Metacompactness in Countable Products

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Abstract In this paper, we present that if $Y$ is a hereditarily metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered metacompact spaces, then the followings are equivalent:

1. $Y \times \prod_{n \in \omega} X_n$ is metacompact,
2. $Y \times \prod_{n \in \omega} X_n$ is countable metacompact,
3. $Y \times \prod_{n \in \omega} X_n$ is orthocompact.

Thereby, this result generalizes Theorem 5.4 in [Tanaka, Tsukuba. J. Math., 1993, 17: 565–587]. In addition, we obtain that if $Y$ is a hereditarily $\sigma$-metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered $\sigma$-metacompact spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is $\sigma$-metacompact.

Keywords metacompact; $\sigma$-metacompact; Čech-scattered

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1. Introduction

The notion of $C$-scattered space was introduced and investigated by Telgarsky [1]. Furthermore, utilizing it to products, he proved the following:

(A) ([1]) If $X$ is a $C$-scattered paracompact space, then the product $X \times Y$ is paracompact for each paracompact space $Y$.

As a generalization of $C$-scattered space, Čech-scattered space introduced by Hohti and Yun [2] plays an important role in study of paracompactness in countable products. Accordingly, the following result is obtained.

(B) ([2]) If $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact spaces, then the product $\prod_{n \in \omega} X_n$ is paracompact.

In 2005, Aoki and Tanaka [3] extended the above result by proving that:

(C) ([3]) If $Y$ is a perfect paracompact space, and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered paracompact spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is paracompact.

Recently, the authors [4] investigated the weak submetacompactness in countable products and obtained that:

(D) ([4]) If $Y$ is hereditarily weakly submetacompact, and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered weakly submetacompact spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is weakly submetacompact.
As well known, the following diagram is easily verified:

\[
\text{paracompact} \rightarrow \text{metacompact} \rightarrow \sigma\text{-metacompact} \rightarrow \text{weakly submetacompact}.
\]

But the reverse is not true [5, 6]. In addition, Zhu [7] showed that there is a first countable, regular separable, Lindelöf space \(X\) such that \(X^n\) is Lindelöf for each \(n \in \omega\), but \(X^n\) is not \(\sigma\)-metacompact. Above all, it is naturally to raise the following question.

**Question 1** Let \(Y\) be a hereditarily metacompact (\(\sigma\)-metacompact) space. Is the product \(Y \times \prod_{n \in \omega} X_n\) metacompact (\(\sigma\)-metacompact) if \(\{X_n : n \in \omega\}\) is a countable collection of Čech-scattered metacompact (\(\sigma\)-metacompact) spaces?

This paper mainly discusses countable products of metacompactness. Firstly, we obtain a group of equivalent conditions, which extend Tanaka’s result in [8], among metacompactness, countable metacompactness and orthocompactness in countable products. Finally, we give an affirmative answer to Question 1 for \(\sigma\)-metacompactness.

Throughout this paper, assume that each space is Tychonoff and \(\omega\) is the set of natural numbers.

2. Preliminaries

In the rest of this section, we stated some notation and basic facts. Undefined terminology can be found in Engelking [9]. A space \(X\) is scattered if every nonempty closed subset \(S\) has an isolated point \(s\). And a space \(X\) is said to be \(C\)-scattered (Čech-scattered) if for every nonempty closed subset \(S\) of \(X\), there exists a point \(s \in S\) which has a compact (Čech-complete) neighborhood in \(S\). Evidently, all of the scattered spaces, locally compact spaces and \(C\)-scattered spaces are Čech-scattered.

For a subset \(S\) of \(X\), \(|S|\) (resp., \(\overline{S}\)) denotes its cardinality (resp., closure). Assume that \(S\) is closed. Put

\[
S^* = \{x \in S : x \text{ has no Čech-complete neighborhood in } S\}.
\]

Let \(S^0 = S\), \(S^{(\alpha + 1)} = (S^{(\alpha)})^*\), and \(S^{(\alpha)} = \bigcap_{\beta < \alpha} S^{(\beta)}\) for a limit ordinal \(\alpha\). Note that each \(S^{(\alpha)}\) is closed in \(X\). Furthermore, a space \(X\) is Čech-scattered if and only if \(X^{(\alpha)} = \emptyset\) for some ordinal \(\alpha\). Obviously, a Čech-scattered space is hereditary for its closed (open) subspace. A closed subset \(S\) of \(X\) is called topped if \(S \cap X^{(\alpha(S))}\) is nonempty Čech-complete and \(S \cap X^{(\alpha(S) + 1)} = \emptyset\) for some ordinal \(\alpha(S)\). Denote \(S \cap X^{(\alpha(S))}\) by \(\text{Top}(S)\). For each \(x \in X\), there is a unique ordinal \(\alpha\) such that \(x \in X^{(\alpha)} \setminus X^{(\alpha + 1)}\). Let \(\text{rank}(x) = \alpha\). Then, there is an open neighborhood base \(\mathcal{V}\) of \(x\) in \(X\) such that for each \(V \in \mathcal{V}\), \(\overline{V}\) is topped in \(X\) and \(\alpha(\overline{V}) = \text{rank}(x)\). A collection \(\mathcal{V}\) of subsets of \(X\) is a refinement of \(\mathcal{U}\) if each member of \(\mathcal{V}\) is contained in some member of \(\mathcal{U}\) and \(\bigcup \mathcal{V} = \bigcup \mathcal{U}\).

