Journal of Mathematical Research with Applications
Nov., 2015, Vol. 35, No. 6, pp. 692–700
DOI:10.3770/j.issn:2095-2651.2015.06.011
Http://jmre.dlut.edu.cn

Metacompactness in Countable Products

Jianjun WANG
Department of Mathematics, Sichuan Agricultural University, Sichuan 625014*, P. R. China*

Abstract In this paper, we present that if Y is a hereditarily metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered metacompact spaces, then the followings are equivalent:
(1) $Y \times \prod_{n\in\omega} X_n$ is metacompact,
(2) $Y \times \prod_{n\in\omega} X_n$ is countable metacompact,
(3) $Y \times \prod_{n\in\omega} X_n$ is orthocompact.
Thereby, this result generalizes Theorem 5.4 in [Tanaka, Tsukuba. J. Math., 1993, 17: 565–587]. In addition, we obtain that if Y is a hereditarily σ-metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered σ-metacompact spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is σ-metacompact.

Keywords metacompact; σ-metacompact; Čech-scattered

MR(2010) Subject Classification 54B10; 54D15; 54D20

1. Introduction

The notion of C-scattered space was introduced and investigated by Telgarsky [1]. Furthermore, utilizing it to products, he proved the following:

(A) ([1]) If X is a C-scattered paracompact space, then the product $X \times Y$ is paracompact for each paracompact space Y.

As a generalization of C-scattered space, Čech-scattered space introduced by Hohti and Yun [2] plays an important role in study of paracompactness in countable products. Accordingly, the following result is obtained.

(B) ([2]) If $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact spaces, then the product $\prod_{n\in\omega} X_n$ is paracompact.

In 2005, Aoki and Tanaka [3] extended the above result by proving that:

(C) ([3]) If Y is a perfect paracompact space, and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact.

Recently, the authors [4] investigated the weak submetacompactness in countable products and obtained that:

(D) ([4]) If Y is hereditarily weakly submetacompact, and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered weakly submetacompact spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is weakly submetacompact.

Received January 27, 2015; Accepted April 27, 2015
Supported by the Scientific Research Fund of Sichuan Provincial Education Department (Grant No. 14ZB0007).
E-mail address: jianjunw55@163.com

As well known, the following diagram is easily verified:

paracompact → metacompact → σ-metacompact → weakly submetacompact.

But the reverse is not true [5,6]. In addition, Zhu [7] showed that there is a first countable, regular separable, Lindelöf space X such that X^n is Lindelöf for each $n \in \omega$, but X^ω is not σ-metacompact. Above all, it is naturally to raise the following question.

Question 1 *Let Y be a hereditarily metacompact (σ-metacompact) space. Is the product $Y \times \prod_{n\in\omega} X_n$ metacompact (σ-metacompact) if $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered metacompact (σ-metacompact) spaces?*

This paper mainly discusses countable products of metacompactness. Firstly, we obtain a group of equivalent conditions, which extend Tanaka's result in [8], among metacompactness, countable metacompactness and orthocompactness in countable products. Finally, we give an affirmative answer to Question 1 for σ-metacompactness.

Throughout this paper, assume that each space is Tychonoff and ω is the set of natural numbers.

2. Preliminaries

In the rest of this section, we stated some notation and basic facts. Undefined terminology can be found in Engelking [9]. A space X is scattered if every nonempty closed subset S has an isolated point s. And a space X is said to be C-scattered (Čech-scattered) if for every nonempty closed subset S of X, there exists a point $s \in S$ which has a compact (Čech-complete) neighborhood in S. Evidently, all of the scattered spaces, locally compact spaces and C-scattered spaces are Čech-scattered.

For a subset S of X, $|S|$ (resp., $\overline{S}$) denotes its cardinality (resp., closure). Assume that S is closed. Put

$$S^* = \{x \in S : x \text{ has no Čech-complete neighborhood in } S.\}$$

Let S^0=S, $S^{(\alpha+1)}$=$(S^{(\alpha)})^*$, and $S^{(\alpha)}$=$\bigcap_{\beta<\alpha} S^{(\beta)}$ for a limit ordinal α. Note that each $S^{(\alpha)}$ is closed in X. Furthermore, a space X is Čech-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. Obviously, a Čech-scattered space is hereditary for its closed (open) subspace. A closed subset S of X is called topped if $S \bigcap X^{(\alpha(S))}$ is nonempty Čech-complete and $S \bigcap X^{(\alpha(S)+1)}$=$\emptyset$ for some ordinal $\alpha(S)$. Denote $S \bigcap X^{(\alpha(S))}$ by Top(S). For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$. Let rank($x$)=$\alpha$. Then, there is an open neighborhood base $\mathcal{V}$ of x in X such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped in X and $\alpha(\overline{V}) = \text{rank}(x)$. A collection $\mathcal{V}$ of subsets of X is a refinement of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$ and $\bigcup\mathcal{V}$=$\bigcup\mathcal{U}$.

To complete our proof, the following definitions and lemmas are useful. Therefore, we briefly state it here.

