Algebraic Properties of Dual Toeplitz Operators on Harmonic Hardy Space over Polydisc

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Abstract In this paper, we introduce the harmonic Hardy space on \( \mathbb{T}^n \) and study some algebraic properties of dual Toeplitz operator on the harmonic Hardy space on \( \mathbb{T}^n \).

Keywords Hardy space; Toeplitz operator; spectrum; semi-commutative

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1. Introduction

Let \( \mathbb{T} \) be the unit circle on the complex plane \( \mathbb{C} \). For a fixed positive integer \( n \geq 1 \), the \( \mathbb{C}^n \) and \( \mathbb{T}^n \) are the cartesian products of \( n \) copies of \( \mathbb{C} \) and \( \mathbb{T} \), respectively. Denote by \( z = (z_1, \ldots, z_n) \) the coordinates on \( \mathbb{C}^n \). Let \( d\sigma \) be the normalized Haar measure on \( \mathbb{T}^n \) and \( L^2(\mathbb{T}^n) \) be the square integral functions with respect to \( d\sigma \). The Hardy space \( H^2(\mathbb{T}^n) \) is the closure of analytic polynomials in \( L^2(\mathbb{T}^n) \), that is

\[
H^2(\mathbb{T}^n) = \overline{\text{clos}\{p(z_1, \ldots, z_n) : p \text{ is analytic polynomials}\}}
\]

In the setting of classical Hardy space on \( \mathbb{T} \), it is well known that \( H^2(\mathbb{T}) + \overline{H^2(\mathbb{T})} = L^2(\mathbb{T}) \), where \( \overline{\cdot} \) is the complex conjugate. However, in the higher dimension \( (n \geq 2) \), the situation is completely different, indeed, \( H^2(\mathbb{T}^n) + \overline{H^2(\mathbb{T}^n)} \) is much smaller than \( L^2(\mathbb{T}^n) \). Denote

\[
h^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) + \overline{H^2(\mathbb{T}^n)}
\]

and call it the harmonic Hardy space on \( \mathbb{T}^n \). The reader may not confuse that \( h^2(\mathbb{T}^n) \) does not contain all harmonic functions. In the whole paper, \( P \) denotes the orthogonal projection from \( L^2(\mathbb{T}^n) \) onto \( h^2(\mathbb{T}^n) \) and \( Q = 1 - P \).

Let \( L^\infty(\mathbb{T}^n) \) be the set of essentially bounded measurable functions on \( \mathbb{T}^n \). For \( \varphi \in L^\infty(\mathbb{T}^n) \), the Toeplitz operator \( T_\varphi \) on \( h^2(\mathbb{T}^n) \) is defined by

\[
T_\varphi f = P(\varphi f), \quad f \in h^2(\mathbb{T}^n).
\]

The Toeplitz operators on analytic and harmonic function spaces have been widely studied [1].
The Hankel operator and dual Toeplitz operator can also be defined as follows:

\[ H_\varphi : h^2(T^n) \rightarrow (h^2(T^n))^\perp, \]
\[ H_\varphi f = Q(\varphi f), \quad f \in h^2(T^n). \]

\[ S_\varphi : (h^2(T^n))^\perp \rightarrow (h^2(T^n))^\perp, \]
\[ S_\varphi f = Q(\varphi f), \quad f \in h^2(T^n)^\perp. \]

One can check that \( H_\varphi^* f = P(\varphi f), \quad f \in h^2(T^n)^\perp, \) and
\[ \| S_\varphi(f) \|_2 = \| Q(\varphi f) \|_2 \leq \| \varphi f \|_2 \leq \| \varphi \|_\infty \| f \|_2, \]
where \( \| \cdot \|_\infty \) is the essential sup norm and \( \| \cdot \|_2 \) is the norm of \( L^2(T^n) \). The following algebraic properties of dual Toeplitz operators are also easy to check. For \( \varphi, \psi \in L^\infty(T^n), \alpha, \beta \in \mathbb{C}, \) we have
\[ S_\alpha \varphi + \beta \psi = \alpha S_\varphi + \beta S_\psi. \]


