ u)\\
&= X(F(\frac{\|u^*\omega\|^2}{2})) - \omega(\mathrm{d}u(X), \alpha_u(\operatorname{grad} F'(\frac{\|u^*\omega\|^2}{2}))) - \omega(\mathrm{d}u(X), J({}^N\nabla H\circ u))-\\
&\quad F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(X), \operatorname{div}\sigma_u) - \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega((\nabla_{e_i}\mathrm{d}u)(X), \alpha_u(e_i))\\
&= F'(\frac{\|u^*\omega\|^2}{2})X(\frac{\|u^*\omega\|^2}{2}) - \omega(\mathrm{d}u(X), \tau_{F,H}(u))-\\
&\quad \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega((\nabla_{e_i}\mathrm{d}u)(X), \alpha_u(e_i))\\
&= \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega((\nabla_X\mathrm{d}u)(e_i), \alpha_u(e_i)) - \omega(\mathrm{d}u(X), \tau_{F,H}(u))-\\
&\quad \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega((\nabla_{e_i}\mathrm{d}u)(X), \alpha_u(e_i)).
\end{aligned}
$$

Since $(\nabla_X \mathrm{d}u)(e_i) = (\nabla_{e_i}\mathrm{d}u)(X)$, we obtain $(\operatorname{div} S_{F,H})(X) = -\omega(\mathrm{d}u(X), \tau_{F,H}(u))$. This completes the proof. □

From the above Proposition, we know that if $u : M \to N$ is an E_F-critical map with potential, we have

$$\operatorname{div} S_{F,H} = 0, \tag{11}$$

that is, u satisfies the $E_{F,H}$-conservation law.

4. Monotonicity formula

Let (M, g) be a complete noncompact Riemannian manifold with a pole x_0. Denote by $r(x)$ the g-distance function relative to the pole x_0, that is $r(x) = \operatorname{dist}_g(x, x_0)$. Set $B(r) = \{x \in M^m : r(x) \le r\}$. It is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $\operatorname{Hess}(r^2)$ associated to eigenvalue 2. Denote by $\lambda_{\max}$ (resp., $\lambda_{\min}$) the maximum (resp., minimal) eigenvalues of $\operatorname{Hess}(r^2) - 2\mathrm{d}r \otimes \mathrm{d}r$

at each point of $M-\{x_0\}$. Let (N^n,J,h) be a Kähler manifold, and H be a smooth function on N.

Let $X\in\Gamma_0(TM)$ be a smooth vector field on M, and let φ_t^X $(-\varepsilon<t<\varepsilon)$ be a 1-parameter family of diffeomorphisms of M for this vector field X.

Theorem 4.1 *Let $u:M\to N$ be a C^2 map. Then we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{F,H}(u\circ\varphi_t^X)|_{t=0}=-\int_M\langle S_{F,H},\frac{1}{2}L_Xg\rangle\mathrm{d}v_g, \tag{12}$$

where L_X is the Lie derivative with respect to the direction X and $\langle S_{F,H},L_Xg\rangle=\sum_{ij}S_{F,H}(e_i,e_j)$ $L_Xg(e_i,e_j)$ for a local orthonormal frame field $\{e_1,\dots,e_m\}$ on M.

Proof This formula follows from the general form (Theorem 2.1) of the first variation formula. Let $u_t=u\circ\varphi_t^X$ and $\mathrm{d}u(X)$ be the vector field for the deformation u_t.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_{F,H}(u\circ\varphi_t^X)|_{t=0}\\
&=-\int_M h(\mathrm{d}u(X),{}^N\nabla H\circ u)\mathrm{d}v_g+\int_M\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\widetilde{\nabla}_{e_i}\mathrm{d}u(X),\alpha_u(e_i))\mathrm{d}v_g\\
&=-\int_M[\mathrm{div}(H\circ uX)-H\circ u\mathrm{div}\,X]\mathrm{d}v_g+\\
&\quad\int_M\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\widetilde{\nabla}_{e_i}\mathrm{d}u(X),\alpha_u(e_i))\mathrm{d}v_g\\
&=\int_M H\circ u\mathrm{div}\,X\mathrm{d}v_g+\int_M\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\widetilde{\nabla}_{e_i}\mathrm{d}u(X),\alpha_u(e_i))\mathrm{d}v_g.
\end{aligned}\tag{13}$$