Definition 2.1 *A space X is said to be metacompact (σ-metacompact, metaLindelöf, orthocompact) if for every open covering $\mathcal{U}$ of X, there is a point finite (σ-point finite, point countable, interior preserving) open refinements $\mathcal{V}$.*

Recall that a space X is said to be hereditarily metacompact (hereditarily σ-metacompact, hereditarily metaLindelöf) if every subspace G of X is metacompact (σ-metacompact, metaLindelöf). And by these definitions, it is easily proved that a space X is hereditarily metacompact (hereditarily σ-metacompact, hereditarily metaLindelöf) if and only if every open subspace G of X is metacompact (σ-metacompact, metaLindelöf).

Lemma 2.2 ([4]) *The product $X \times Y$ is Čech-scattered if X and Y are Čech-scattered spaces.*

Lemma 2.3 ([9]) *A Tychonoff space X is Čech-complete if and only if there exists a countable family $\{\mathcal{A}_i\}_{i\in\omega}$ of open covers of the space X with the property that any family $\mathcal{F}$ of closed subsets of X, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_i$ for $i \in \omega$, has nonempty intersection.*

Note that the intersection $\bigcap \mathcal{F}$ is countable compact in Lemma 2.3.

Lemma 2.4 ([10]) *A space X is λ-paracompact if and only if for every directed open cover $\mathcal{U}$ of X with cardinality $\leq \lambda$, there is a locally finite open cover $\mathcal{V}$ of X such that $\{\overline{V} : V \in \mathcal{V}\}$ refines $\mathcal{U}$. A space X is countably paracompact if and only if λ=ω.*

Lemma 2.5 ([11]) *If a product space X=$\prod_{\alpha\in\kappa} X_\alpha$ is orthocompact, then X is κ-metacopmact.*

3. metacompactness in countable products

In [12], the authors proved that the product of a countable collection of Čech-scattered metacompact spaces is metacompact. By Burke [13], every perfect metacompact space is hereditarily metacompact. Now we discuss countable products of metacompactness again.

Theorem 3.1 *If Y is a hereditarily metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered metacompact spaces, then the followings are equivalent.*

(a) *$Y \times \prod_{n\in\omega} X_n$ is metacompact,*

(b) *$Y \times \prod_{n\in\omega} X_n$ is countable metacompact,*

(c) *$Y \times \prod_{n\in\omega} X_n$ is orthocompact.*

Proof (a)⇒ (c) holds obviously.

(c)⇒ (b). By Lemma 2.5, it is clear.

(b)⇒ (a). This proof is a modification of [8, Theorem 5.4]. Using the proof of [14, Theorem], we may assume that for each $n \in \omega$, X_n=X and Top(X)=$\{a\}$ for some $a \in X$. To complete our proof, it suffices to show that $Y \times X^\omega$ is metacompact.

Let $\mathcal{G}$ be an arbitrary open covering of $Y \times X^\omega$ and closed under finite unions. We are going to find a point finite open refinement of $\mathcal{G}$.

Let $\mathcal{B}$ be a base of $Y \times X^\omega$, consisting of all sets of the form D=$\widetilde{D} \times \prod_{i\in\omega} D_i$ and for each $i \in \omega$, $\overline{D_i}$ is topped, i.e., Top($\overline{D_i}$) is Čech-complete. Then, there is a sequence $\{\mathcal{W}_{i,m}(D) : m \in \omega\}$ of open covers of Top($\overline{D_i}$), such that if $\mathcal{F}$ is a collection of nonempty closed subset of Top($\overline{D_i}$) with the finite intersection property such that for each $m \in \omega$, there are $F_m \in \mathcal{F}$ and $W_m \in \mathcal{W}_{i,m}(D)$

with $F_m \subset W_m$, then the intersection $\bigcap \mathcal{F}$ is nonempty. Let $n(D) = \inf\{i : D_j = X, \text{ for } j \geq i\}$. And define $\mathcal{C}$ as follows:

$(*)$ $(D, \mathcal{W}_{i,m}(D)) \in \mathcal{C}$, $m \in \omega$, if $D{=}\widetilde{D} \times \prod_{i\in\omega} D_i \in \mathcal{B}$ and $\mathcal{W}_{i,m}(D)$ is an open cover of $\mathrm{Top}(\overline{D_i})$, satisfying the conditions described above.

For each $m \in \omega$, let $(D, \mathcal{W}_{i,m}(D)) \in \mathcal{C}$. In case of that $i < n(D)$, let m=1. Then for each $W \in \mathcal{W}_{i,1}(D)$, there is an open subset W' of $\overline{D_i}$ such that $W = W' \bigcap \mathrm{Top}(\overline{D_i})$. Moreover, $\{W' : W \in \mathcal{W}_{i,1}(D)\} \bigcup \{\overline{D_i} - \mathrm{Top}(\overline{D_i})\}$ covers $\overline{D_i}$, hence, it follows from [12, Lemma 2] that there is an open covering $\mathcal{A}_i(D)$ of D_i such that:

(a) $\mathcal{A}_i(D)$ is point finite,

(b) for each $A \in \mathcal{A}_i(D)$, $\overline{A}$ is topped and contained in some member of $\{W' : W \in \mathcal{W}_{i,m}(D)\} \bigcup \{\overline{D_i} - \mathrm{Top}(\overline{D_i})\}$.

