A New Class of Harmonic Multivalent Functions Defined by Subordination

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Abstract In the present paper, we introduce some new subclasses of harmonic multivalent functions defined by generalized Dziok-Srivastava operator. Sufficient coefficient conditions, distortion bounds and extreme points for functions of these classes are obtained.

Keywords harmonic multivalent functions; Dziok-Srivastava operator; subordination; extreme points; distortion bounds

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1. Introduction and preliminaries

A continuous function \( f = u + iv \) is a complex valued harmonic function in a complex domain \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). In any simply connected domain \( D \subset \mathbb{C} \), we can write \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic in \( D \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \). A necessary and sufficient condition for \( f \) to be locally univalent and sense preserving in \( D \) is that \( |h'(z)| > |g'(z)| \) in \( D \) (see [1]).

Let \( H_m \) \((m \geq 1)\) denote the family of functions \( f = h + \overline{g} \) that are multivalent harmonic and orientation preserving functions in \( D \) with the normalization \( h(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=m}^{\infty} b_k z^k \) \((|b_m| < 1)\). Ahuja and Jahangiri [2,3] introduced and studied certain subclasses of the family \( H_m \).

Denote by \( H_p \) the class of \( p \)-valent harmonic functions \( f \) that are sense preserving in \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( f \) of the form

\[
    f = h + \overline{g},
\]

where

\[
    h(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=p+1}^{\infty} b_k z^k. \tag{1.1}
\]

Obvious \( H_p \subset H_m \).

Also, we denote by \( \mathcal{P}(p) \) the class of \( p \)-valent harmonic functions \( f \in H_p \) and

\[
    h(z) = z^p - \sum_{k=p+1}^{\infty} |a_k| z^k \quad \text{and} \quad g(z) = -\sum_{k=p+1}^{\infty} |b_k| z^k. \tag{1.2}
\]

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Let $F$ be fixed multivalent harmonic function given by
\[ F = H(z) + G(z) = z^p + \sum_{k=p+1}^{\infty} A_k z^k + \sum_{k=p+1}^{\infty} B_k z^k. \] (1.4)

We define the Hadamard product (or convolution) of $F$ and $f$ by
\[ (F \ast f)(z) := z^p + \sum_{k=p+1}^{\infty} a_k A_k z^k + \sum_{k=p+1}^{\infty} b_k B_k z^k = (f \ast F)(z). \] (1.5)

For positive real values of $\alpha_i$ ($i = 1, \ldots, l$) and $\beta_j$ ($j = 1, \ldots, m$), the generalized hypergeometric function $_pF_m$ (with $l$ numerator and $m$ denominator parameters) is defined by
\[ _pF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{(\beta_1)_k \cdots (\beta_m)_k} \frac{z^k}{k!}, \]
where $l \leq m + 1$; $l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$, and $(\lambda)_n$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by
\[ (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & n \in \mathbb{N}. \end{cases} \]

Corresponding to the function
\[ h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = z^{-p}F_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)(z), \]
the linear operator $H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : H_p \rightarrow H_p$ is defined by using the following Hadamard product (or convolution):
\[ H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) \ast f(z). \]

For a function $f$ of the form (1.1), we have
\[ H_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} a_k z^k + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_l)_k}{k!(\beta_1)_k \cdots (\beta_m)_k} b_k z^k \]
\[ := H_{p,l,m}[\alpha_1]f(z). \] (1.6)

The above-defined operator $H_{p,l,m}[\alpha_1]$ ($p = 1$) was introduced by the Dziok-Srivastava operator [4,5]. Using the same methods of [6], we introduce the generalized Dziok-Srivastava operator in $H_p$ as follows:
\[ L^{1,\alpha_1}_{\lambda,l,m}f(z) = (1 - \lambda)H_{p,l,m}[\alpha_1]f(z) + \frac{\lambda}{p}z(H_{p,l,m}[\alpha_1]f(z))' \]
\[ := L^{\alpha_1}_{\lambda,l,m}f(z), \quad \lambda \geq 0, \]
where
\[ z(H_{p,l,m}[\alpha_1]f(z))' = z(H_{p,l,m}[\alpha_1]h(z))' - \overline{z(H_{p,l,m}[\alpha_1]g(z))'}. \]