We choose a locally orthonormal frame $\{e_1,\dots,e_m\}$ on M, such that $\nabla_{e_i}e_j|_P=0$, where $P\in M$. At P, we compute

$$\begin{aligned}
&\sum_i\omega(\widetilde{\nabla}_{e_i}\mathrm{d}u(X),\alpha_u(e_i))\\
&=\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega((\nabla_{e_i}\mathrm{d}u)(X),\alpha_u(e_i))+\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X),\alpha_u(e_i))\\
&=\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega((\nabla_X\mathrm{d}u)(e_i),\alpha_u(e_i))+\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X),\alpha_u(e_i))\\
&=\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\widetilde{\nabla}_X\mathrm{d}u(e_i),\alpha_u(e_i))+\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X),\alpha_u(e_i))\\
&=\sum_i F'(\frac{\|u^*\omega\|^2}{2})\nabla_X(\frac{\|u^*\omega\|^2}{2})+\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X),\alpha_u(e_i))\\
&=L_XF(\frac{\|u^*\omega\|^2}{2})+\sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X),\alpha_u(e_i)).
\end{aligned}\tag{14}$$

From (13) and (14), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{F,H}(u\circ\varphi_t^X)|_{t=0}$$

$$
\begin{aligned}
&= \int_M H\circ u \operatorname{div} X \mathrm{d}v_g + \int_M \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\widetilde{\nabla}_{e_i}\mathrm{d}u(X), \alpha_u(e_i))\mathrm{d}v_g \\
&= \int_M H\circ u \operatorname{div} X \mathrm{d}v_g + \int_M [L_X F(\frac{\|u^*\omega\|^2}{2}) + \\
&\quad \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X), \alpha_u(e_i))]\mathrm{d}v_g \\
&= \int_M H\circ u \operatorname{div} X \mathrm{d}v_g - \int_M F(\frac{\|u^*\omega\|^2}{2}) L_X(\mathrm{d}v_g) + \\
&\quad \int_M \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X), \alpha_u(e_i))\mathrm{d}v_g \\
&= -\int_M [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\operatorname{div} X \mathrm{d}v_g + \\
&\quad \int_M \sum_i F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(\nabla_{e_i}X), \alpha_u(e_i))\mathrm{d}v_g \\
&= -\int_M [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\sum_{ij} g(e_i,e_j) g(\nabla_{e_i}X, e_j)\mathrm{d}v_g + \\
&\quad \int_M \sum_{ij} F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(e_j), \alpha_u(e_i)) g(\nabla_{e_i}X, e_j)\mathrm{d}v_g \\
&= -\int_M \sum_{ij} S_{F,H}(e_i,e_j) g(\nabla_{e_i}X, e_j)\mathrm{d}v_g \\
&= -\frac{1}{2}\int_M \sum_{ij} S_{F,H}(e_i,e_j)[g(\nabla_{e_i}X, e_j) + g(\nabla_{e_j}X, e_i)]\mathrm{d}v_g \\
&= -\int_M \sum_{ij} \langle S_{F,H}, \frac{1}{2}L_X g\rangle \mathrm{d}v_g .
\end{aligned}
$$

This completes the proof. □

Definition 4.2 *Let u be a smooth map (M,g) into (N,J,h). We call it weakly E_F-critical map with potential for $E_{F,H}$ if*

$$\frac{\mathrm{d}}{\mathrm{d}t} E_{F,H}(u\circ \varphi_t^X)|_{t=0} = 0 \tag{15}$$

for all $X \in \Gamma_0(TM)$.