In case of that $i = n(D)$, we can also take a point finite open covering $\mathcal{A}_{n(D)}(D)$ of D_i such that for each $A \in \mathcal{R}_i(D)$, $\overline{A}$ is topped. And there exists a proper member $A_0 \in \mathcal{A}_{n(D)}(D)$ with $a \in A_0$ and for each $A^* \in \mathcal{A}_{n(D)}(D) - \{A_0\}$, $a \notin A^*$.

By construction of each $\mathcal{A}_i(D)$, let $\mathcal{R}(D) = \prod_{i\leq n(D)} \mathcal{A}_i(D)$. Clearly, $\mathcal{R}(D)$ is a point finite open covering of $\prod_{i\leq n(D)} D_i$.

Let $R = \prod_{i\leq n(D)} R_i \in \mathcal{R}(D)$ with $\mathrm{Top}(\overline{R}) \bigcap \mathrm{Top}(\prod_{i\leq n(D)} \overline{D_i}) \neq \emptyset$. Then, $\mathrm{Top}(\overline{R_i}) \bigcap \mathrm{Top}(\overline{D_i}) \neq \emptyset$ for each $i \leq n(R)$. Observe that $\mathrm{Top}(\overline{R_i}) \bigcap \mathrm{Top}(\overline{D_i}) = \overline{R_i} \bigcap \mathrm{Top}(\overline{D_i}) = \mathrm{Top}(\overline{R_i})$. Hence, by (ii), $\mathrm{Top}(\overline{R_i}) \subset W$ for some $W \in \mathcal{W}_{i,1}(D)$. For $R \in \mathcal{R}(D)$, define $P(R) = R \times X \times \cdots$. Then $\mathrm{Top}(\overline{P(R)}) = \mathrm{Top}(\overline{R}) \times \{a\} \times \cdots$. Correspondingly, define $R_{\widetilde{D}} = \widetilde{D} \times P(R) = \widetilde{D} \times \prod_{i\leq n(D)} R_i \times X \times \cdots$. Thereby, for each $y \in \widetilde{D}$, define $\widehat{R}_y = \{y\} \times \mathrm{Top}(\overline{P(R)})$. Namely, $\widehat{R}_y$ is Čech-complete. Now define $\widehat{R}_y$ satisfying $(**)$ as follows:

$(**)$ If there are some basic open subsets E_1 and E_2 in $Y \times X^\omega$ and some $G \in \mathcal{G}$ such that $\widehat{R}_y \subset E_1 \subset \overline{E_1} \subset E_2 \subset \overline{E_2} \subset G$.

By this way, we say that R holds $(**)$ if there exists a $y \in \widetilde{D}$ such that $\widehat{R}_y$ satisfies $(**)$. Fix a $y \in \widetilde{D}$. Suppose that $\widehat{R}_y$ satisfies the condition $(**)$. Let $k(R, y) = \inf\{n(E_1) : E_1$ and E_2 are some basic open subsets in $Y \times X^\omega$ with $n(E_1) = n(E_2)$ such that $\widehat{R}_y \subset E_1 \subset \overline{E_1} \subset E_2 \subset \overline{E_2} \subset G$ for some $G \in \mathcal{G}\}$. Then, there are some basic open subsets $E_1(R, y) = \widetilde{E_1(R, y)} \times \prod_{i\in\omega} E_1(R, y)_i$ and $E_2(R, y) = \widetilde{E_2(R, y)} \times \prod_{i\in\omega} E_2(R, y)_i$ in $Y \times X^\omega$ and some $G(R, y) \in \mathcal{G}$ such that:

(1) (a) $\widehat{R}_y \subset E_1(R, y) \subset \overline{E_1(R, y)} \subset E_2(R, y) \subset \overline{E_2(R, y)} \subset G(R, y)$;

(b) $k(R, y){=}n(E_1(R, y))$.

Let $r(R, y){=}\max\{n(D) + 1, k(R, y)\}$. Define $H(R, y)$ as follows:

$$H(R, y) = \widetilde{H(R, y)} \times \prod_{i<r(R,y)} P(R)_i \bigcap E_1(R, y)_i \times X \times \cdots = \widetilde{H(R, y)} \times \prod_{i\in\omega} H(R, y)_i.$$

By the definition of $H(R, y)$, we may assume that:

(2) (a) for $i \in \omega$ with $k(R, y) \leq i < r(R, y)$, let $H(R, y)_i = P(R)_i$;

(b) for $i \in \omega$ with $i < k(R, y)$ and $i < n(D)$, let $H(R, y)_i = P(R)_i \bigcap E_1(R, y)_i$;

(c) for $i \in \omega$ with $n(D) \leq i < k(R, y)$, let $H(R, y)_i = \{a\}$;

(d) in case of that $r(R, y) = n(D) + 1$, let $H(R, y)_i = X$ for each $i \geq n(D) + 1$; in case of that $r(R, y) = k(R, y) > n(D) + 1$, let $H(R, y)_i = X$ for $i \geq k(R, y)$;

(e) $\widetilde{H(R,y)} = \bigcap_{i=1}^{2} \widetilde{E_i(R,y)}$.