In general,
\[ L^{\tau,\alpha_1}_{\lambda,l,m}f(z) = L^{\alpha_1}_{\lambda,l,m}(L^{-1,\alpha_1}_{\lambda,l,m}f(z)), \quad l \leq m + 1; \ l, m \in \mathbb{N}_0, \tau \in \mathbb{N}, \]
(1.7)
where
\[
L_{\lambda, l, m}^{\tau,\alpha_1} f(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{1 + k\lambda}{p} \right) \left( \frac{1 + k\lambda}{1 + \frac{k\lambda}{p}} \right) a_k z^k,
\]
and \( \lambda \geq 0, \tau \in \mathbb{N}. \)

For \( \mu > 0 \) and \( \tau \in \mathbb{N}, \) we introduce the following linear operator \( J_\tau^\mu : H_\mu \rightarrow H_\mu, \) defined by
\[
J_\tau^\mu f(z) = J_\tau^\mu (z) * f(z) = J_\tau^\mu (z) * h(z) + \overline{J_\tau^\mu (z) * g(z)}, \quad z \in U,
\]
where \( J_\tau^\mu (z) \) is the function defined as follows:
\[
J_\tau^\mu(z) = \frac{z^\mu}{(1 - z)^\mu}, \quad \mu > 0, z \in U,
\]
and
\[
L_{\lambda, l, m}^{\tau,\alpha_1} J_\tau^\mu(z) = z^p + \sum_{k=p+1}^{\infty} \left( \frac{1 + k\lambda}{1 + \frac{k\lambda}{p}} \right) a_k z^k, \quad \mu > 0, z \in U,
\]
combining (1.9)–(1.12), we obtain
\[
J_\tau^\mu(z) = \sum_{k=p+1}^{\infty} \left( \frac{k!}{(1 + \frac{k\lambda}{p})} \right) a_k z^k, \quad \mu > 0, z \in U.
\]
If \( f \) is given by (1.1), then we find from (1.9) and (1.13) that
\[
J_\tau^\mu f(z) = J_\tau^\mu h(z) + \overline{J_\tau^\mu g(z)} = z^p + \sum_{k=p+1}^{\infty} \Phi_k a_k z^k + \sum_{k=p+1}^{\infty} \Phi_k b_k z^k
\]
\[
\Phi_k = \left( \frac{k!(1 + \frac{k\lambda}{p})(1 + \frac{k\lambda}{1 + \frac{k\lambda}{p}})}{k!} \right) a_k z^k, \quad \mu > 0.
\]

Let \( f_1 \) and \( f_2 \) be two analytic functions in the open unit disk \( U. \) We say that the function \( f_1 \) is subordinate to \( f_2 \) in \( U, \) and write \( f_1(z) \prec f_2(z) \) \((z \in U), \) if there exists a Schwarz function \( \omega, \) which is analytic in \( U \) with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) \((z \in U), \) such that \( f_1(z) = f_2(\omega(z)) \) \((z \in U) \) (see [7]).

By making use of the principle of subordination between analytic functions, we introduce the class \( H_\mu(A, B; \mu, \tau, \alpha, \delta). \)

**Definition 1.1** A function \( f(z) \in H_\mu \) of the form (1.1) is said to be in the class \( H_\mu(A, B; \mu, \tau, \alpha, \delta) \) if and only if
\[
\chi_{\delta, \mu}(f(z)) - \alpha |(\chi_{\delta, \mu}(f(z)) - 1| < \frac{1 + A z}{1 + B z},
\]
(1.15)
where
\[
\chi_{\delta, \mu}(f(z)) = (1 - \delta) \frac{J_p^\mu f(z)}{z^p} + \frac{\delta}{\mu 2p - 1} (J_p^\mu f(z))' \tag{1.17}
\]
and \(J_p^\mu f(z)\) is defined by (1.14) and \(p \in \mathbb{N}; A, B \in \mathbb{R}, A \neq B, |B| \leq 1; \tau \in \mathbb{N}, \mu > 0, \alpha \geq 0, \delta \geq 0.\)