The Toeplitz operator, Hankel operator and dual Toeplitz operator have close relationships through the multiplication operators on \( L^2(T^n) \). Under the decomposition
\[ L^2(T^n) = h^2(T^n) \oplus (h^2(T^n))^\perp, \]
the multiplication operator \( M_\varphi, \varphi \in L^\infty(T^n) \) can be represented as follows
\[ M_\varphi = \begin{pmatrix} T_\varphi & H_\varphi \\ H_\varphi^* & S_\varphi \end{pmatrix}. \]

For \( \varphi, \psi \in L^\infty(T^n), \) the identity \( M_\varphi M_\psi = M_{\varphi \psi} = M_\psi M_\varphi \) implies that
\[ S_{\varphi \psi} = S_\varphi S_\psi + H_\varphi H_\psi^* = S_\psi S_\varphi + H_\psi H_\varphi^*. \]
Equation (1) will be used frequently.

2. Properties of spectrum

Characterizations of spectrum is one of important properties for bounded linear operators.

For \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_n^+, \) we denote \( z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}. \)

Lemma 2.1 Let \( \psi \in C(T^n), \) where \( C(T^n) \) is the set of continuous functions on \( T^n. \) Then for
Proof Firstly, let us show that
\[
\frac{1}{\lambda_m} \int_{\mathbb{T}^n} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \rightarrow 0, \quad m \rightarrow \infty,
\]
for any neighborhood of $z$ with following form,
\[
V_\varepsilon(z) = \{ \zeta \in \mathbb{T}^n : |1 - z^{\alpha} \langle \zeta, z \rangle | < \varepsilon \}.
\]
On $\mathbb{T}^n \setminus V_\varepsilon(z)$,
\[
|z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} \leq \sup_{\zeta \in \mathbb{T}^n \setminus V_\varepsilon(z)} |1 + z^{\alpha} \langle \zeta, z \rangle | = \rho < 2,
\]
whence
\[
\int_{\mathbb{T}^n \setminus V_\varepsilon(z)} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \leq \rho^{2m} \sigma(\mathbb{T}^n \setminus V_\varepsilon(z)).
\]
If $0 < \delta < 2 - \rho < 1$, considering another neighborhood $V_\varepsilon(\delta)$ of $z$, we can get
\[
\lambda_m \geq \int_{V_\varepsilon(\delta)} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \geq (1 - \delta)^2 (2 - \delta)^{2m} \sigma(V_\delta).
\]
From inequalities (3)–(5), we see that, as $m \rightarrow \infty$,
\[
\frac{1}{\lambda_m} \int_{\mathbb{T}^n \setminus V_\varepsilon(z)} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \leq c \left( \frac{\rho}{2 - \delta} \right)^{2m} \rightarrow 0.
\]
Therefore,
\[
\frac{1}{\lambda_m} \int_{V_\varepsilon(z)} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \rightarrow 1, \quad m \rightarrow \infty.
\]
Since $\lambda_m$ is independent of $z$ in $\mathbb{T}^n$, we have
\[
\left| \frac{1}{\lambda_m} \int_{\mathbb{T}^n} \psi(\zeta) \left| z^{\alpha} \langle \zeta, z \rangle \right|^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) - \psi(z) \right|
\leq \sup_{\zeta \in \mathbb{T}^n \setminus V_\varepsilon(z)} |\psi(\zeta) - \psi(z)| \left| \frac{1}{\lambda_m} \int_{\mathbb{T}^n \setminus V_\varepsilon(z)} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \right|
+ \sup_{\zeta \in V_\varepsilon(z)} |\psi(\zeta) - \psi(z)| \left| \frac{1}{\lambda_m} \int_{V_\varepsilon(z)} |z^{\alpha} \langle \zeta, z \rangle |^2 |1 + z^{\alpha} \langle \zeta, z \rangle |^{2m} d\sigma(\zeta) \right|.
\]
The continuity of $\psi$ with conclusion (2) and (6) yields the desired result. □

Lemma 2.2 Let $\varphi \in L^\infty(\mathbb{T}^n)$. If $S_{\varphi}$ is invertible in $(h^2(\mathbb{T}^n))^\perp$, then $\varphi$ is invertible in $L^\infty(\mathbb{T}^n)$.