Distinctly, $H(R,y) \in \mathcal{B}$ with $\widehat{R}_y \subset H(R,y)$, and contained in some member of $\mathcal{G}$ and for each $i \in \omega$, $\overline{H(R,y)_i}$ is topped.

For each $k \in \omega$, let $\mathcal{H}(R,k) = \{\widetilde{H(R,y)} \bigcap \widetilde{D} : k(R,y) \le k\}$. For $k \in \omega$, let $L(R,k) = \{y \in \widetilde{D} : k(R,y) \le k\}$. Clearly, $L(R,k) \subset L(R,k+1)$ and $L(R,k) = \bigcup \mathcal{H}(R,k)$. By the hereditary metacompactness of Y, there exists a collection $\mathcal{L}(R)$=$\bigcup_{k\in\omega} \mathcal{L}(R,k)$ of open subsets in Y such that:

(3) (a) $L(R,k)$=$\bigcup \mathcal{L}(R,k)$;

(b) $\mathcal{L}(R,k)$ refines $\mathcal{H}(R,k)$;

(c) each $\mathcal{L}(R,k)$ is point-finite in Y.

For each $L \in \mathcal{L}(R,k)$, there exists a $y(L) \in L(R,k)$ such that $L \subset \widetilde{H(R,y(L))} \bigcap \widetilde{D}$. So $k(R,y(L)) \le k$. Now, define $O(R,L)$ as follows: $O(R,L) = L \times \prod_{i\in\omega} H(R,y(L))_i$. By the definition, $O(R,L) \in \mathcal{B}$ and $L \times \mathrm{Top}(\overline{P(R)}) \subset O(L,R)$.

Put $\mathcal{N}(R,L)$=$\mathcal{P}(\{0,1,\ldots,r(R,y(L))-1\})$. Fix an $A \in \mathcal{N}(R,L)$. Define $D_A(R,L) = L \times \prod_{i\in\omega} D_A(R,L)_i$ as follows:

(4) (a) if $i \in A$ with $i < n(D)$, let $D_A(R,L)_i = P(R)_i - \overline{P(R)_i \bigcap E_2(R)_i}$;

(b) if $i \in A$ with $r(R,y(L)) = k(R,y(L)) > i \ge n(D)$, let $D_A(R,L)_i = X - \{a\}$;

(c) if $i < r(R,y(L))$ with $i \notin A$, let $D_A(R,L)_i = P(R)_i \bigcap E_1(R,y(L))_i$;

(d) for each i with $i \ge r(R,y(L))$, let $D_A(R,L)_i = X$.

Clearly, if i satisfies (4) (c) or (d), then $\overline{D_A(R,L)_i}$ is topped. And if $i \in A$ with $k(R,y(L)) \le i < r(R,y(L))$, then $D_A(R,L)_i = \emptyset$. Now, we consider the other cases:

(i) if $i \in A$ with $i < \min\{n(D), k(R,y(L))\}$;

(ii) if $i \in A$ with $r(R,y(L)) = k(R,y(L)) > i \ge n(D)$;

(iii) if $i = r(R,y(L))$.

If i satisfies the conditions (i) or (ii), then $\overline{D_A(R,L)_i}$ does not need to be topped and hence, there is an open covering $\mathcal{B}(\overline{D_A(R,L)_i})$ of $\overline{D_A(R,L)_i}$ such that for each $B \in \mathcal{B}(\overline{D_A(R,L)_i})$, $\overline{B}$ is topped. Then there is a point finite, open refinement $\mathcal{D}_{A,i}(R,L)$ of $\mathcal{B}(\overline{D_A(R,L)_i})$, covering $D_A(R,L)_i$ and for each $D_i^* \in \mathcal{D}_{A,i}(R,L)$, $\overline{D_i^*}$ is topped. If i satisfies (iii), there is a proper point finite, open covering $\mathcal{D}_{A,r(R,y(L))}(R)$ of X and for each $D_i^* \in \mathcal{D}_{A,r(R,y(L))}(R,L)$, $\overline{D_i^*}$ is topped. Next, define the colletion $\mathcal{D}_A^*(R,L)$ as follows:

(5) $D^*(L) = L \times \prod_{i\in\omega} D_i^* \in \mathcal{D}_A^*(R,L)$ if for each $i \in \omega$,

(a) if $i \in A$ with $k(R,y(L)) \le i < n(D)$, let $D_i^* = \emptyset$;

(b) if i satisfies one of the conditions (i), (ii) and (iii), let $D_i^* \in \mathcal{D}_{A,i}(R,L)$;

(c) if $i \notin A$ with $i < r(R,y(L))$, let $D_i^* = D_{A,i}(R,L)$;

(d) let $D_i^* = X$ for each $i > r(R,y(L))$.