For \(\delta = 0\), we obtain the following new subclass:

A function \(f \in H_p\) of the form (1.1) is said to be in the class \(L_p(A, B; \mu, \tau, \alpha)\) if and only if
\[
\frac{J_p^\mu f(z)}{z^p} - |\alpha| \frac{J_p^\mu f(z)}{z^p} - 1| < \frac{1 + Az}{1 + Bz}, \tag{1.18}
\]
where \(J_p^\mu f(z)\) is defined by (1.14) and \(p \in \mathbb{N}; A, B \in \mathbb{R}, A \neq B, |B| \leq 1; \tau \in \mathbb{N}, \mu > 0, \alpha \geq 0.\)

We also let
\[
\mathcal{P}_p(A, B; \mu, \tau, \alpha, \delta) = \mathcal{P}_p \cap H_p(A, B; \mu, \tau, \alpha, \delta)
\]
and
\[
\mathcal{L}_p(A, B; \mu, \tau, \alpha) = \mathcal{P}_p \cap L(A, B; \mu, \tau, \alpha).
\]

In this paper, we aim to introduce some new subclasses of harmonic multivalent functions defined by generalized Dziok-Srivastava operator and obtain some results including sufficient coefficient conditions, distortion bounds and extreme points for functions of these classes.

2. Main results

Lemma 2.1 ([8]) Let \(\alpha \geq 0\) and \(A, B \in \mathbb{R}, A \neq B, |B| \leq 1.\) If \(\omega(z)\) is an analytic function with \(\omega(0) = 1\), then we have
\[
\omega(z) - \alpha |\omega(z) - 1| < \frac{1 + Az}{1 + Bz} \iff \omega(z)(1 - a e^{-i\phi}) + a e^{-i\phi} < \frac{1 + Az}{1 + Bz}, \quad \phi \in \mathbb{R}. \tag{2.1}
\]

Using Lemma 2.1 and (1.18), we get that \(f(z) \in H_p(A, B; \mu, \tau, \alpha, \delta)\) if and only if
\[
\chi_{\delta, \mu}(f(z))(1 - a e^{-i\phi}) + a e^{-i\phi} < \frac{1 + Az}{1 + Bz}, \tag{2.2}
\]
where \(\chi_{\delta, \mu}(f(z))\) is given by (1.17).

Theorem 2.2 Let \(f = h + \overline{g}\) be such that \(h\) and \(g\) are given by (1.2). Also, suppose that \(p \in \mathbb{N}, A, B \in \mathbb{R}\) and \(A \neq B, |B| \leq 1.\) If
\[
\sum_{k=p+1}^{\infty} (1 + |B|) (1 + \alpha) (|\xi^\mu_k| |a_k| + |\eta^\mu_k| |b_k|) \leq |A - B|, \tag{2.3}
\]
where
\[
\xi^\mu_k = (1 - \delta + \frac{\delta k}{p}) \Phi^\mu_k \quad \text{and} \quad \eta^\mu_k = (1 - \delta - \frac{\delta k}{p}) \Phi^\mu_k \tag{2.4}
\]
and \(\Phi^\mu_k\) is given by (1.15), then \(f \in H_p(A, B; \mu, \tau, \alpha, \delta).\)

Proof We first show that if the inequality (2.3) holds for the coefficients of \(f = h + \overline{g}\), then the required condition (2.2) is satisfied. In view of (2.2), we need to prove that \(p(z) \prec \frac{1 + Az}{1 + Bz}\), where
\[
p(z) = \chi_{\delta, \mu}(f(z))(1 - a e^{-i\phi}) + a e^{-i\phi}. \tag{2.5}
\]
Using the fact that \( p(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k \), it suffices to show that
\[
|1 - p(z)| - |Bp(z) - A| \leq 0.
\]
(2.6)