Theorem 4.3 *Let $u:(M,g)\to(N,J,h)$ be a weakly E_F-critical map with potential. If $H \le 0$ (or $H|_{u(M)} \le 0$) and*

$$1 + \frac{1}{2}(m-1)\lambda_{\min} - 2\mathrm{d}_F \max\{2, \lambda_{\max}\} \ge C, \tag{16}$$

then

$$\frac{\int_{B(\rho_1)}[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g}{\rho_1^C} \le \frac{\int_{B(\rho_2)}[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g}{\rho_2^C}$$

for any $0 < \rho_1 \le \rho_2$. In particular, if $\int_{B(R)}[F(\frac{|\mathrm{d}u|^2}{2}) - H\circ u]\mathrm{d}v_g = o(R^C)$, then u is isotropic, where C is a positive constant and d_F is defined as follows: $\mathrm{d}_F = \sup_{t\ge 0}\frac{tF'(t)}{F(t)}$ (see [8,19]).

Proof We take $X=\xi(r)r\frac{\partial}{\partial r}=\frac{1}{2}\xi(r)\nabla r^2$, where ∇ denotes the covariant derivative determined by g and $\xi(r)$ is a nonnegative function determined later. Let $\{e_i\}_{i=1}^m$ be an orthonormal basis with respect to g and $e_m=\frac{\partial}{\partial r}$. We may assume that $\mathrm{Hess}(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}_{i=1}^m$.

Now we compute

$$
\begin{aligned}
&\langle S_{F,H},L_{\xi(r)r\frac{\partial}{\partial r}}g\rangle=\sum_{i,j}S_{F,H}(e_i,e_j)(L_{\xi(r)r\frac{\partial}{\partial r}}g)(e_i,e_j)\\
&=\sum_{i,j}[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]g(e_i,e_j)(L_{\xi(r)r\frac{\partial}{\partial r}}g)(e_i,e_j)-\\
&\quad\sum_{i,j}F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(e_i),\alpha_u(e_j))(L_{\xi(r)r\frac{\partial}{\partial r}}g)(e_i,e_j)\\
&=\sum_{i}[F(\frac{\|u^*\omega\|^2}{2})-H\circ u](L_{\xi(r)r\frac{\partial}{\partial r}}g)(e_i,e_i)-\\
&\quad\sum_{i,j}F'(\frac{\|u^*\omega\|^2}{2})\omega(\mathrm{d}u(e_i),\alpha_u(e_j))(L_{\xi(r)r\frac{\partial}{\partial r}}g)(e_i,e_j)\\
&=\xi(r)\sum_{i}[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]\mathrm{Hess}(r^2)(e_i,e_i)+2[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]r\xi'(r)-\\
&\quad\xi(r)\sum_{i,j}F'(\frac{\|u^*\omega\|^2}{2})h(\mathrm{d}u(e_i),\alpha_u(e_j))\mathrm{Hess}(r^2)(e_i,e_j)-\\
&\quad 2F'(\frac{\|u^*\omega\|^2}{2})r\xi'(r)\omega(\mathrm{d}u(e_m),\alpha_u(e_m))\\
&\ge\xi(r)[F(\frac{|\mathrm{d}u|^2}{2})-H\circ u][2+(m-1)\lambda_{\min}]-\\
&\quad\xi(r)F'(\frac{|\mathrm{d}u|^2}{2})\max\{2,\lambda_{\max}\}2\|u^*\omega\|^2+\\
&\quad 2[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]r\xi'(r)-2F'(\frac{\|u^*\omega\|^2}{2})r\xi'(r)\omega(\mathrm{d}u(e_m),\alpha_u(e_m))\\
&\ge\xi(r)[F(\frac{\|u^*\omega\|^2}{2})-H\circ u][2+(m-1)\lambda_{\min}]-\\
&\quad\xi(r)4\mathrm{d}_F\max\{2,\lambda_{\max}\}[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]+\\
&\quad 2[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]r\xi'(r)-2F'(\frac{\|u^*\omega\|^2}{2})r\xi'(r)\omega(\mathrm{d}u(e_m),\alpha_u(e_m))\\
&\ge\xi(r)[F(\frac{\|u^*\omega\|^2}{2})-H\circ u][2+(m-1)\lambda_{\min}-4\mathrm{d}_F\max\{2,\lambda_{\max}\}]+\\
&\quad 2[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]r\xi'(r)-2F'(\frac{\|u^*\omega\|^2}{2})r\xi'(r)\omega(\mathrm{d}u(e_m),\alpha_u(e_m)).
\end{aligned}\tag{17}
$$