With that, let $\mathcal{D}_A(R,L) = \{D^* \in \mathcal{D}_A^*(R,L) : D^* \ne \emptyset\}$. Thus, we infer that

(6) the collection $\mathcal{D}_A(R,L)$ is point finite in $Y \times X^\omega$.

Further on, let $\mathcal{D}(R,L) = \bigcup\{\mathcal{D}_A(R,L) : A \in \mathcal{N}(R,L)\}$. Therefore, by (6) and the definitions of $P(R)$, $O(R,L)$, collection $\mathcal{D}(R,L)$ satisfies the following:

(7) (a) collection $\mathcal{D}(R,L)$ is point finite in $Y \times X^\omega$ and $L \times P(R) = O(R,L) \bigcup (\bigcup \mathcal{D}(R,L))$;

for each $D^* \in \mathcal{D}(R,L)$,

(b) $n(D^*) = r(R, y(L))$ and $n(D^*) > n(D)$;

(c) for each $i \in \omega$, $\alpha(\overline{D_i^*}) \leq \alpha(\overline{D_i})$;

(d) if $i \leq n(D)$ with $\alpha(\overline{D_i^*}) = \alpha(\overline{D_i})$, then $\mathrm{Top}(\overline{D_i^*}) \subset \mathrm{Top}(\overline{R_i}) \subset \mathrm{Top}(\overline{D_i})$, and $\mathcal{W}_{i,m}(D^*) = \{W \bigcap \overline{D_i^*} : W \in \mathcal{W}_{i,m+1}(D)\}$, $m \in \omega$. Thus $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$;

(e) if $k(R, y(L)) < n(D)$, there is an $i < k(R, y(L))$ such that $\alpha(\overline{D_i^*}) < \alpha(\overline{D_i})$.

By constructions above, let $\mathcal{L}(R) = \bigcup_{k\in\omega} \mathcal{L}(R,k)$. And let

$$\mathcal{Z}(D,R) = \{O(R,L) : L \in \mathcal{L}(R)\}, \mathcal{D}(D,R) = \bigcup\{\mathcal{D}(R,L) : L \in \mathcal{L}(R)\}.$$

When R does not hold $(**)$ or $\mathrm{Top}(\overline{R}) \bigcap \mathrm{Top}(\prod_{i\leq n(D)} \overline{D_i}) = \emptyset$, let $\mathcal{Z}(D,R) = \{\emptyset\}$, $\mathcal{D}(D,R) = \{D^*\}$, where $D^* = R \times X \times \cdots$. We can also take some proper sequence $\{\mathcal{W}_{i,m}(D^*) : m \in \omega\}$ such that $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$, $m \in \omega$, as the ones described before.

Summing up discussions above, we let

$$\mathcal{Z}(D) = \bigcup\{\mathcal{Z}(D,R) : R \in \mathcal{R}(D)\}, \mathcal{D}(D) = \bigcup\{\mathcal{D}(D,R) : R \in \mathcal{R}(D)\}.$$

Thereby, the following statements are straightforward by (6) and (7).

(8) (a) $\mathcal{Z}(D)$ is a point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $\mathcal{Z}(D)$ is contained in some member of $\mathcal{G}$;

(b) collection $\mathcal{D}(D)$ is a point finite collection of basic open subsets of $Y \times X^\omega$;

(c) $D = \bigcup \mathcal{Z}(D) \bigcup (\bigcup \mathcal{D}(D))$;

for each $D^* = \widetilde{D^*} \times \prod_{i\in\omega} D_i^* \in \mathcal{D}(D,R)$, $R = \prod_{i\leq n(D)} R_i \in \mathcal{R}(D)$,

(d) $n(D^*) > n(D)$ and for each $i \in \omega$, $\alpha(\overline{D_i^*}) \leq \alpha(\overline{D_i})$;

(e) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \leq n(D)$, if $\alpha(\overline{D_i^*}) = \alpha(\overline{D_i})$, then $\mathrm{Top}(\overline{D_i^*}) \subset \mathrm{Top}(\overline{R_i})$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \bigcap \overline{D_i^*} : W \in \mathcal{W}_{i,m+1}(D)\}$;

(f) if R satisfies $(**)$ and $D^* = L \times \prod_{i\in\omega} D_i^*$ for some $L \in \mathcal{L}(R)$, with $k(R, y(L)) < n(D)$, then there is an $i < k(R, y(L))$ such that $\alpha(\overline{D_i^*}) < \alpha(\overline{D_i})$.