Therefore, we get
\[
|1 - p(z)| - |Bp(z) - A| = \left| (1 - \alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} \left[ \frac{\eta_k}{k} a_k z^{k-p} + \frac{\eta_k}{k} b_k z^{-p} \right] \right| -
\]
\[
|B - B(1 - \alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} \left[ \frac{\eta_k}{k} a_k z^{k-p} + \frac{\eta_k}{k} b_k z^{-p} \right] - A|
\]
\[
\leq \left| (1 + \alpha) \sum_{k=p+1}^{\infty} \left[ |\xi_k| |a_k| |z|^{k-p} + |\eta_k| |b_k| |z|^{k-p} \right] \right| -
\]
\[
(\|A - B\| - \|B\|) (1 + \alpha) \sum_{k=p+1}^{\infty} \left[ |\xi_k| |a_k| |z|^{k-p} + |\eta_k| |b_k| |z|^{k-p} \right] - |A - B|
\]
\[
\leq \sum_{k=p+1}^{\infty} (1 + \|B\|) (1 + \alpha) \left[ |\xi_k| |a_k| + |\eta_k| |b_k| \right] - |A - B| \leq 0.
\]

By hypothesis the last expression is non-positive. Thus the proof is completed. The coefficient bound (2.3) is sharp for the function
\[
f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{|A - B|}{(1 + |B|)(1 + \alpha)} \left( \frac{1}{|\xi_k|} Y_k z^k + \frac{1}{|\eta_k|} \sum_{i=1}^{\infty} Y_i z^i \right),
\]
where \( \sum_{k=p+1}^{\infty} (|X_k| + |Y_k|) = 1 \).

**Corollary 2.3** Let \( f = h + \eta \) be such that \( h \) and \( g \) are given by (1.2), \( \xi_k \) and \( \eta_k \) are given by (2.4). Also, suppose that \( p \in N \) and \( A, B \in R \). Then,

(i) For \(-1 \leq B < A \leq 1, B \leq 0\), if
\[
\sum_{k=p+1}^{\infty} (1 - B) (1 + \alpha) \left( |\xi_k| |a_k| + |\eta_k| |b_k| \right) \leq A - B,
\]
then \( f \in H_p(A, B; \mu, \tau, \alpha, \delta) \).

(ii) For \(-1 \leq A < B \leq 1, B > 0\), if
\[
\sum_{k=p+1}^{\infty} (1 + B) (1 + \alpha) \left( |\xi_k| |a_k| + |\eta_k| |b_k| \right) \leq B - A,
\]
then \( f \in H_p(A, B; \mu, \tau, \alpha, \delta) \).

**Corollary 2.4** Let \( f = h + \eta \) be such that \( h \) and \( g \) are given by (1.2). Also, suppose that \( p \in N, A, B \in R \) and \( A \neq B, |B| \leq 1 \). If
\[
\sum_{k=p+1}^{\infty} (1 + |B|) (1 + \alpha) \left( |\xi_k| |a_k| + |\eta_k| |b_k| \right) \leq |A - B|,
\]
where $\Phi_k^0$ is given by (1.15), then $f \in L_p(A, B; \mu, \tau, \alpha)$. 

**Theorem 2.5** Let $f = h + g$ be such that $h$ and $g$ are given by (1.2), $\xi_k^\mu$ and $\eta_k^\mu$ are given by (2.4). Also, suppose that $p \in \mathbb{N}$, $A, B \in \mathbb{R}$ and $A \neq B$, $|B| \leq 1$, $0 \leq \delta < \frac{B}{|B|+1}$. Then 

(i) For $-1 \leq B < A \leq 1$, $B < 0$, $f \in \overline{P}_p(A, B; \mu, \tau, \alpha, \delta)$ if and only if 

$$
\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(\xi_k^\mu|a_k| + \eta_k^\mu|b_k|) \leq A - B. \quad (2.8)
$$

(ii) For $-1 \leq A < B \leq 1$, $B > 0$, $f \in \overline{P}_p(A, B; \mu, \tau, \alpha, \delta)$ if and only if 

$$
\sum_{k=p+1}^{\infty} (1+B)(1+\alpha)(\xi_k^\mu|a_k| + \eta_k^\mu|b_k|) \leq B - A. \quad (2.9)
$$

**Proof** Since $\overline{P}_p(A, B; \mu, \tau, \alpha, \delta) \subset H_p(A, B; \mu, \tau, \alpha, \delta)$. According to Corollary 2.3, we only need to prove the “only if” part of the theorem. 