To complete our proof, the following definitions and lemmas are useful. Therefore, we briefly state it here.

**Definition 2.1** A space \(X\) is said to be metacompact (\(\sigma\)-metacompact, metaLindelöf, orthocompact) if for every open covering \(\mathcal{U}\) of \(X\), there is a point finite (\(\sigma\)-point finite, point countable, interior preserving) open refinements \(\mathcal{V}\).
Recall that a space $X$ is said to be hereditarily metacompact (hereditarily $\sigma$-metacompact, hereditarily metaLindelöf) if every subspace $G$ of $X$ is metacompact ($\sigma$-metacompact, metaLindelöf). And by these definitions, it is easily proved that a space $X$ is hereditarily metacompact (hereditarily $\sigma$-metacompact, hereditarily metaLindelöf) if and only if every open subspace $G$ of $X$ is metacompact ($\sigma$-metacompact, metaLindelöf).

**Lemma 2.2** ([4]) The product $X \times Y$ is Čech-scattered if $X$ and $Y$ are Čech-scattered spaces.

**Lemma 2.3** ([9]) A Tychonoff space $X$ is Čech-complete if and only if there exists a countable family $\{A_i\}_{i \in \omega}$ of open covers of the space $X$ with the property that any family $F$ of closed subsets of $X$, which has the finite intersection property and contains sets of diameter less than $A_i$ for $i \in \omega$, has nonempty intersection.

Note that the intersection $\bigcap F$ is countable compact in Lemma 2.3.

**Lemma 2.4** ([10]) A space $X$ is $\lambda$-paracompact if and only if for every directed open cover $U$ of $X$ with cardinality $\leq \lambda$, there is a locally finite open cover $V$ of $X$ such that $\{V : V \in V\}$ refines $U$. A space $X$ is countably paracompact if and only if $\lambda = \omega$.

**Lemma 2.5** ([11]) If a product space $X = \prod_{\alpha \in \kappa} X_\alpha$ is orthocompact, then $X$ is $\kappa$-metacompact.

### 3. metacompactness in countable products

In [12], the authors proved that the product of a countable collection of Čech-scattered metacompact spaces is metacompact. By Burke [13], every perfect metacompact space is hereditarily metacompact. Now we discuss countable products of metacompactness again.

**Theorem 3.1** If $Y$ is a hereditarily metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered metacompact spaces, then the followings are equivalent.

(a) $Y \times \prod_{n \in \omega} X_n$ is metacompact,
(b) $Y \times \prod_{n \in \omega} X_n$ is countable metacompact,
(c) $Y \times \prod_{n \in \omega} X_n$ is orthocompact.

**Proof** (a) $\Rightarrow$ (c) holds obviously.

(c) $\Rightarrow$ (b). By Lemma 2.5, it is clear.

(b) $\Rightarrow$ (a). This proof is a modification of [8, Theorem 5.4]. Using the proof of [14, Theorem], we may assume that for each $n \in \omega$, $X_n = X$ and $\text{Top}(X) = \{a\}$ for some $a \in X$. To complete our proof, it suffices to show that $Y \times X^\omega$ is metacompact.

Let $G$ be an arbitrary open covering of $Y \times X^\omega$ and closed under finite unions. We are going to find a point finite open refinement of $G$.

Let $B$ be a base of $Y \times X^\omega$, consisting of all sets of the form $D = \tilde{D} \times \prod_{i \in \omega} D_i$ and for each $i \in \omega$, $\tilde{D}_i$ is topped, i.e., $\text{Top}(\tilde{D}_i)$ is Čech-complete. Then, there is a sequence $\{W_{i,m}(D) : m \in \omega\}$ of open covers of $\text{Top}(\tilde{D}_i)$, such that if $F$ is a collection of nonempty closed subset of $\text{Top}(\tilde{D}_i)$ with the finite intersection property such that for each $m \in \omega$, there are $F_m \in F$ and $W_{i,m}(D)$
with $F_m \subset W_m$, then the intersection $\bigcap F$ is nonempty. Let $n(D) = \inf\{i : D_j = X, \text{ for } j \geq i\}$. And define $C$ as follows:

(*) $(D, W_{i,m}(D)) \in C$, $m \in \omega$, if $D = \bigcap D_i \times \prod_{i \in \omega} D_i \in C$ and $W_{i,m}(D)$ is an open cover of $\text{Top}(D_i)$, satisfying the conditions described above.