From (16) and (17), we have

$$
\begin{aligned}
\langle S_{F,H},\frac{1}{2}L_Xg\rangle\ge&C\xi(r)[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]+[F(\frac{\|u^*\omega\|^2}{2})-H\circ u]r\xi'(r)-\\
&F'(\frac{\|u^*\omega\|^2}{2})r\xi'(r)\omega(\mathrm{d}u(e_m),\alpha_u(e_m)).
\end{aligned}\tag{18}
$$

From (12), (15) and (18), we have

$$0 \geq \int_M [C\xi(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u] + [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]r\xi'(r) - F'(\frac{\|u^*\omega\|^2}{2})r\xi'(r)\omega(\mathrm{d}u(e_m), \alpha_u(e_m))]\mathrm{d}v_g. \quad (19)$$

Take and fix a positive number ε, and let φ be a smooth function on $[0, \infty)$ such that

$$\varphi(r) = \varphi_\varepsilon(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq 1; \\ 0, & \text{if } 1+\varepsilon \leq r, \end{cases} \quad (20)$$

and $\frac{\mathrm{d}\varphi(r)}{\mathrm{d}r} \leq 0$. We define

$$\xi(r) = \xi_\rho(r) = \varphi(\frac{r}{\rho}) \quad (21)$$

and we can verify

$$\xi'(r)r = -\rho\frac{\mathrm{d}\xi_\rho(r)}{\mathrm{d}\rho}, \quad \text{and} \quad \xi'(r) = \frac{1}{\rho}\varphi'(\frac{r}{\rho}) \leq 0. \quad (22)$$

From (19) and (22), we have

$$\begin{aligned} 0 &\geq \int_M [C\xi_\rho(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u] + [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]r\xi'_\varrho(r)]\mathrm{d}v_g \\ &= C\int_M \xi_\rho(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g - \rho\frac{\mathrm{d}}{\mathrm{d}\rho}\int_M \xi_\rho(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g, \end{aligned}$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}\rho}[\rho^{-C}\int_M \xi_\rho(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g] \geq 0.$$

Therefore

$$\rho_1^{-C}\int_M \xi_{\rho_1}(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g \leq \rho_2^{-C}\int_M \xi_{\rho_2}(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g$$

for any $0 < \rho_1 \leq \rho_2$. Because $\mathrm{Supp}\xi_\rho \subseteq B((1+\varepsilon)\rho)$, we have

$$\frac{\int_{B((1+\varepsilon)\rho_1)} \xi_{\rho_1}(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g}{\rho_1^C} \leq \frac{\int_{B((1+\varepsilon)\rho_2)} \xi_{\rho_2}(r)[F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g}{\rho_2^C}.$$

Letting $\varepsilon \to 0$ and noting that $\xi_\rho(r) = 1$ on $B(\rho)$, we have

$$\frac{\int_{B(\rho_1)} [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g}{\rho_1^C} \leq \frac{\int_{B(\rho_2)} [F(\frac{\|u^*\omega\|^2}{2}) - H\circ u]\mathrm{d}v_g}{\rho_2^C}$$

for any $0 < \rho_1 \leq \rho_2$. This completes the proof. □

Lemma 4.4 ([8,9,17,20–22]) *Let (M^m, g) be a complete Riemannian manifold with a pole x_0. Denote by K_r the radial curvature of M.*