(i) Let $f \in \overline{P}_p(A, B; \mu, \tau, \alpha, \delta)$, $-1 \leq B < A \leq 1$, $B < 0$. Then 

$$
\frac{1}{Bp(z) - A} < 1, \quad (2.10)
$$

where $p(z)$ is defined by (2.5). Clearly, (2.10) is equivalent to 

$$
\left| \frac{(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} (\xi_k^\mu a_k z^{k-p} + \eta_k^\mu b_k z^{-p}z^k)}{B - B(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} (\xi_k^\mu a_k z^{k-p} + \eta_k^\mu b_k z^{-p}z^k) - A} \right| < 1. \quad (2.11)
$$

From (2.11), we have 

$$
\left\{ \frac{(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} (\xi_k^\mu a_k z^{k-p} + \eta_k^\mu b_k z^{-p}z^k)}{A - B + B(1-\alpha e^{-i\phi}) \sum_{k=p+1}^{\infty} (\xi_k^\mu a_k z^{k-p} + \eta_k^\mu b_k z^{-p}z^k)} \right\} < 1. \quad (2.12)
$$

Taking $z = r \ (0 < r < 1)$ and $\phi = \pi$, then (2.12) gives 

$$
\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)(\xi_k^\mu|a_k| + \eta_k^\mu|b_k|) r^{k+p} \leq A - B. \quad (2.13)
$$

Letting $r \to 1$ in (2.13), we will get (2.8). 

(ii) Similar to the proof of (2.8), we can prove (2.9). \square 

**Corollary 2.6** Let $f = h + g$ be such that $h$ and $g$ are given by (1.2), $\Phi_k^0$ is given by (1.15). Also, suppose that $p \in \mathbb{N}$, $A, B \in \mathbb{R}$ and $A \neq B$, $|B| \leq 1$. Then 

(i) For $-1 \leq B < A \leq 1$, $B < 0$, $f \in \overline{L}(A, B; \mu, \tau, \alpha)$ if and only if 

$$
\sum_{k=p+1}^{\infty} (1-B)(1+\alpha)\Phi_k^\mu(|a_k| + |b_k|) \leq A - B. 
$$

(ii) For $-1 \leq A < B \leq 1$, $B > 0$, $f \in \overline{L}(A, B; \mu, \tau, \alpha)$ if and only if 

$$
\sum_{k=p+1}^{\infty} (1+B)(1+\alpha)\Phi_k^\mu(|a_k| + |b_k|) \leq B - A. 
$$
Theorem 2.7 Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by (1.3), \( \xi_k^\mu \) and \( \eta_k^\mu \) are given by (2.4). Also, suppose that \( \mu > 1, 0 \leq \delta < \frac{p}{2p+1} \). Then

(i) For \( -1 \leq B < A \leq 1, B < 0 \), if \( f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta) \), then

\[
 r^p - \frac{A - B}{(1 - B)(1 + \alpha)} \eta_{p+1}^{\mu} r^{p+1} \leq |f(z)| \leq r^p + \frac{A - B}{(1 - B)(1 + \alpha)} \eta_{p+1}^{\mu} r^{p+1}. \tag{2.14}
\]

(ii) For \( -1 \leq A < B \leq 1, B > 0 \), if \( f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta) \), then

\[
 r^p - \frac{B - A}{(1 + B)(1 + \alpha)} \eta_{p+1}^{\mu} r^{p+1} \leq |f(z)| \leq r^p + \frac{B - A}{(1 + B)(1 + \alpha)} \eta_{p+1}^{\mu} r^{p+1}. \tag{2.15}
\]