For each $m \in \omega$, let $(D, W_{i,m}(D)) \in C$. In case of that $i < n(D)$, let $m = 1$. Then for each $W \in W_{i,1}(D)$, there is an open subset $W'$ of $D_i$ such that $W = W' \cap \text{Top}(D_i)$. Moreover, $\{W' : W \in W_{i,1}(D)\} \cup \{\text{Top}(D_i)\}$ covers $D_i$, hence, it follows from [12, Lemma 2] that there is an open covering $A_i(D)$ of $D_i$ such that:

(a) $A_i(D)$ is point finite,
(b) for each $A \in A_i(D)$, $\overline{A}$ is topped and contained in some member of $\{W' : W \in W_{i,m}(D)\} \cup \{\text{Top}(D_i)\}$.

In case of that $i = n(D)$, we can also take a point finite open covering $A_{n(D)}(D)$ of $D_i$ such that for each $A \in \mathcal{R}(D)$, $\overline{A}$ is topped. And there exists a proper member $A_0 \in A_{n(D)}(D)$ with $a \in A_0$ and for each $A' \in A_{n(D)}(D) - \{A_0\}$, $a \notin A'$.

By construction of each $A_i(D)$, let $\mathcal{R}(D) = \prod_{i \leq n(D)} A_i(D)$. Clearly, $\mathcal{R}(D)$ is a point finite open covering of $\prod_{i \leq n(D)} D_i$.

Let $R = \prod_{i \leq n(D)} R_i \in \mathcal{R}(D)$ with $\text{Top}(R) \cap \text{Top}(\prod_{i \leq n(D)} D_i) \neq \emptyset$. Then, $\text{Top}(R_i) \cap \text{Top}(D_i) \neq \emptyset$ for each $i \leq n(R)$. Observe that $\text{Top}(R_i) \cap \text{Top}(D_i) = R_i \cap \text{Top}(D_i) = \text{Top}(R_i)$. Hence, by (ii), $\text{Top}(R_i) \subset W$ for some $W \in W_{i,1}(D)$. For $R \in \mathcal{R}(D)$, define $P(R) = R \times X \times \cdots$. Then $\text{Top}(P(R)) = \text{Top}(R) \times \{1\} \times \cdots$. Correspondingly, define $R_D = \bigcap P(R) = \bigcap R_i \times X \times \cdots$. Thereby, for each $y \in D$, define $\mathcal{R}_y = \{y\} \times \text{Top}(P(R))$. Namely, $\mathcal{R}_y$ is Čech-complete.

Now define $\mathcal{R}_y$ satisfying (***) as follows:

(***) If there are some basic open subsets $E_1$ and $E_2$ in $Y \times X^\omega$ and some $G \in \mathcal{G}$ such that $\mathcal{R}_y \subset E_1 \subset \mathcal{E}_1 \subset E_2 \subset \mathcal{E}_2 \subset G$.

By this way, we say that $R$ holds (***) if there exists a $y \in D$ such that $\mathcal{R}_y$ satisfies (**). Fix a $y \in D$. Suppose that $\mathcal{R}_y$ satisfies the condition (**). Let $k(R, y) = \inf\{n(E_1) : E_1 \text{ and } E_2 \text{ are basic open subsets in } Y \times X^\omega \text{ with } e_1 = n(E_2) \text{ such that } \mathcal{R}_y \subset E_1 \subset \mathcal{E}_1 \subset E_2 \subset \mathcal{E}_2 \subset G \text{ for some } G \in \mathcal{G}\}$. Then, there are some basic open subsets $E_1(R, y) = E_1(R, y) \times \prod_{i \in \omega} E_1(R, y)_i$ and $E_2(R, y) = E_2(R, y) \times \prod_{i \in \omega} E_2(R, y)_i$ in $Y \times X^\omega$ and some $G(R, y) \in \mathcal{G}$ such that:

1. (a) $\mathcal{R}_y \subset E_1(R, y) \subset \mathcal{E}_1(R, y) \subset E_2(R, y) \subset \mathcal{E}_2(R, y) \subset G(R, y)$.
2. (b) $k(R, y) = n(E_1(R, y))$.

Let $r(R, y) = \max\{n(D) + 1, k(R, y)\}$. Define $H(R, y)$ as follows:

$$H(R, y) = H(R, y) \times \prod_{i < r(R, y)} P(R)_i \cap \bigcap_{i \in \omega} E_1(R, y)_i \times X \times \cdots = H(R, y) \times \prod_{i \in \omega} H(R, y)_i.$$
(e) \( \hat{H}(R, y) = \bigcap_{i=1}^{n} E_i(R, y) \).

Distinctly, \( H(R, y) \in B \) with \( \hat{R}_y \subset H(R, y) \), and contained in some member of \( G \) and for each \( i \in \omega \), \( H(R, y)_i \) is topped.