(i) If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha \geq \beta > 0$ and $(m-1)\beta - 4\mathrm{d}_F\alpha > 0$, then

$$[(m-1)\lambda_{\min} + 2 - 4\mathrm{d}_F\max\{2, \lambda_{\max}\}] \geq 2(m - \frac{2\mathrm{d}_F\alpha}{\beta});$$

(ii) If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$ and $0 \leq B < 2\varepsilon$, then

$$[(m-1)\lambda_{\min} + 2 - 4\mathrm{d}_F\max\{2, \lambda_{\max}\}] \geq 2[1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4\mathrm{d}_F e^{\frac{A}{2\varepsilon}}];$$

(iii) If $-\frac{a^2}{c^2+r^2} \le K_r \le \frac{b^2}{c^2+r^2}$ *with* $a^2 \ge 0$, $b^2 \in [0, \frac{1}{4}]$ *and* $c^2 \ge 0$, *then*

$$[(m-1)\lambda_{\min} + 2 - 4\mathrm{d}_F \max\{2, \lambda_{\max}\}] \ge 2[1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\mathrm{d}_F \frac{1+\sqrt{1+4a^2}}{2}].$$

Corollary 4.5 *Let* (M, g) *be an* m*-dimensional complete manifold with a pole* x_0. *Assume that the radial curvature* K_r *of* M *satisfies one of the following three conditions:*

(i) If $-\alpha^2 \le K_r \le -\beta^2$ *with* $\alpha \ge \beta > 0$ *and* $(m-1)\beta - 4\mathrm{d}_F\alpha \ge 0$;

(ii) If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\varepsilon}}$ *with* $\varepsilon > 0$, $A \ge 0$, $0 \le B < 2\varepsilon$ *and* $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4\mathrm{d}_F e^{\frac{A}{2\varepsilon}} > 0$;

(iii) If $-\frac{a^2}{c^2+r^2} \le K_r \le \frac{b^2}{c^2+r^2}$ *with* $a^2 \ge 0$, $b^2 \in [0, \frac{1}{4}]$, $c^2 \ge 0$ *and* $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\mathrm{d}_F \frac{1+\sqrt{1+4a^2}}{2} > 0$.

If $u : (M, g) \to (N, J, h)$ *is a weakly* E_F*-critical map with potential, where* $H \in C^\infty(M)$ *and* $H \le 0$, *(or* $H|_{u(M)} \le 0$*), then*

$$\frac{\int_{B(\rho_1)}[F(\frac{\|u^*\omega\|^2}{2}) - H \circ u]\mathrm{d}v_g}{\rho_1^\Lambda} \le \frac{\int_{B(\rho_2)}[F(\frac{\|u^*\omega\|^2}{2}) - H \circ u]\mathrm{d}v_g}{\rho_2^\Lambda} \tag{23}$$

for any $0 < \rho_1 \le \rho_2$, *where*

$$\Lambda = \begin{cases} m - \frac{4\mathrm{d}_F\alpha}{\beta}, & \text{if } K_r \text{ satisfies (i);} \\ 1 + (m-1)(1 - \frac{B}{2\varepsilon}) - 4\mathrm{d}_F e^{\frac{A}{2\varepsilon}}, & \text{if } K_r \text{ satisfies (ii);} \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\mathrm{d}_F \frac{1+\sqrt{1+4a^2}}{2}, & \text{if } K_r \text{ satisfies (iii).} \end{cases}$$

In particular, if $\int_{B(R)}[F(\frac{\|u^*\omega\|^2}{2}) - H \circ u]\mathrm{d}v_g = o(R^\Lambda)$, *then* u *is isotropic.*

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