Proof Since \( f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta) \), by using Theorem 2.5, we have

\[
 (1 - B)(1 + \alpha) \eta_{p+1}^\mu \sum_{k=p+1}^\infty (|a_k| + |b_k|) \leq \sum_{k=p+1}^\infty (1 - B)(1 + \alpha)(\xi_k^\mu |a_k| + \eta_k^\mu |b_k|) \leq A - B, \tag{2.16}
\]

which implies that

(i) If \( -1 \leq B < A \leq 1 \) and \( B < 0 \), then from (2.16) we obtain

\[
 \sum_{k=p+1}^\infty (|a_k| + |b_k|) \leq \frac{A - B}{(1 - B)(1 + \alpha) \eta_{p+1}^\mu}. \tag{2.17}
\]

On the other hand,

\[
 |f(z)| \leq r^p + \sum_{k=p+1}^\infty (|a_k| + |b_k|) r^k \leq r^p + r^{p+1} \sum_{k=p+1}^\infty (|a_k| + |b_k|)
\]

\[
 \leq r^p + \frac{A - B}{(1 - B)(1 + \alpha) \eta_{p+1}^\mu} r^{p+1}
\]

and

\[
 |f(z)| \geq r^p - \frac{A - B}{(1 - B)(1 + \alpha) \eta_{p+1}^\mu} r^{p+1}.
\]

Hence (2.14) follows. The case for (ii) \(-1 \leq A < B \leq 1 \) and \( B > 0 \) can be proved in the same manner and hence we omit it. \( \square \)

Corollary 2.8 Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by (1.3), \( \xi_k^\mu \) and \( \eta_k^\mu \) are given by (2.4). Also, suppose that \( \mu > 1, 0 \leq \delta < \frac{p}{2p+1} \). Then

(i) For \( -1 \leq B < A \leq 1, B < 0 \), if \( f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta) \), then

\[
 \{ w : |w| < 1 - \frac{A - B}{(1 - B)(1 + \alpha) \eta_{p+1}^\mu} \} \subset f(U).
\]

(ii) For \( -1 \leq A < B \leq 1, B > 0 \), if \( f \in \overline{H}_p(A, B; \mu, \tau, \alpha, \delta) \), then

\[
 \{ w : |w| < 1 - \frac{B - A}{(1 + B)(1 + \alpha) \eta_{p+1}^\mu} \} \subset f(U).
\]

Corollary 2.9 Let \( f = h + \overline{g} \) be such that \( h \) and \( g \) are given by (1.3), \( \Phi_k^\mu \) is given by (1.15). Also, suppose that \( |z| = r < 1, \mu > 1 \). Then

(i) For \( -1 \leq B < A \leq 1, B < 0 \), if \( f \in \overline{T}_p(A, B; \mu, \tau, \alpha) \), then

\[
 r^p - \frac{A - B}{(1 - B)(1 + \alpha) \Phi_{p+1}^\mu} r^{p+1} \leq |f(z)| \leq r^p + \frac{A - B}{(1 - B)(1 + \alpha) \Phi_{p+1}^\mu} r^{p+1}.
\]
Putting

Consequently, using Theorem 2.5, we have

\[ f \leq \frac{B - A}{(1 + B)(1 + \alpha)} \Phi_{p+1}^p r^{p+1} \leq |f(z)| \leq r^p + \frac{B - A}{(1 + B)(1 + \alpha)} \Phi_{p+1}^p r^{p+1}. \]