For each \( k \in \omega \), let \( H(R, k) = \{ \hat{H}(R, y) \cap \hat{D} : k(R, y) \leq k \} \). For \( k \in \omega \), let \( L(R, k) = \{ y \in \hat{D} : k(R, y) \leq k \} \). Clearly, \( L(R, k) \subset L(R, k + 1) \) and \( L(R, k) = \bigcup H(R, k) \). By the hereditary metacompactness of \( Y \), there exists a collection \( \mathcal{L}(R) = \bigcup_{k \in \omega} L(R, k) \) of open subsets in \( Y \) such that:

(3) (a) \( L(R, k) = \bigcup \mathcal{L}(R, k) \);
(b) \( \mathcal{L}(R, k) \) refines \( H(R, k) \);
(c) each \( \mathcal{L}(R, k) \) is point-finite in \( Y \).

For each \( L \in \mathcal{L}(R, k) \), there exists a \( y(L) \in L(R, k) \) such that \( L \subset H(R, y(L)) \bigcap \hat{D} \). So \( k(R, y(L)) \leq k \). Now, define \( O(R, L) \) as follows: \( O(R, L) = L \times \prod_{i \in \omega} H(R, y(L)_i) \). By the definition, \( O(R, L) \in B \) and \( L \times \text{Top}(P(R)) \subset O(L, R) \).

Put \( \mathcal{N}(R, L) = P(\{0, 1, \ldots, r(R, y(L)) - 1\}) \). Fix an \( A \in \mathcal{N}(R, L) \). Define \( D_A(R, L) = L \times \prod_{i \in \omega} D_A(R, L)_i \) as follows:

(4) (a) if \( i \in A \) with \( i < n(D) \), let \( D_A(R, L)_i = P(R)_i - \overline{P(R)_i \bigcap E_i(R)} \);
(b) if \( i \in A \) with \( r(R, y(L)) = k(R, y(L)) > i \geq n(D) \), let \( D_A(R, L)_i = X - \{ a \} \);
(c) if \( i < r(R, y(L)) \) with \( i \notin A \), let \( D_A(R, L)_i = P(R)_i \bigcap E_i(R, y(L)_i) \);
(d) for each \( i \) with \( i \geq r(R, y(L)) \), let \( D_A(R, L)_i = X \).

Clearly, if \( i \) satisfies (4) (c) or (d), then \( \overline{D_A(R, L)}_i \) is topped. And if \( i \in A \) with \( k(R, y(L)) \leq i < r(R, y(L)) \), then \( D_A(R, L)_i = \emptyset \). Now, we consider the other cases:

(i) if \( i \in A \) with \( i < \min\{n(D), k(R, y(L))\} \);
(ii) if \( i \in A \) with \( r(R, y(L)) = k(R, y(L)) > i \geq n(D) \);
(iii) if \( i = r(R, y(L)) \).

If \( i \) satisfies the conditions (i) or (ii), then \( \overline{D_A(R, L)}_i \) does not need to be topped and hence, there is an open covering \( \mathcal{B}(D_A(R, L)_i) \) of \( \overline{D_A(R, L)}_i \) such that for each \( B \in \mathcal{B}(\overline{D_A(R, L)}_i) \), \( \overline{B} \) is topped. Then there is a point finite, open refinement \( \mathcal{D}_{A,i}(R, L) \) of \( \mathcal{D}_{A,R}(R, L)_i \), covering \( D_A(R, L) \), and for each \( D_{A,i}^* \subset \mathcal{D}_{A,i}(R, L) \), \( \overline{D_{A,i}^*} \) is topped. If \( i \) satisfies (iii), there is a proper point finite, open covering \( \mathcal{D}_{A,i}(r(R, y(L))) \) of \( X \) and for each \( D_{A,i}^* \subset \mathcal{D}_{A,i}(r(R, y(L))) \), \( \overline{D_{A,i}^*} \) is topped. Next, define the collection \( \mathcal{D}_{A}(R, L) \) as follows:

(5) \( D^*(L) = L \times \prod_{i \in \omega} D_{A,i}^* \subset \mathcal{D}_{A}(R, L) \) if for each \( i \in \omega \),
(a) if \( i \in A \) with \( k(R, y(L)) \leq i < n(D) \), let \( D_{A,i}^* = \emptyset \);
(b) if \( i \) satisfies one of the conditions (i), (ii) and (iii), let \( D_{A,i}^* \subset \mathcal{D}_{A,i}(R, L) \);
(c) if \( i \notin A \) with \( i < r(R, y(L)) \), let \( D_{A,i}^* = \mathcal{D}_{A,i}(R, L) \);
(d) let \( D_{A,i}^* = X \) for each \( i > r(R, y(L)) \).

With that, let \( \mathcal{D}_A(R, L) = \{ D^* \subset \mathcal{D}_A(R, L) : D^* \neq \emptyset \} \). Thus, we infer that
(6) the collection \( \mathcal{D}_A(R, L) \) is point finite in \( Y \times X^\omega \).