Proof The assumption that $S_{\varphi}$ is invertible implies that there is a constant $k > 0$ satisfying $\|S_{\varphi} f\| \geq k \| f \|$, $f \in (h^2(\mathbb{T}^n))^\perp$. Considering the projection has norm 1, we can see
\[
\| \varphi f \| \geq k \| f \|, \quad f \in (h^2(\mathbb{T}^n))^\perp.
\]
Particularly, for
\[ f(z) = z^{\alpha} \frac{\langle \zeta, \lambda \rangle}{n+1} (1 + z^{\alpha} \frac{\langle \zeta, \lambda \rangle}{n+1})^m, \quad \zeta \in \mathbb{T}^n, \ m \geq 1, \ \alpha = (2, 0, \ldots, 0), \]
which is clearly an element of \((h^2(\mathbb{T}^n))^\perp\), the inequality (7) yields
\[ \int_{\mathbb{T}^n} |\varphi(z)|^2 |z^{\alpha} \frac{\langle \zeta, \lambda \rangle}{n+1}|^2 |1 + z^{\alpha} \frac{\langle \zeta, \lambda \rangle}{n+1}|^{2m} d\sigma(z) \geq k^2 \lambda_m. \]
It follows that for any nonnegative \(\psi \in C(\mathbb{T}^n)\), one has
\[ \frac{1}{\lambda_m} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} |\varphi(z)|^2 \psi(\zeta) \left| z^{\alpha} \frac{\langle \zeta, \lambda \rangle}{n+1} \right|^2 |1 + z^{\alpha} \frac{\langle \zeta, \lambda \rangle}{n+1}|^{2m} d\sigma(z) d\sigma(\zeta) \geq k^2 \int_{\mathbb{T}^n} \psi(\zeta) d\sigma(\zeta). \]
Hence, invoking Lemma 2.1, we obtain
\[ \int_{\mathbb{T}^n} (|\varphi(z)|^2 - k^2) \psi(z) d\sigma(z) \geq 0, \]
which implies that \(|\varphi(z)| \geq k > 0\) a.e., in \(\mathbb{T}^n\), hence \(\varphi(z)\) is invertible in \(L^\infty(\mathbb{T}^n)\). □

An immediate consequence is the following spectral inclusion theorem. But firstly, let us denote by \(R(f)\) the essential range of the essentially bounded function \(f\), and by \(\sigma(T), r(T)\) respectively the spectrum and the spectral radius of an operator \(T\).

**Theorem 2.3** If \(\varphi\) is in \(L^\infty(\mathbb{T}^n)\), then \(R(\varphi) \subseteq \sigma(S_\varphi)\).

### 3. Semi-commuting dual Toeplitz operator

When will the product of two dual Toeplitz operators be semi-commutative? This question has been solved well in Hardy space and harmonic Bergman space [2–5]. But the crucial question is what are the conditions we need if the product of two dual Toeplitz operators with the special form we have discussed is a dual Toeplitz operator in harmonic Hardy space. The following several theorems have been observed.

Equation (1) suggests that \(S_\varphi\) and \(S_\psi\) commute if \(\varphi\) or \(\psi\) is constant, that is \(H^*_\varphi = 0\) or \(H^*_\psi = 0\). If a non-trivial linear combination of \(\varphi\) and \(\psi\) is constant, they do commute as well.

**Lemma 3.1** If \(\varphi \in H^\infty(\mathbb{D}^n)\), then \(H^*_\varphi((h^2)^\perp) \subseteq H^2, \ H^*_\varphi((h^2)^\perp) \subseteq \overline{H^2}.\)

**Proof** Let \(f = \sum_{k=1}^\infty \sum_{|\alpha|,|\beta|=k} a_{\alpha,\beta} z^\alpha \overline{\beta}^{\beta}\), the parameters \(\alpha, \beta \in \mathbb{Z}^n_+.\) To satisfy the condition \(f \in (h^2)^\perp\), for any positive integer \(k\), there exist \(i\) and \(j\) such that \(\alpha_i > \beta_i, \alpha_j < \beta_j\). Let \(H^*_\varphi(f) = g + h\), \(g\) be analytic, and \(h\) be co-analytic, then we can get \(h\) is constant (If not, then \(\alpha_i + c \leq \beta_i, c \geq 0\), this is a contradiction), which yields the first desired result. Similarly, the second result can be proved. □

Combining this lemma with Eq. (1), we get the following proposition.