**Theorem 2.10** Let \( f = h + g \) be such that \( h \) and \( g \) are given by (1.2), \( \xi_k^p \) and \( \eta_k^p \) are given by (2.4). Also, suppose that \( p \in \mathbb{N}, A, B \in \mathbb{R} \) and \( A \neq B, |B| \leq 1, 0 \leq \delta < \frac{p}{2p+1} \). Then \( f \in \text{cleo} \mathcal{P}_p(A, B; \mu, \tau, \alpha, \delta) \) if and only if

\[ f(z) = \sum_{k=p}^{\infty} X_k h_k + \sum_{k=p+1}^{\infty} Y_k (h_k + g_k), \quad z \in U^*, \quad \text{(2.18)} \]

where

\[ h_p = z^p, \]

\[ h_k = \begin{cases} z^p - \frac{A-B}{(1-B)(1+\alpha)} \xi_k^p z^k, & k \geq p + 1, -1 \leq B < A \leq 1, B < 0, \\
\end{cases} \]

\[ g_k = \begin{cases} -\frac{A-B}{(1-B)(1+\alpha)} \eta_k^p z^k, & k \geq p + 1, -1 \leq B < A \leq 1, B > 0, \\
\end{cases} \]

\[ g_k = \begin{cases} z^p - \frac{A-B}{(1-B)(1+\alpha)} \eta_k^p z^k, & k \geq p + 1, -1 \leq A < B \leq 1, B > 0, \\
\end{cases} \]

and

\[ X_p = 1 - \sum_{k=p+1}^{\infty} (X_k + Y_k), \quad X_k \geq 0, Y_k \geq 0. \]

In particular, the extreme points of \( \mathcal{P}_p(A, B; \mu, \tau, \alpha) \) are \( h_k \) and \( g_k \).

**Proof** Let \(-1 \leq B < A \leq 1, B < 0\). We get

\[ f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{A-B}{(1-B)(1+\alpha)} \left( \frac{1}{\xi_k^p} X_k z^k + \frac{1}{\eta_k^p} Y_k z^k \right). \quad \text{(2.19)} \]

Since \( 0 \leq X_k \leq 1, \quad (k = p + 1, \ldots), \) we obtain

\[ \sum_{k=p+1}^{\infty} \frac{(1-B)(1+\alpha)}{A-B} \frac{A-B}{(1-B)(1+\alpha)} \xi_k^p X_k + \frac{(1-B)(1+\alpha)}{A-B} \frac{A-B}{(1-B)(1+\alpha)} \eta_k^p Y_k \]

\[ = \sum_{k=p+1}^{\infty} (X_k + Y_k) = 1 - X_p \leq 1. \]

Consequently, using Theorem 2.5, we have \( f \in \mathcal{P}_p(A, B; \mu, \tau, \alpha, \delta) \).

Conversely, if \( f \in \mathcal{P}_p(A, B; \mu, \tau, \alpha, \delta) \), then

\[ |a_k| \leq \frac{A-B}{(1-B)(1+\alpha)} \xi_k^p, \quad |b_k| \leq \frac{A-B}{(1-B)(1+\alpha)} \eta_k^p. \quad \text{(2.20)} \]

Putting

\[ X_k = \frac{(1-B)(1+\alpha)}{A-B} |a_k|, \quad Y_k = \frac{(1-B)(1+\alpha)}{A-B} |b_k| \quad \text{(2.21)} \]
and $X_p = 1 - \sum_{k=p+1}^{\infty} (X_k + Y_k) \geq 0$, we obtain

$$f(z) = z^p - \sum_{k=p+1}^{\infty} |a_k|z^k - \sum_{k=p+1}^{\infty} |b_k|z^k$$

$$= (X_p + \sum_{k=p+1}^{\infty} (X_k + Y_k))z^p - \sum_{k=p+1}^{\infty} A - B (1 - B)(1 + \alpha)\xi_k z^k X_k z^k -$$

$$\sum_{k=p+1}^{\infty} A - B (1 - B)(1 + \alpha)\eta_k Y_k z^k$$

$$= X_k z^p + \sum_{k=p+1}^{\infty} h_k(z) X_k + \sum_{k=p+1}^{\infty} (z^p + g_k(z)) Y_k$$

$$= X_p h_p + \sum_{k=p+1}^{\infty} h_k X_k + \sum_{k=p+1}^{\infty} (h_p + g_k) Y_k$$

$$= \sum_{k=p}^{\infty} h_k X_k + \sum_{k=p+1}^{\infty} (h_p + g_k) Y_k.$$  

Thus $f$ can be expressed in the form (2.18). The case for $-1 \leq A < B \leq 1, B > 0$ can be proved in the same manner and hence we omit it. □