Further on, let \( \mathcal{D}(R, L) = \bigcup \{ \mathcal{D}_A(R, L) : A \in \mathcal{N}(R, L) \} \). Therefore, by (6) and the definitions of \( P(R), O(R, L) \), collection \( \mathcal{D}(R, L) \) satisfies the following:

(7) (a) collection \( \mathcal{D}(R, L) \) is point finite in \( Y \times X^\omega \) and \( L \times P(R) = O(R, L) \cup (\bigcup \mathcal{D}(R, L)) \);
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for each $D^* \in \mathcal{D}(R, L)$,
(b) $n(D^*) = r(R, y(L))$ and $n(D^*) > n(D)$;
(c) for each $i \in \omega$, $\alpha(\overline{D^*_i}) \leq \alpha(\overline{D}_i)$;
(d) if $i \leq n(D)$ with $\alpha(\overline{D^*_i}) = \alpha(\overline{D}_i)$, then $\text{Top}(\overline{D^*_i}) \subset \text{Top}(\overline{D}_i) \subset \text{Top}(\overline{D})$, and $\mathcal{W}_{i,m}(D^*) = \{W \cap \overline{D^*_i} : W \in \mathcal{W}_{i,m+1}(D), m \in \omega\}$. Thus $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$;
(e) if $k(R, y(L)) < n(D)$, there is an $i < k(R, y(L))$ such that $\alpha(\overline{D^*_i}) < \alpha(\overline{D}_i)$.

By constructions above, let $\mathcal{L}(R) = \bigcup_{k \in \omega} \mathcal{L}(R, k)$. And let

$$Z(D, R) = \{O(R, L) : L \in \mathcal{L}(R)\}, \mathcal{D}(D, R) = \bigcup\{\mathcal{D}(R, L) : L \in \mathcal{L}(R)\}.$$ 

When $R$ does not hold $(\ast)$ or $\text{Top}(\overline{R}) \cap \text{Top}(\prod_{i \leq n(D)} \overline{D}_i) = \emptyset$, let $Z(D, R) = \{\emptyset\}, \mathcal{D}(D, R) = \{D^*\}$, where $D^* = R \times X \times \cdots$. We can also take some proper sequence $\{\mathcal{W}_{i,m}(D^*) : m \in \omega\}$ such that $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$, $m \in \omega$, as the ones described before.

Summing up discussions above, we let

$$Z(D) = \bigcup\{Z(D, R) : R \in \mathcal{R}(D)\}, \mathcal{D}(D) = \bigcup\{\mathcal{D}(D, R) : R \in \mathcal{R}(D)\}.$$ 

Thereby, the following statements are straightforward by (6) and (7).

(8) (a) $Z(D)$ is a point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $Z(D)$ is contained in some member of $\mathcal{G}$;
(b) collection $\mathcal{D}(D)$ is a point finite collection of basic open subsets of $Y \times X^\omega$;
(c) $D = \bigcup Z(D) \bigcup (\bigcup \mathcal{D}(D))$;
for each $D^* = \overline{D^*_i} \times \prod_{i \in \omega} D^*_i \in \mathcal{D}(D, R)$, $R = \prod_{i \leq n(D)} R_i \in \mathcal{R}(D)$,
(d) $n(D^*) > n(D)$ and for each $i \in \omega$, $\alpha(\overline{D^*_i}) \leq \alpha(\overline{D}_i)$;
(e) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \leq n(D)$, if $\alpha(\overline{D^*_i}) = \alpha(\overline{D}_i)$, then $\text{Top}(\overline{D^*_i}) \subset \text{Top}(\overline{D}_i)$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \cap \overline{D^*_i} : W \in \mathcal{W}_{i,m+1}(D)\}$;
(f) if $R$ satisfies $(\ast)$ and $D^* = L \times \prod_{i \in \omega} D^*_i$ for some $L \in \mathcal{L}(R)$, with $k(R, y(L)) < n(D)$, then there is an $i < k(R, y(L))$ such that $\alpha(\overline{D^*_i}) < \alpha(\overline{D}_i)$.

Proceeding by induction on $n \in \omega$, we define two families $Z_n$ and $D_n$ as follows. Let $Z_0 = \{\emptyset\}, D_0 = \{D(0)\}$, where $D(0) = Y \times X^\omega$. Put $\mathcal{W}_{i,m} = \{\{a\}\}$ for each $i, m \in \omega$. Now assume that we are given two families $Z_n$ and $D_n$ of basic open subsets of $Y \times X^\omega$ if $n = m$.