**Proposition 3.2** If the symbols \(\varphi\) and \(\psi\) are both analytic or co-analytic, then the dual Toeplitz operators \(S_\varphi\) and \(S_\psi\) are commutative, i.e., \(S_\varphi S_\psi = S_\psi S_\varphi\).
Proof If \( \varphi \) and \( \psi \) are both analytic, by Lemma 3.1, for all \( f \in (h^2(\mathbb{T}^2))^\perp \), there exists \( g \in H^2 \) such that \( H_f H_\varphi^* (f) = H_\varphi (g) \). Since the product of two analytic functions is still analytic, \( H_\varphi (g) = Q(\varphi g) \equiv 0 \), namely, \( H_\varphi H_\varphi^* \equiv 0 \), which is equivalent to the fact that \( S_{\varphi \psi} = S_{\varphi} S_{\psi} = S_{\psi} S_{\varphi} \). The same result also can be proved when \( \varphi \) and \( \psi \) are both co-analytic through a similar method. \( \square \)

**Theorem 3.3** Suppose that \( \varphi = f + \pi^{m_1} \pi^{n_1}, \psi = g + \pi^{m_2} \pi^{n_2} \), where \( f, g \in H^\infty(\mathbb{D}^2) \), and the parameters \( m_1, m_2, n_1, n_2 \) are non-zero positive integers. Then \( S_{\varphi} S_{\psi} = S_{\varphi \psi} \) if and only if both \( f \) and \( g \) are constants.

**Proof** By the definition of dual Toeplitz operator, we can get that

\[
S_{\varphi} S_{\psi} = S_{f} S_{g} + S_{f} S_{m_2 n_2} + S_{m_1 n_1} S_{g} + S_{m_1 n_1} S_{m_2 n_2} = S_{\varphi \psi} = S_{f} S_{g} + f S_{m_2 n_2} + g S_{m_1 n_1} + S_{m_2 n_2} S_{m_1 n_1}.
\]

(8)

Equation (1) and Proposition 3.2 can reduce Eq. (8) to the following formula:

\[
S_{f} S_{m_2 n_2} = H_f H_{\varphi} \pi^{m_1 n_1} S_{g} + S_{m_1 n_1} S_{g} - H_{\pi^{m_1 n_1}} H_f^* S_{\varphi} \pi^{m_2 n_2}
\]

(9)

which is equivalent to

\[
H_f H_{\pi^{m_2 n_2}} + H_{\pi^{m_1 n_1}} H_f^* = 0.
\]

(10)

For the reason of the Eqs. (8)–(10), \( S_{\varphi} \) and \( S_{\psi} \) are semi-commutative if and only if Eq. (10) is set up on \( (h^2(\mathbb{T}^2))^\perp \). Let

\[
f = \sum_{i,j \geq 0} a_{ij} z^i w^j, \quad g = \sum_{k,l \geq 0} b_{kl} z^k w^l, \quad (z, w) \in \mathbb{T}^2.
\]

Firstly, assume that \( f \) and \( g \) are both constants. It is clear that \( H_f = H_f^* = H_f^* G_f^* \), and the Eq. (10) is set up, whence \( S_{\varphi} S_{\psi} = S_{\varphi \psi} \). Now we assume that \( S_{\varphi} S_{\psi} = S_{\varphi \psi} \). A little more computation gives that

\[
H_f H_{\pi^{m_2 n_2}}(z^\alpha \pi^{\beta} \pi) = \begin{cases} 0, & \alpha > m_2, \\
\sum_{i,j \geq 0, i \leq m_2 - \alpha} a_{ij} z^i w^{m_2 + n_2 - j} + \sum_{\alpha \leq \beta < m_1 - \alpha} a_{ij} z^i w^{m_2 - \alpha - i} w^{\beta - n_2}, \alpha \leq m_2, \\
\sum_{\beta \leq \beta < m_1 - \alpha} b_{kl} z^i w^{m_1 - \alpha - k} w^{\beta - n_1 - l}, & \alpha > m_1,
\end{cases}
\]