**Corollary 2.11** Let $f = h + g$ be such that $h$ and $g$ are given by (1.2), $\Phi_k^p$ is given by (1.15). Also, suppose that $p \in \mathbb{N}, A, B \in \mathbb{R}$ and $A \neq B, |B| \leq 1$. Then $f \in \text{clco}(\mathcal{T}_p(A, B; \mu, \tau, \alpha))$ if and only if

$$f(z) = \sum_{k=p}^{\infty} X_k h_k + \sum_{k=p+1}^{\infty} Y_k (h_p + g_k), \quad z \in U^*,$$

where

$$h_p = z^p,$$

$$h_k = \begin{cases} z^p - \frac{A - B}{(1 - B)(1 + \alpha)}\Phi_k^p z^k, & k \geq p + 1, -1 \leq B < A \leq 1, B < 0, \\ z^p - \frac{B - A}{(1 + B)(1 + \alpha)}\Phi_k^p z^k, & k \geq p + 1, -1 \leq A < B \leq 1, B > 0, \\ - \frac{A - B}{(1 - B)(1 + \alpha)}\Phi_k^p z^k, & k \geq p + 1, -1 \leq B < A \leq 1, B < 0, \\ - \frac{B - A}{(1 + B)(1 + \alpha)}\Phi_k^p z^k, & k \geq p + 1, -1 \leq A < B \leq 1, B > 0, \end{cases}$$

and

$$X_p \equiv 1 - \sum_{k=p+1}^{\infty} (X_k + Y_k).$$

In particular, the extreme points of $\mathcal{T}_p(A, B; \mu, \tau, \alpha)$ are $h_k$ and $g_k$.

**Theorem 2.12** The class $\mathcal{F}_p(A, B; \mu, \tau, \alpha, \delta)$ ($0 \leq \delta < \frac{p}{p+2}$) is closed under convex combinations.
Proof For $j = 1, 2$, let the functions $f_j$ given by
\[ f_j(z) = z^p - \sum_{k=p+1}^{\infty} |a_{jk}|z^k - \sum_{k=p+1}^{\infty} |b_{jk}|z^k, \tag{2.22} \]
be in the class $\mathcal{H}_p(A, B; \mu, \tau, \alpha, \delta)$.

For $\lambda_j, \sum_{j=1}^{\infty} \lambda_j = 1$, the convex combinations can be expressed in the form
\[ \sum_{j=1}^{\infty} \lambda_j f_j = z^p - \sum_{k=p+1}^{\infty} (\sum_{j=1}^{\infty} \lambda_j |a_{jk}|)z^k - \sum_{k=p+1}^{\infty} (\sum_{j=1}^{\infty} \lambda_j |b_{jk}|)z^k. \tag{2.23} \]

(i) For $-1 < B < A \leq 1, B < 0$, from (2.8), (2.22) and (2.23), we get
\[ \sum_{k=p+1}^{\infty} (1 - B)(1 + \alpha) \left( \sum_{j=1}^{\infty} \lambda_j (\xi^p_k |a_{jk}| + \eta^p_k |b_{jk}|) \right) \]
\[ = \sum_{j=1}^{\infty} \lambda_j \left[ \sum_{k=p+1}^{\infty} (1 - B)(1 + \alpha) (\xi^p_k |a_{jk}| + \eta^p_k |b_{jk}|) \right] \]
\[ \leq \sum_{j=1}^{\infty} \lambda_j (A - B) = A - B. \]

That is, $\sum_{j=1}^{\infty} \lambda_j f_j \in \mathcal{H}_p(A, B; \mu, \tau, \alpha, \delta)$. The case for (ii) $-1 \leq A < B \leq 1, B > 0$ can be proved in the same manner and hence we omit it. \( \Box \)

Corollary 2.13 The class $\mathcal{H}_p(A, B; \mu, \tau, \alpha, \delta)$ is closed under convex combinations.

References