And both of families $Z_n$ and $D_n$ satisfy the following:

(9) (a) $Z_n = \bigcup\{Z(D) : D \in \mathcal{D}_{n-1}\}$ is a point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $Z_n$ is contained in some member of $\mathcal{G}$;
(b) $D_n = \bigcup\{D(D) : D \in \mathcal{D}_{n-1}\}$ is a point finite collection of basic open subsets of $Y \times X^\omega$;
for each $D = \overline{D_i} \times \prod_{i \in \omega} D^*_i \in \mathcal{D}_{n-1}$, $D^* = \overline{D^*_i} \times \prod_{i \in \omega} D^*_i \in \mathcal{D}(D, R)$, $R = \prod_{i \leq n(D)} R_i \in \mathcal{R}(D)$,
(c) $(D, \mathcal{W}_{i,m}(D)) \in \mathcal{C}$,
(d) $D = \bigcup Z(D) \bigcup (\bigcup D(D))$,
(e) $n(D^*) > n(D)$,
(f) for each $i \in \omega$, $\alpha(\overline{D^*_i}) \leq \alpha(\overline{D}_i)$,
(g) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \leq n(D)$, if $\alpha(\overline{D^*_i}) = \alpha(\overline{D}_i)$, then $\text{Top}(\overline{D^*_i}) \subset \text{Top}(\overline{D}_i)$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \cap \overline{D^*_i} : W \in \mathcal{W}_{i,m+1}(D)\}$,
(h) $Y(D, R) = \{ y \in \tilde{D} : \tilde{R}_y \text{ satisfies } (***) \}$ for $R \in \mathcal{R}(D)$ and $Y(n - 1) = \bigcup \{ Y(D, R) : D \in \mathcal{D}_{n-1}, R \in \mathcal{R}(D) \}$.

(i) if $y \in Y(D, R)$, $R \in \mathcal{R}(D)$ with $k(R, y) < n(D)$, then there is an $i < k(R, y)$ such that $\alpha(\overline{D_i}) < \alpha(\overline{D_y})$.

By above all constructions, we can easily check that the families $Z_{n+1}$ and $\mathcal{D}_{n+1}$ satisfy the consequents of (9) (a) ~ (i). Let $Z = \bigcup_{n \in \omega} Z_n$. Our proof will be completed if the following claim is true.

**Claim** $Z$ is a $\sigma$-point finite open refinement of $G$.

By (9) (a), (b) and the induction, $Z$ is a $\sigma$-point finite collection of open sets in $Y \times X^\omega$. It suffices to show that $Z$ covers $Y \times X^\omega$. To show this, assume the contrary. Let $(y, (x_k)) \in Y \times X^\omega - \bigcup Z$. By (8) and (9) repeatedly, there are some collections $\{ R(m) : m \geq 1 \}, \{ D(m) : m \geq 1 \}$, where $D(0) = Y \times X^\omega, \{ y(m) : m \geq 1 \}$ satisfying for each $m \geq 1$,

\[(9) \quad \begin{align*}
& (\alpha) \quad (y, (x_k)) \in D(m) = D(m) \times \prod_{i \in \omega} D(m)_j \in D(D(m - 1), R(m)), \\
& (b) \quad n(D(m)) > n(D(m - 1)) \text{ and } \alpha(\overline{D(m)_j}) \leq \alpha(\overline{D(m - 1)_i}), \\
& (c) \quad \text{for } j \in \omega, \text{ if } \alpha(\overline{D(m)_j}) = \alpha(\overline{D(m - 1)_i}), \text{ then } \text{Top}(\overline{D(m)_j}) \subset \text{Top}(\overline{D(m - 1)_i}) \\
& \text{and for each } j \in \omega, W_{i,j}(D(m)) = \{ W \cap D(m)_j : W \in W_{i,j+1}(D(m - 1)) \}, \\
& (d) \quad \text{if } \text{Top}(\overline{D(m - 1)_j}) \text{ satisfies } (***) \text{ with } k(\overline{D(m - 1)_j}, y(m - 1)) < n(D(m - 1)), \text{ then there is an } i < k(\overline{D(m - 1)_j}, y(m - 1)) \text{ such that } \alpha(\overline{D(m)_j}) < \alpha(\overline{D(m - 1)_j}).
\end{align*}
\]

Fix an $i \in \omega$. By (10) (b), $n(D(m)) > n(D(m - 1))$ for each $m \geq 1$. Then there is an $s_i \in \omega$ such that $i < n(D(s_i))$. Let $s_i^* = \inf \{ m \in \omega : i < n(D(m)) \}$. And then, $n(D(m)) > i$ for each $m \geq s_i^*$. In addition, by (10) (b), $\alpha(\overline{D(m)_j}) \leq \alpha(\overline{D(m - 1)_i})$ for each $m \geq 1$. So, there is a $t_i \in \omega$ such that $\alpha(\overline{D(t_i)_j}) = \alpha(\overline{D(t_i)_j})$ for each $t \geq t_i$. Let $m_i^* = \max \{ s_i^*, t_i \} + 1$. Thus, $i < n(D(m))$ and $\alpha(\overline{D(m)_j}) = \alpha(\overline{D(m_i^*)})$ for $m \geq m_i^*$. Moreover, by (10) (c), $\text{Top}(\overline{D(m)_j}) \subset \text{Top}(\overline{D(m - 1)_i})$ for $m \geq m_i^*$. Then there is a sequence $\{ W(m - 1) : m \geq m_i^* \}$ of open subsets of $X$ such that for each $m \geq m_i^*, W(m - 1) \in W_i, m_i^* + 1(D(m_i^* - 1))$ and $\text{Top}(\overline{D(m_i^*)_j}) \subset W(m - 1)$.