(11)

\[
H_{\pi^{m_1 n_1}} H_f^* (z^\alpha \pi^{\beta} \pi) = \begin{cases} 0, & \beta > n_1, \\
\sum_{i,j \geq 0, i \leq n_2 - \beta} a_{ij} z^i w^{m_2 - \beta - j} + \sum_{\beta \leq \beta < n_2 - \beta} a_{ij} z^i w^{m_2 - \alpha - i} w^{\beta - n_2 + j}, \beta \leq n_2, \\
\sum_{\beta \leq \beta < m_1 - \alpha} b_{kl} z^i w^{m_1 - \alpha - k} w^{\beta - n_1 - l} + \sum_{\alpha \leq \beta < m_1 - \alpha} b_{kl} z^i w^{m_1 - \alpha - k} w^{\beta - n_1 - l}, & \alpha \leq m_1.
\end{cases}
\]

(12)

\[
H_f H_{\pi^{m_2 n_2}}(z^\alpha \pi^{\beta} \pi) = \begin{cases} 0, & \beta > n_2, \\
\sum_{i,j \geq 0, i \leq n_2 - \beta} a_{ij} z^i w^{m_2 - \beta - j} + \sum_{\beta \leq \beta < n_2 - \beta} a_{ij} z^i w^{m_2 - \alpha - i} w^{\beta - n_2 + j}, \beta \leq n_2, \\
\sum_{\beta \leq \beta < m_1 - \alpha} b_{kl} z^i w^{m_1 - \alpha - k} w^{\beta - n_1 - l} + \sum_{\alpha \leq \beta < m_1 - \alpha} b_{kl} z^i w^{m_1 - \alpha - k} w^{\beta - n_1 - l}, & \alpha \leq m_1.
\end{cases}
\]

(13)
\[ H_{\alpha, \beta} \left( z^\alpha w^\beta \right) = \begin{cases} \sum_{m_1 \leq k < \alpha + m_1} b_{kl} z^{\alpha + m_1 - k} w^{\beta - n_1 + l}, & \beta > n_1, \\
\sum_{k > m_1 + l \geq \alpha + m_1} b_{kl} z^{k - \alpha - m_1} \overline{w}^{- \beta - l} \sum_{\alpha < k + l \geq \beta} b_{kl} z^{\alpha + m_1 - k} w^{\beta - n_1 + l}, & \beta \leq n_1. \end{cases} \]

(14)

We distinguish several cases.

**Case 1** \( m_1 = m_2 = m \geq 1, n_1 = n_2 = n \geq 1. \) For \( \beta > n, \) since (13) + (14) = 0, we can get

\[ 0 + \sum_{m_1 \leq k < \alpha + m_1} b_{kl} z^{\alpha + m_1 - k} w^{\beta - n_1 + l} = 0. \]

By the linear independence and the arbitrariness of \( \alpha, \) it follows

\[ b_{kl} = 0, \quad k > 0, \quad l \geq 0, \]

(15)

which means that \( g \) is only about \( w. \) For \( \beta = n, \) (13) + (14) = 0 implies

\[ \sum_{0 \leq i \leq m + \alpha - 1} a_{ij} z^i \overline{w}^j = 0, \]

hence

\[ a_{ij} = 0, \quad i \geq 0, \quad j > 0, \]

(16)

which means that \( f \) is only about \( z. \)

If \( \alpha = m \) in the case (11) + (12) = 0, together with the conclusion (16),

\[ \sum_{j > 0} \sum_{i > 0} a_{i0} z^i \overline{w}^j = 0, \]

hence

\[ a_{i0} = 0, \quad i \geq 1. \]

(17)

If \( \alpha > m \) in the case (11) + (12) = 0, together with the conclusion (16), then by the same way, we can get

\[ 0 + \sum_{k > 0} b_{kl} z^{\alpha - m - k} \overline{w}^{\beta - n_1 + l} = 0, \]

so

\[ b_{kl} = 0, \quad l > 0. \]

(18)

Considering the conclusions (15)–(18), both \( f \) and \( g \) are constants.