Proceeding by induction on $n \in \omega$, we define two families $\mathcal{Z}_n$ and $\mathcal{D}_n$ as follows. Let $\mathcal{Z}_0 = \{\emptyset\}$, $\mathcal{D}_0 = \{D(0)\}$, where $D(0) = Y \times X^\omega$. Put $\mathcal{W}_{i,m} = \{\{a\}\}$ for each $i, m \in \omega$. Now assume that we are given two families $\mathcal{Z}_n$ and $\mathcal{D}_n$ of basic open subsets of $Y \times X^\omega$ if $n = m$. And both of families $\mathcal{Z}_n$ and $\mathcal{D}_n$ satisfy the following:

(9) (a) $\mathcal{Z}_n = \bigcup\{\mathcal{Z}(D) : D \in \mathcal{D}_{n-1}\}$ is a point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $\mathcal{Z}_n$ is contained in some member of $\mathcal{G}$;

(b) $\mathcal{D}_n = \bigcup\{\mathcal{D}(D) : D \in \mathcal{D}_{n-1}\}$ is a point finite collection of basic open subsets of $Y \times X^\omega$;

for each $D = \widetilde{D} \times \prod_{i\in\omega} D_i \in \mathcal{D}_{n-1}$, $D^* = \widetilde{D^*} \times \prod_{i\in\omega} D_i^* \in \mathcal{D}(D,R)$, $R = \prod_{i\leq n(D)} R_i \in \mathcal{R}(D)$,

(c) $(D, \mathcal{W}_{i,m}(D)) \in \mathcal{C}$,

(d) $D = \bigcup \mathcal{Z}(D) \bigcup (\bigcup \mathcal{D}(D))$,

(e) $n(D^*) > n(D)$,

(f) for each $i \in \omega$, $\alpha(\overline{D_i^*}) \leq \alpha(\overline{D_i})$,

(g) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \leq n(D)$, if $\alpha(\overline{D_i^*}) = \alpha(\overline{D_i})$, then $\mathrm{Top}(\overline{D_i^*}) \subset \mathrm{Top}(\overline{R_i})$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \bigcap \overline{D_i^*} : W \in \mathcal{W}_{i,m+1}(D)\}$,

(h) $Y(D,R)=\{y\in\widetilde{D}:\widehat{R}_y$ satisfies $(**)\}$ for $R\in\mathcal{R}(D)$ and $\mathrm{Y}(n-1)=\bigcup\{Y(D,R):D\in\mathcal{D}_{n-1},R\in\mathcal{R}(D)\}$.

(i) if $y\in Y(D,R)$, $R\in\mathcal{R}(D)$ with $k(R,y)<n(D)$, then there is an $i<k(R,y)$ such that $\alpha(\overline{D_i^*})<\alpha(\overline{D_i})$.

By above all constructions, we can easily check that the families $\mathcal{Z}_{n+1}$ and $\mathcal{D}_{n+1}$ satisfy the consequents of (9) (a) $\sim$ (i). Let $\mathcal{Z}=\bigcup_{n\in\omega}\mathcal{Z}_n$. Our proof will be completed if the following claim is true.

Claim $\mathcal{Z}$ is a σ-point finite open refinement of $\mathcal{G}$.

By (9) (a), (b) and the induction, $\mathcal{Z}$ is a σ-point finite collection of open sets in $Y\times X^\omega$. It suffices to show that $\mathcal{Z}$ covers $Y\times X^\omega$. To show this, assume the contrary. Let $(y,(x_k))\in Y\times X^\omega-\bigcup\mathcal{Z}$. By (8) and (9) repeatedly, there are some collections $\{R(m):m\geq 1\}$, $\{D(m):m\geq 1\}$, where $D(0)=Y\times X^\omega$, $\{y(m):m\geq 1\}$ satisfying for each $m\geq 1$,

(10) (a) $(y,(x_k))\in D(m)=\widetilde{D(m)}\times\prod_{i\in\omega}D(m)_i\in\mathcal{D}(D(m-1),R(m))$, and $R(m)=\prod_{i\leq n(D(m-1))}R(m)_i\in\mathcal{R}(D(m-1))$, $y(m-1)\in Y(m-1)$,

(b) $n(D(m))>n(D(m-1))$ and $\alpha(\overline{D(m)_i})\leq\alpha(\overline{D(m-1)_i})$,

(c) for $i\leq n(D(m-1))$, if $\alpha(\overline{D(m)_i})=\alpha(\overline{D(m-1)_i})$, then $\mathrm{Top}(\overline{D(m)_i})\subset\mathrm{Top}(\overline{R(m-1)_i})$ and for each $j\in\omega$, $\mathcal{W}_{i,j}(D(m))=\{W\bigcap\overline{D(m)_i}:W\in\mathcal{W}_{i,j+1}(D(m-1))\}$,

(d) if $\widehat{R(m-1)}_{y(m-1)}$ satisfies $(**)$ with $k(R(m-1),y(m-1))<n(D(m-1))$, then there is an $i<k(R(m-1),y(m-1))$ such that $\alpha(\overline{D(m)_i})<\alpha(\overline{D(m-1)_i})$.