Let $K_1 = \cap_{m \geq m_i^*} \text{Top}(\overline{D(m)_j})$. Clearly $K_1 \subset \cap_{m \geq m_i^*} \text{Top}(\overline{D(m - 1)_i})$. It follows fromLemma 2.3 that $K_1$ is nonempty and compact. And then, define $K = \{ y \} \times \prod_{i \in \omega} K_i$. Obviously, $K$ is compact. By Wallace theorem in Engelking [9], there exists some $G \in \mathcal{G}$ such that $K \subset G$.

Let $p = \inf \{ n(V) : K \subset V \subset \bigcup \mathcal{V} \subset G \}$, where $V = \bigcup \prod_{i \in \omega} V_i$ is an open subset of $Y \times X^\omega$. Then, there exists an $m_0 \in \omega$ such that $p < n(D(m_0))$. Again let $m_1 = \max \{ m_i^* : i < p \}$ and $m^* = \max \{ m_0, m_1 \}$. Therefore, we infer that $p < n(D(m^*)) < n(D(m^*))$ and for each $i < p$, $m_i^* \leq m^*$ and $\text{Top}(\overline{D(m^*)_j}) \subset V_i$. Thus, $\text{Top}(\overline{D(m^*)_j}) \subset V_i$. Namely, $\text{Top}(\overline{D(m^*)_j})$ satisfies (**). Again by (10) (d), since $k(\overline{D(m^*)_j}, y(m^*)) \leq p < n(D(m^*))$, there is an $i < k(\overline{D(m^*)_j}, y(m^*))$ such that $\alpha(\overline{D(m^*_j + 1)}) < \alpha(\overline{D(m^*_j)})$. This is a contradiction.

Thereby the Claim is true.

For each $n \in \omega$, let $Z_n = \bigcup Z_n$. Then $\{ Z_n : n \in \omega \}$ is a countable covering of $Y \times X^\omega$. By the countable metacompactness of $Y \times X^\omega$, there is a point finite open refinements $\{ G_n : n \in \omega \}$ of $\{ Z_n : n \in \omega \}$. Observe that the collection $\{ G_n \bigcap Z : Z \in Z_n, n \in \omega \}$ is a point finite refinements of $\mathcal{G}$. Hence, $Y \times X^\omega$ is metacompact.
4. Countable products of $\sigma$-metacompact spaces

By the definitions of Čech-scattered and $\sigma$-metacompact space, the following lemma can be easily checked.

**Lemma 4.1** If $X$ is a Čech-scattered $\sigma$-metacompact space, then for every open cover $\mathcal{U}$ of $X$, there exists a $\sigma$-point finite open cover $\mathcal{V} = \bigcup_{n \in \omega} V_n$ of $X$ such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped and is contained in some element of $\mathcal{U}$.

Since point finite is $\sigma$-point finite, we are wandering $\sigma$-metacompactness in countable products. The following theorem is a modification of [15, Theorem 3.4], and for completeness, we briefly state its proof here.

**Theorem 4.2** If $Y$ is a hereditarily $\sigma$-metacompact space and $\{X_n : n \in \omega\}$ is a countable collection of Čech-scattered $\sigma$-metacompact spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is $\sigma$-metacompact.

**Proof** Let $\mathcal{G}$ be an arbitrary open covering of $Y \times X^\omega$ and closed under finite unions. We are going to find a $\sigma$-point finite open refinement of $\mathcal{G}$.

Let $B, D = D \times \prod_{n \in \omega} D_n$, $\mathcal{C}, \mathcal{R}(D)$ and $\mathcal{W}_{i,m}(D)$, $m \in \omega$ be the same ones described in Theorem 3.1. By the same manners as Theorem 3.1, we can construct two collections $Z_i(D)$ and $D_i(D)$, $i \in \omega$, such that:

1' (a) $Z(D) = \bigcup_{i \in \omega} Z_i(D)$ is a $\sigma$-point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $Z(D)$ is contained in some member of $\mathcal{G}$,

(b) $D(D) = \bigcup_{i \in \omega} D_i(D)$ is a $\sigma$-point finite collection of basic open subsets of $Y \times X^\omega$,

(c) $D = \bigcup Z(D) \cup (\bigcup D(D))$,

for each $D^* = D^* \times \prod_{i \in \omega} D_i^* \in D(D, R)$, $D = \prod_{i \in \omega} D_i^* \in \mathcal{R}(D)$,