**Case 2** \( m_1 = m_2 = m \geq 1, n_2 \neq n_1. \) Without loss of generality, we can assume \( n_2 > n_1 \geq 1. \)

For \( \beta > n_2 > n_1, \) it follows from (13) + (14) = 0 that

\[ 0 + \sum_{m_1 \leq k < \alpha + m_1} b_{kl} z^{\alpha + m_1 - k} w^{\beta - n_1 + l} = 0, \]

so

\[ b_{kl} = 0, \quad k > 0, \quad l \geq 0. \]

(19)
For $\beta = n_2 > n_1$,
\[
\sum_{\alpha \leq k < n_2 + \alpha \atop \beta \geq j > 0} a_{ij} z^{\alpha + n + a - i} w^j + \sum_{\alpha \leq k < n_2 + \alpha \atop \beta \geq j > 0} b_{kl} z^{\alpha + m - k} w^{\beta - n_1 + l} = \sum_{\alpha \leq k < n_2 + \alpha \atop \beta \geq j > 0} a_{ij} z^{\alpha + n - i} w^j = 0,
\]
which means that
\[a_{ij} = 0, \quad i \geq 0, \quad j > 0. \tag{20}\]
We also need to consider the condition $(11) + (12) = 0$.

For $\alpha = m$, together with $(19)$ and $(20)$, the equation can be reduced to
\[\sum_{i > 0 \atop j > 0} a_{i0} z^{i \alpha + n_2} + 0 = 0,\]
hence
\[a_{i0} = 0, \quad i > 0. \tag{21}\]
For $\alpha > m \geq 1$,
\[b_{0l} = 0, \quad l > 0. \tag{22}\]
By the conclusions $(19)$–$(22)$, the result that $f$ and $g$ are both constants can be observed immediately.

**Case 3** $m_1 \neq m_2$, $n_1 = n_2 = n \geq 1$. It is easy to complete the proof by the similar way used in Case 2.

**Case 4** $m_1 \neq m_2$, $n_1 \neq n_2$. Without loss of generality, we can assume that $m_2 > m_1 \geq 1$, $n_1 > n_2 \geq 1$. For $\alpha > m_2 > m_1$,
\[0 + \sum_{k \geq 0 \atop 0 \leq \beta \leq n_1} b_{kl} z^{\alpha - m_1 + k} w^{\beta + n_1 - l} = 0,
\]
which means that
\[b_{kl} = 0, \quad k \geq 0, \quad l > 0. \tag{23}\]
For $\alpha = m_2 > m_1$, together with $(23)$, the following equation is checked easily:
\[0 + \sum_{j > 0 \atop 0 \leq \beta \leq n_2} a_{ij} z^{i \beta + n_2 - j} = 0,
\]
which implies that
\[a_{ij} = 0, \quad i > 0, \quad j \geq 0. \tag{24}\]
Through a similar discussion of $\beta$, we can get
\[a_{0j} = 0, \quad j > 0; \quad b_{k0} = 0, \quad k > 0. \tag{25}\]
Therefore, the proof can be completed together with the conclusions $(23)$–$(25)$.

In Theorem 3.3, each parameter should be non-zero positive integer. Next theorem considers the case of $m_1 = m_2 = 0$. 
Theorem 3.4 Suppose that $\varphi = f + w^{n_1}$, $\psi = g + w^{n_2}$, where $f, g \in H^{\infty}(\mathbb{D}^2)$, and the parameters $n_1, n_2$ are non-zero positive integers. Then the following conclusions can be established:

(i) If $n_1 = n_2$, $S_\varphi S_\psi = S_{\varphi \psi}$ if and only if $f$ and $g$ are both only about $z$, and $f + g = a + bz$;

(ii) If $n_1 \neq n_2$, $S_\varphi S_\psi = S_{\varphi \psi}$ if and only if $f = a_0 + a_1 z$, $g = b_0 + b_1 z$.

Proof Let $m_1 = m_2 = 0$ in the equation that (11) + (12) = (13) + (14) = 0. Then the theorem can be proved by a discussion of $\alpha$ and $\beta$ as we have done before in this paper. □

References