Fix an $i\in\omega$. By (10) (b), $n(D(m))>n(D(m-1))$ for each $m\geq 1$. Then there is an $s_i\in\omega$ such that $i<n(D(s_i))$. Let $s_i^*=\inf\{m\in\omega:i<n(D(m))\}$. And then, $n(D(m))>i$ for each $m\geq s_i^*$. In addition, by (10) (b), $\alpha(\overline{D(m)_i})\leq\alpha(\overline{D(m-1)_i})$ for each $m\geq 1$. So, there is a $t_i\in\omega$ such that $\alpha(\overline{D(t)_i})=\alpha(\overline{D(t_i)_i})$ for each $t\geq t_i$. Let $m_i^*=\max\{s_i^*,t_i\}+1$. Thus, $i<n(D(m))$ and $\alpha(\overline{D(m)_i})=\alpha(\overline{D(m_i^*)_i})$ for $m\geq m_i^*$. Moreover, by (10) (c), $\mathrm{Top}(\overline{D(m)_i})\subset\mathrm{Top}(\overline{R(m-1)_i})$ for $m\geq m_i^*$. Then there is a sequence $\{W(m-1):m\geq m_i^*\}$ of open subsets of X such that for each $m\geq m_i^*$, $W(m-1)\in\mathcal{W}_{i,m-m_i^*+1}(D(m_i^*-1))$ and $\mathrm{Top}(\overline{R(m-1)_i})\subset W(m-1)$.

Let $K_i=\bigcap_{m\geq m_i^*}\mathrm{Top}(\overline{D(m)_i})$. Clearly $K_i\subset\bigcap_{m\geq m_i^*}\mathrm{Top}(\overline{R(m-1)_i})$. It follows from Lemma 2.3 that K_i is nonempty and compact. And then, define $K=\{y\}\times\prod_{i\in\omega}K_i$. Obviously, K is compact. By Wallace theorem in Engelking [9], there exists some $G\in\mathcal{G}$ such that $K\subset G$. Let $p=\inf\{n(V):K\subset V\subset\overline{V}\subset G\}$, where $V=\widetilde{V}\times\prod_{i\in\omega}V_i$ is an open subset of $Y\times X^\omega$. Then, there exists an $m_0\in\omega$ such that $p<n(D(m_0))$. Again let $m_1=\max\{m_i^*:i<p\}$ and $m^*=\max\{m_0,m_1\}$. Therefore, we infer that $p<n(D(m^*))<n(D(m^*))$ and for each $i<p$, $m_i^*\leq m^*$ and $\mathrm{Top}(\overline{D(m^*)_i})\subset V_i$. So, $\mathrm{Top}(\overline{R(m^*)_i})\subset V_i$. Thus $\widehat{R(m^*)}_y\subset V$. Namely, $\widehat{R(m^*)}_y$ satisfies $(**)$. Again by (10) (d), since $k(R(m^*),y(m^*))=k(R(m^*),y)\leq p<n(D(m^*))$, there is an $i<k(R(m^*),y(m^*))$ such that $\alpha(\overline{D(m^*+1)_i})<\alpha(\overline{D(m^*)_i})$. This is a contradiction.

Thereby the Claim is true.

For each $n\in\omega$, let $Z_n=\bigcup\mathcal{Z}_n$. Then $\{Z_n:n\in\omega\}$ is a countable covering of $Y\times X^\omega$. By the countable metacompactness of $Y\times X^\omega$, there is a point finite open refinements $\{G_n:n\in\omega\}$ of $\{Z_n:n\in\omega\}$. Observe that the collection $\{G_n\bigcap Z:Z\in\mathcal{Z}_n,n\in\omega\}$ is a point finite refinements of $\mathcal{G}$. Hence, $Y\times X^\omega$ is metacompact.

4. Countable products of σ-metacompact spaces

By the definitions of Čech-scattered and σ-metacompact space, the following lemma can be easily checked.

Lemma 4.1 *If X is a Čech-scattered σ-metacompact space, then for every open cover $\mathcal{U}$ of X, there exists a σ-point finite open cover $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{V}_n$ of X such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped and is contained in some element of $\mathcal{U}$.*

Since point finite is σ-point finite, we are wandering σ-metacompactness in countable products. The following theorem is a modification of [15, Theorem 3.4], and for completeness, we briefly state its proof here.

Theorem 4.2 *If Y is a hereditarily σ-metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered σ-metacompact spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is σ-metacompact.*

Proof Let $\mathcal{G}$ be an arbitrary open covering of $Y \times X^\omega$ and closed under finite unions. We are going to find a σ-point finite open refinement of $\mathcal{G}$.

Let $\mathcal{B}$, $D = \widetilde{D} \times \prod_{i\in\omega} D_i$, $n(D)$, $\mathcal{C}$, $\mathcal{R}(D)$ and $\mathcal{W}_{i,m}(D)$, $m \in \omega$ be the same ones described in Theorem 3.1. By the same manners as Theorem 3.1, we can construct two collections $\mathcal{Z}_i(D)$ and $\mathcal{D}_i(D)$, $i \in \omega$, such that:

(1') (a) $\mathcal{Z}(D) = \bigcup_{i\in\omega} \mathcal{Z}_i(D)$ is a σ-point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $\mathcal{Z}(D)$ is contained in some member of $\mathcal{G}$,

(b) $\mathcal{D}(D) = \bigcup_{i\in\omega} \mathcal{D}_i(D)$ is a σ-point finite collection of basic open subsets of $Y \times X^\omega$,