(d) $n(D^*) > n(D)$ and for each $i \in \omega$, $\alpha(D_i^*) \leq \alpha(D_i^*)$,

(e) $(D^*, \mathcal{W}_{i,m}(D^*)) \in \mathcal{C}$ such that for each $i \leq n(D)$, if $\alpha(D_i^*) = \alpha(D_i^*)$, then $\text{Top}(D_i^*) \subset \text{Top}(D_i^*)$ and for each $m \in \omega$, $\mathcal{W}_{i,m}(D^*) = \{W \cap D_i^* : W \in \mathcal{W}_{i,m+1}(D)\}$,

(f) if $R$ satisfies (**) of Theorem 3.1 and $D^* = L \times \prod_{i \in \omega} D_i^*$ for some $L \in \mathcal{L}(R)$, with $k(R, y(L)) < n(D)$, then there is an $i < k(R, y(L))$ such that $\alpha(D_i^*) < \alpha(D_i^*)$.

Now, proceeding by induction on $n \in \omega$, we define two families $Z_n$ and $D_n$ as follows. Let $Z_0 = \{\}$, $D_0 = \{D(0)\}$, where $D(0) = Y \times X^\omega$. Put $\mathcal{W}_{i,m} = \{\{a\}\}$ for each $i, m \in \omega$. Now assume that when $n = m$, both of the families $Z_n$ and $D_n$ of basic open subsets of $Y \times X^\omega$ are given and satisfy the following:

2' (a) $Z_n = \bigcup Z(D) : D \in D_{n-1}$ is a $\sigma$-point finite collection of basic open subsets of $Y \times X^\omega$ such that every member of $Z_n$ is contained in some member of $\mathcal{G}$,

(b) $D_n = \bigcup D(D) : D \in D_{n-1}$ is a $\sigma$-point finite collection of basic open subsets of $Y \times X^\omega$,

for each $D = D \times \prod_{i \in \omega} D_i \in D_{n-1}$, $D^* = D^* \times \prod_{i \in \omega} D_i^* \in D(D, R)$, $R = \prod_{i \leq n(D)} R_i$.

(e) $(D, \mathcal{W}_{i,m}(D)) \in \mathcal{C}$

(d) $D = \bigcup Z(D) \cup (\bigcup D(D))$. 


(e) \( n(D^*) > n(D) \),
(f) for each \( i \in \omega \), \( \alpha(\mathcal{D}^*_i) \leq \alpha(\mathcal{D}_i) \),
(g) \( (D^*, W_{i,m}(D^*)) \in \mathcal{C} \) such that for each \( i \leq n(D) \), if \( \alpha(\mathcal{D}^*_i) = \alpha(\mathcal{D}_i) \), then \( \text{Top}(\mathcal{D}^*_i) \subset \text{Top}(\mathcal{D}_i) \) and for each \( m \in \omega \), \( W_{i,m}(D^*) = \{ W \cap \mathcal{D}^*_i : W \in W_{i,m+1}(D) \} \),
(h) \( Y(D, R) = \{ y \in D : \tilde{R}_y \text{ satisfies } (**) \} \) of Theorem 3.1 for \( R \in \mathcal{R}(D) \) and \( Y(n - 1) = \bigcup \{ Y(D, R) : D \in \mathcal{D}_{n-1}, R \in \mathcal{R}(D) \} \).

(i) if \( y \in Y(D, R), R \in \mathcal{R}(D) \) with \( k(R, y) < n(D) \), then there is an \( i < k(R, y) \) such that \( \alpha(\mathcal{D}^*_i) < \alpha(\mathcal{D}_i) \).

By above constructions, we infer that the families \( Z_{n+1} \) and \( D_{n+1} \) satisfy the consequents of (2') (a) \( \sim \) (i). Let \( Z = \bigcup_{n \in \omega} Z_n \). By the analogous way of proof of Claim in Theorem 3.1, we have that \( Z \) is a \( \sigma \)-point finite open refinement of \( G \). And hence the proof is completed. \( \square \)

Similarly, the following theorem is direct.

**Theorem 4.3** If \( Y \) is a hereditary metaLindelöf space and \( \{ X_n : n \in \omega \} \) is a countable collection of Čech-scattered metaLindelöf spaces, then the product \( Y \times \prod_{n \in \omega} X_n \) is metaLindelöf.

Consequently, combining Theorems 4.2 and 4.3, we have the following result.

**Corollary 4.4** If \( Y \) is a hereditarily \( \sigma \)-metacompact (metaLindelöf) space and \( \{ X_n : n \in \omega \} \) is a countable collection of C-scattered \( \sigma \)-metacompact (metaLindelöf) spaces, then the product \( Y \times \prod_{n \in \omega} X_n \) is \( \sigma \)-metacompact (metaLindelöf).

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**References**