(c) $D = \bigcup \mathcal{Z}(D) \bigcup (\bigcup \mathcal{D}(D))$,

for each $D^* = \widetilde{D^*} \times \prod_{i\in\omega} D_i^* \in \mathcal{D}(D, R)$, $R = \prod_{i\leq n(D)} R_i \in \mathcal{R}(D)$,

(d) $n(D^*) > n(D)$ and for each $i \in \omega$, $\alpha(\overline{D_i^*}) \leq \alpha(\overline{D_i})$,

(e) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \leq n(D)$, if $\alpha(\overline{D_i^*}) = \alpha(\overline{D_i})$, then $\mathrm{Top}(\overline{D_i^*}) \subset \mathrm{Top}(\overline{R_i})$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \bigcap \overline{D_i^*} : W \in \mathcal{W}_{i,m+1}(D)\}$,

(f) if R satisfies $(**)$ of Theorem 3.1 and $D^* = L \times \prod_{i\in\omega} D_i^*$ for some $L \in \mathcal{L}(R)$, with $k(R, y(L)) < n(D)$, then there is an $i < k(R, y(L))$ such that $\alpha(\overline{D_i^*}) < \alpha(\overline{D_i})$.

Now, proceeding by induction on $n \in \omega$, we define two families $\mathcal{Z}_n$ and $\mathcal{D}_n$ as follows. Let $\mathcal{Z}_0 = \{\emptyset\}$, $\mathcal{D}_0 = \{D(0)\}$, where $D(0) = Y \times X^\omega$. Put $\mathcal{W}_{i,m} = \{\{a\}\}$ for each $i, m \in \omega$. Now assume that when $n = m$, both of the families $\mathcal{Z}_n$ and $\mathcal{D}_n$ of basic open subsets of $Y \times X^\omega$ are given and satisfy the following:

(2') (a) $\mathcal{Z}_n = \bigcup\{\mathcal{Z}(D) : D \in \mathcal{D}_{n-1}\}$ is a σ-point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $\mathcal{Z}_n$ is contained in some member of $\mathcal{G}$,

(b) $\mathcal{D}_n = \bigcup\{\mathcal{D}(D) : D \in \mathcal{D}_{n-1}\}$ is a σ-point finite collection of basic open subsets of $Y \times X^\omega$,

for each $D = \widetilde{D} \times \prod_{i\in\omega} D_i \in \mathcal{D}_{n-1}$, $D^* = \widetilde{D^*} \times \prod_{i\in\omega} D_i^* \in \mathcal{D}(D, R)$, $R = \prod_{i\leq n(D)} R_i \in \mathcal{R}(D)$,

(c) $(D, \mathcal{W}_{i,m}(D)) \in \mathcal{C}$

(d) $D = \bigcup \mathcal{Z}(D) \bigcup (\bigcup \mathcal{D}(D))$,

(e) $n(D^*) > n(D)$,

(f) for each $i \in \omega$, $\alpha(\overline{D_i^*}) \le \alpha(\overline{D_i})$,

(g) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \le n(D)$, if $\alpha(\overline{D_i^*}) = \alpha(\overline{D_i})$, then $\text{Top}(\overline{D_i^*}) \subset \text{Top}(\overline{R_i})$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \bigcap \overline{D_i^*} : W \in \mathcal{W}_{i,m+1}(D)\}$,

(h) $Y(D,R) = \{y \in \widetilde{D} : \widehat{R}_y$ satisfies $(**)\}$ of Theorem 3.1 for $R \in \mathcal{R}(D)$ and $\text{Y}(n-1) = \bigcup\{Y(D,R) : D \in \mathcal{D}_{n-1}, R \in \mathcal{R}(D)\}$.

(i) if $y \in Y(D,R)$, $R \in \mathcal{R}(D)$ with $k(R,y) < n(D)$, then there is an $i < k(R,y)$ such that $\alpha(\overline{D_i^*}) < \alpha(\overline{D_i})$.

By above constructions, we infer that the families $\mathcal{Z}_{n+1}$ and $\mathcal{D}_{n+1}$ satisfy the consequents of (2') (a) $\sim$ (i). Let $\mathcal{Z} = \bigcup_{n\in\omega} \mathcal{Z}_n$. By the analogous way of proof of Claim in Theorem 3.1, we have that $\mathcal{Z}$ is a σ-point finite open refinement of $\mathcal{G}$. And hence the proof is completed. □

Similarly, the following theorem is direct.

Theorem 4.3 *If Y is a hereditary metaLindelöf space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered metaLindelöf spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is metaLindelöf.*

Consequently, combining Theorems 4.2 and 4.3, we have the following result.

Corollary 4.4 *If Y is a hereditarily σ-metacompact (metaLindelöf) space and $\{X_n : n \in \omega\}$ is a countable collection of C-scattered σ-metacompact (metaLindelöf) spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is σ-metacompact (metaLindelöf).*

Acknowledgements The authors are very grateful to the referees for their helpful comments and suggestions.

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